Honam Mathematical J. **27** (2005), No. 4, pp. 561–570

$R(L)$-MAPS AND POSITIVE IMPLICATIVITY IN SUBTRACTION ALGEBRAS

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Abstract. In this paper, we introduce the notion of positive implicative subtraction algebras and study some relations between $R(L)$-maps and positive implicativity in subtraction algebras.

1. Introduction

B. M. Schein ([8]) considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka ([9]) discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. H. Kim and H. S. Kim ([6]) showed that a subtraction algebra is equivalent to an implicative BCK-algebra, and a subtraction semigroup is a special case of a BCI-semigroup which is a generalization of a ring. Y. B. Jun et al. ([4]) introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set and discussed related results.

Received September 25, 2005. Accepted December 14, 2005.

2000 Mathematics Subject Classification : 06F35, 06A06.

Key words and phrases : (positive implicative) subtraction algebra, $R(L)$-maps.

Y. B. Jun and K. H. Kim ([5]) introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also obtained an another condition for an ideal to be a prime/irreducible ideal. In this paper, we introduce the notion of positive implicative subtraction algebras and study some relations between $R(L)$-maps and positive implicativity in subtraction algebras.

2. Preliminaries

A *subtraction algebra* ([8]) is defined as an algebra $(X; -)$ with a binary operation "$-$" satisfying the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [3,4]):

(a1) $(x - y) - y = x - y$.

(a2) $x - 0 = x$ and $0 - x = 0$.

(a3) $(x - y) - x = 0$.

(a4) $x - (x - y) \leq y$.

(a5) $(x - y) - (y - x) = x - y$.

(a6) $x - (x - (x - y)) = x - y$.

(a7) $(x - y) - (z - y) \leq x - z$.

(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1. ([4]) A non-empty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in A$,

(I2) $(\forall x \in X)(\forall y \in A)(x - y \in A \implies x \in A)$.

Lemma 2.2. ([2]) Let $(X; -)$ be a subtraction algebra. Then $(X; -)$ is a poset.

Let $X := (X; -_X, \leq)$ and $Y := (Y; -_Y, \leq')$ be subtraction algebras. A mapping $f : X \to Y$ is called a (*subtraction*) *homomorphism* if $f(x -_X y) = f(x) -_Y f(y)$ for any $x, y \in X$. A homomorphism f is called a *monomorphism* (resp., an *epimorphism*) if it is injective (resp., surjective). Two subtraction algebras X and Y are said to be *isomorphic*, written by $X \cong Y$, if there exists a bijective homomorphism $f : X \to Y$. For any homomorphism $f : X \to Y$, the set $\{x \in X | f(x) = 0_Y\}$ is called the *kernel* of f, denoted by $Ker(f)$, and the set $\{f(x) | x \in X\}$ is called the *image* of f, denoted by Imf.
Note that $f(0_X) = 0_Y$, since $x - x = 0$. If a mapping $f : X \to Y$ is a homomorphism of subtraction algebras, then it is order preserving, i.e., $x \leq y$ implies $0_Y = f(x - y) = f(x) - f(y)$, i.e., $f(x) \leq f(y)$. Define the trivial homomorphism 0 as $0(x) = 0$ for all $x \in X$.

Suppose that $f : X \to Y$ is a homomorphism of subtraction algebras. For any $x, y \in X$, we define $x \sim y$ if and only if $f(x) = f(y)$. Now we prove that $\sim$ is an equivalence relation on X. It is easy to show that $\sim$ is a tolerance relation. If $x \sim y$ and $y \sim z$, then $f(x) = f(y)$ and $f(y) = f(z)$ and so $f(x) = f(z)$. Therefore $x \sim z$, i.e., $\sim$ is transitive. Thus $\sim$ is an equivalence relation on X. Furthermore we have the following:

Lemma 2.3. If $x \sim y$ and $u \sim v$, then $x - u \sim y - v$, hence $\sim$ is a congruence relation on a subtraction algebra X.

Proof. Let $x \sim y$ and $u \sim v$. Then $f(x) = f(y)$ and $f(u) = f(v)$. So $f(x-u) = f(x) - f(u) = f(y) - f(v) = f(y-v)$. Hence $x - u \sim y - v$.□

We denote $[x]_f := \{y \in X | x \sim y\} = \{y \in X | f(x) = f(y)\}$ by the equivalence class of x determined by the homomorphism f. Then $[0]_f = Kerf$. In fact, if $y \in [0]_f = \{y \in X | 0 \sim y\}$, then $f(0) = f(y)$. Since $f(0) = 0'$, $f(y) = 0'$ and so $y \in Kerf$. Conversely, if $y \in Kerf$, then $f(y) = 0'$. Since $f(0) = 0'$, $f(0) = f(y)$ and so $0 \sim y$. Hence $y \in [0]_f$.

Denote $X/f := \{[x]_f | x \in X\}$ and define that

$$[x]_f \ominus [y]_f := [x - y]_f.$$

Since $\sim$ is a congruence relation on X, the operation "$\ominus$" is well-defined. In what follows, we will prove that $(X/f; \ominus)$ is a subtraction algebra. Let $[x]_f, [y]_f$ and $[z]_f \in X/f$. Then we have the following properties:

(1) $[x]_f \ominus ([y]_f \ominus [x]_f) = [x]_f$;

(2) $[x]_f \ominus ([x]_f \ominus [y]_f) = [y]_f \ominus ([y]_f \ominus [x]_f)$;

(3) $([x]_f \ominus [y]_f) \ominus [z]_f = ([x]_f \ominus [z]_f) \ominus [y]_f$.

Summarizing the above facts we have:

Theorem 2.4. Let $f : X \to Y$ be a homomorphism of subtraction algebras. Then X/f is a subtraction algebra with $[0]_f = Kerf$. It is called a *quotient subtraction algebra determined by f*.

3. R-maps and L-maps in subtraction algebras

In this section, we define R-maps and L-maps in subtraction algebras and investigate several properties in subtraction algebras.

Definition 3.1. Let $(X; -)$ be a subtraction algebra. For a fixed $a \in X$, we define a map $R_a : X \to X$ such that $R_a(x) := x - a$ for all $x \in X$, and we call R_a a *right map* on X. The set of all right maps on X is denoted by $\mathbb{R}(X)$. A *left map* is defined by a similar way, and denoted by $\mathbb{L}(X)$.

We define a binary operation on $\mathbb{R}(X)$ as follows: for any $R_a, R_b \in \mathbb{R}(X)$ and for any $x \in X$,

$$(R_a \circ R_b)(x) := R_a(R_b(x)).$$

Definition 3.2. A right map R_a is called an *idempotent* if $R_a \circ R_a = R_a$, i.e., $(x - a) - a = x - a$ for all $x \in X$.

Definition 3.3. A subtraction algebra $(X; -)$ is said to be *positive implicative* if it satisfies for all x, y and $z \in X$,

$$(x - y) - z = (x - z) - (y - z).$$

Example 3.4. Let $X := \{0, 1, 2, 3, \}$ be a set with the following table:

$-$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then $(X; -)$ is a positive implicative subtraction algebra. Also for each $a \in X$, $(x - a) - a = x - a$ for all $x \in X$ and so a right map R_a is an idempotent.

Theorem 3.5. A subtraction algebra X is positive implicative if and only if every right map on X is an endomorphism of X.

Proof. If X is positive implicative, then for each $a \in X$, $(x - y) - a = (x - a) - (y - a)$, i.e., $R_a(x - y) = R_a(x) - R_a(y)$. Hence R_a is an endomorphism. The converse follows immediately. □

Proposition 3.6. Let X be a subtraction algebra. Then $(\mathbb{R}(X);\circ)$ is a commutative semigroup with zero element R_0.

Proof. Let $R_a, R_b, R_c \in \mathbb{R}(X)$. Since $[(R_a \circ R_b) \circ R_c](x) = [R_a \circ (R_b \circ R_c)](x)$ for all $x \in X$, the associative law holds. Since $R_a \circ R_b(x) = R_b \circ R_a(x)$, the commutative law is true. Since $R_a \circ R_0 = R_0 \circ R_a = R_a$, R_0 is a zero element. This completes the proof. □

Let $End(X)$ denote the set of all right maps which is a homomorphism, i.e.,

$$End(X) = \{R_a \in \mathbb{R}(X) \mid R_a \text{ is a homomorphism}\}.$$

Corollary 3.7. Let X be a positive implicative subtraction algebra. Then $(End(X);\circ)$ is a commutative semigroup with zero element R_0.

We define the followings:
(i) $R_a \leq R_b$ if and only if $R_a(x) \leq R_b(x)$ for all $x \in X$;
(ii) $R_a = R_b$ if and only if $R_a \leq R_b$ and $R_b \leq R_a$.
The following properties are true in $\mathbb{R}(X)$.
(I) For $a, b \in X$, $R_a \circ R_b = R_b \circ R_a$ (by (S3));
(II) $R_a \circ R_a = R_a$;
(III) $R_0 \circ R_a = R_a = R_a \circ R_0$;
(IV) If $a \leq b$, then $R_b \leq R_a$;
(V) If $a \leq b$, then $R_a \circ R_b = R_b$.

Theorem 3.8. Let X be a positive implicative subtraction algebra. Then $(End(X);\leq,\circ)$ is a semi-lattice.

Proof. It is easy to show that $End(X)$ is a partially ordered set. In addition to properties (I), (II) and the associativity, the property required to form a semi-lattice is:

$$R_x \leq R_y \qquad \text{if and only if} \qquad R_x \circ R_y = R_x.$$

If $R_x \leq R_y$, then $R_x(t) \leq R_y(t)$ for all $t \in X$ and so $t - x \leq t - y$. By (a9), we have $(t - x) - x \leq (t - y) - x$. Using (a1), we obtain $t - x \leq (t - y) - x$ and so $R_x(t) \leq R_x(R_y(t)) = R_x \circ R_y(t)$. This means $R_x \leq R_x \circ R_y$. By (a3), we know that $(t - x) - y \leq t - x$ and so $(R_x \circ R_y)(t) \leq R_x(t)$. Hence $R_x \circ R_y \leq R_x$.

Conversely, if $R_x = R_x \circ R_y$, then $t-x = (t-x)-y = (t-y)-(x-y) \leq t - y$, since X is positive implicative. Hence $R_x(t) \leq R_y(t)$. This completes the proof. □

Proposition 3.9. Let $f : X \to Y$ be a homomorphism of subtraction algebra. Then f is injective if and only if $Ker f = \{0\}$.

Proof. Straightforward. □

For a positive implicative subtraction algebra X we define an operation "$\ominus$" on $\mathbb{L}(X)$ as follows: for any $L_a, L_b \in \mathbb{L}(X)$ and any $x \in X$,

$$(L_a \ominus L_b)(x) := L_a(x) - L_b(x).$$

Using positive implicativity of X we know

$$(L_a \ominus L_b)(x) = L_a(x) - L_b(x) = (a - x) - (b - x) = (a - b) - x = L_{a-b}(x).$$

So $L_a \ominus L_b \in \mathbb{L}(X)$.

The next theorem gives a characterization of a positive implicative subtraction algebra by its left maps.

Theorem 3.10. If X is a positive implicative subtraction algebra, then $\mathbb{L}(X)$ is a positive implicative subtraction algebra.

Proof. It is easy to check that $(\mathbb{L}(X); \ominus)$ is a subtraction algebra. For any $x \in X$, by positive implicativity of X, we have

$$\begin{aligned}((L_a \ominus L_b) \ominus L_c)(x) &= ((a - x) - (b - x)) - (c - x)\\ &= ((a - x) - (c - x)) - ((b - x) - (c - x))\\ &= ((L_a \ominus L_c)(x)) - ((L_b \ominus L_c)(x))\\ &= ((L_a \ominus L_c) \ominus (L_a \ominus L_c))(x),\end{aligned}$$

which implies $(L_a \ominus L_b) \ominus L_c = (L_a \ominus L_c) \ominus (L_a \ominus L_c)$. Hence $\mathbb{L}(X)$ is a positive implicative subtraction algebra. □

Corollary 3.11. If $f : X \to \mathbb{L}(X)$ is a homomorphism of subtraction algebra and $\sim$ is the equivalence relation on X defined by $x \sim y \Longleftrightarrow f(x) = f(y)$, i.e., $x \sim y \Longleftrightarrow L_x = L_y$, then the quotient subtraction algebra X/f is isomorphic to the image of f, i.e., $X/f \cong \mathbb{L}(X)$, with $[0]_f = Kerf$.

Proof. We show that the map $f : X \to \mathbb{L}(X)$ defined by $f(x) := L_x$ is an epimorphism. Since for any $t \in X$, $f(x-y)(t) = L_{x-y}(t) = (x-y)-t = (x-t)-(y-t) = L_x(t)-L_y(t) = f(x)(t)-f(y)(t) = [f(x)-f(y)](t)$, it follows that f is a homomorphism. Suppose that $f(x) = f(y)$ for any $x, y \in X$, i.e., $L_x = L_y$. Then, for any $t \in X$, $L_x(t) = L_y(t)$ and hence $x - t = y - t$. If we put $t := y$, then we obtain $x - y = y - y = 0$. Similarly, $y - x = 0$. Hence $x = y$. This proves that f is injective. Clearly, f is surjective. Therefore we obtain $X/f \cong \mathbb{L}(X)$. It follows that $Kerf = \{x \in X | f(x) = L_0\} = \{x \in X | L_x = L_0\} = [0]_f$. □

Proposition 3.12. A subtraction algebra X is positive implicative if and only if $L_{a-b} = L_a \ominus L_b$ for any $a, b \in X$.

Proof. For any $x \in X$, we have $L_{a-b}(x) = (a-b)-x = (a-x)-(b-x) = L_a(x) - L_b(x) = (L_a \ominus L_b)(x)$. □

Let A be an ideal in a subtraction algebra X. Denote by $\mathbb{L}_A$ the set of all left map L_a, $a \in A$.

Theorem 3.13. Let X be a positive implicative subtraction algebra and $A \subseteq X$. A is an ideal in X if and only if $\mathbb{L}_A$ is an ideal in $\mathbb{L}(X)$.

Proof. Suppose that A is an ideal in X. Since $0 \in A$, $L_0 \in \mathbb{L}_A$. Let $L_b \in \mathbb{L}(X)$ and $L_a \in L_A$ with $L_b \ominus L_a \in \mathbb{L}_A$. Then $L_b \ominus L_a = L_c$ for some $c \in A$. Hence $(L_b \ominus L_a)(x) = L_c(x)$ for all $x \in X$. Therefore $(b-x)-(a-x) = c-x$. Since X is positive implicative, we obtain

$(b - a) - x = c - x$. If we put $x := a$ in the above identity, then $(b - a) - a = c - a$. By (a1), we have $b - a = c - a \leq c$. So $b - a \in A$. Since $a \in A$ and A is an ideal in X, we obtain $b \in A$. Hence we have $L_b \in \mathbb{L}_A$. Thus $\mathbb{L}_A$ is an ideal of $\mathbb{L}(X)$.

Conversely, assume that $\mathbb{L}_A$ is an ideal in $\mathbb{L}(X)$. Then $L_0 \in \mathbb{L}_A$. Let $L_0 = L_a$ for some $a \in A$. So $L_0(x) = L_a(x)$ for all $x \in X$. Hence $0 - x = a - x$. By (a2), $a - x = 0$. Put $x = 0$. Then $0 = a \in A$. Next, let $b - a = c$ for some $c \in A$. Then $(b - a) - x = c - x$ for all $x \in X$. Since X is positive implicative, we have $(b - x) - (a - x) = c - x$. Hence $L_b(x) - L_a(x) = L_c(x)$ and so $(L_b \ominus L_a)(x) = L_c(x)$ for all $x \in X$. Therefore $L_b \ominus L_a = L_c \in \mathbb{L}_A$. Since $L_a \in \mathbb{L}_A$ and $\mathbb{L}_A$ is an ideal, we obtain $L_b \in \mathbb{L}_A$. Hence $L_b = L_d$ for some $d \in A$ and so $b - x = d - x$ for all $x \in X$. Put $x = d$ in the above identity. Then $b - d = 0$ and so $b = d \in A$, i.e., A is an ideal. This completes the proof. □

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