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INTUITIONISTIC FUZZY IDEALS OF A SEMIGROUP

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Abstract. We give the characterization of an intuitionistic fuzzy ideal[resp. intuitionistic fuzzy left ideal, an intuitionistic fuzzy right ideal and an intuitionistic fuzzy bi-ideal] generated by an intuitionistic fuzzy set in a semigroup without any condition. And we prove that every intuitionistic fuzzy ideal of a semigroup S is the union of a family of intuitionistic fuzzy principle ideals of S. Finally, we investigate the intuitionistic fuzzy ideal generated by an intuitionistic fuzzy set in S^1.

0. Introduction

In his pioneering paper[21], Zadeh introduced the notion of a fuzzy set in a set X as a mapping from X into the closed unit interval $[0, 1]$. Since then, some researchers[16,17,19,20] applied this notion to semigroup and group theory.

In 1986, Atanassov[2] introduced the concept of intuitionistic fuzzy sets as the generalization of fuzzy sets. Recently Çoker and his colleagues[6,7,8], Hur and his colleagues [13] , and Lee and Lee[18] introduced the concept of intuitionistic fuzzy topological spaces using intuitionistic fuzzy sets and investigated some of their properties. In 1989, Biswas[3] introduced the concept of intuitionistic fuzzy subgroups and

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studied some of it's properties. In 2003, Banerjee and Basnet[2] investigated intuitionistic fuzzy subrings and intuitionistic fuzzy ideals using intuitionistic fuzzy sets. Also, Hur and his colleagues[1,9-11, 14, 15] applied the notion of intuitionistic fuzzy sets to algebra. Moreover, Hur and his colleagues[12] applied one to topological group.

In this paper, we give the characterization of an intuitionistic fuzzy ideal[resp. intuitionistic fuzzy left ideal, an intuitionistic fuzzy right ideal and an intuitionistic fuzzy bi-ideal] generated by an intuitionistic fuzzy set in a semigroup without any condition. And we prove that every intuitionistic fuzzy ideal of a semigroup S is the union of a family of intuitionistic fuzzy principle ideals of S. Finally, we investigate the intuitionistic fuzzy ideal generated by an intuitionistic fuzzy set in S^1.

1. Preliminaries

We will list some concept and one result needed in the later sections.

For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0, 1]$ as I.

Definition 1.1[2, 6]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set*(in short, IFS) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership(namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in a set X defined by $0_\sim(x) = (0, 1)$ and $1_\sim(x) = (1, 0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 1.2[2]. Let X be a nonempty sets and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be an IFSs in X. Then

(1) $A \subset B$ if and only if $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ if and only if $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

(6) $[\,]A = (\mu_A, 1 - \mu_A)$, $<> A = (1 - \nu_A, \nu_A)$.

Definition 1.3[6]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(1) $\bigcap A_i = (\wedge \mu_{A_i}, \vee \nu_{A_i})$.

(2) $\bigcup A_i = (\vee \mu_{A_i}, \wedge \nu_{A_i})$.

Definition 1.4[18]. Let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. An *intuitionistic fuzzy point*(in short, *IFP*) $x_{(\lambda,\mu)}$ of X is an IFS in a set X defined by for each $y \in X$

$$x_{(\lambda,\mu)}(y) = \begin{cases} (\lambda, \mu) & \text{if } y = x, \\ (0, 1) & \text{otherwise.} \end{cases}$$

In this case, x is called the *support* of $x_{(\lambda,\mu)}$ and λ and μ are called the *value* and the *nonvalue* of $x_{(\lambda,\mu)}$, respectively. An IFP $x_{(\lambda,\mu)}$ is said to *belong* to an IFS $A = (\mu_A, \nu_A)$ in X, denoted by $x_{(\lambda,\mu)} \in A$, if $\lambda \leq \mu_A(x)$ and $\mu \geq \nu_A(x)$.

Clearly an intuitionistic fuzzy point can be represented by an ordered pair of fuzzy points as follows:

$$x_{(\lambda,\mu)} = (x_\lambda, 1 - x_{1-\mu})$$

We will denote the set of all IFPs in a set X as $\mathrm{IF}_P(X)$.

Definition 1.5[9]. Let A be an IFS in a set X and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then the set $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu\}$ is called a *(λ, μ)-level subset* of A.

Result 1.A[18, Theorem 2.4]. Let X be a set and let $A \in \mathrm{IFS}(X)$. Then

$$A = \bigcup\{x_{(\lambda,\mu)} : x_{(\lambda,\mu)} \in A\}.$$

In fact, it is not difficult to see that

$$A = \bigcup_{x \in A^{(0,1)}} x_{A(x)}.$$

2. Intuitionistic ideals generated by intuitionistic fuzzy sets

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset of A of S such that

$$A^2 \subset A$$

and by a *left* [resp. *right] ideal* of S we mean a non-empty subset A of S such that

$$SA \subset A \text{ [resp. } AS \subset A].$$

By *two-sided ideal* or simply *ideal* we mean a subset A of S which is both a left and a right ideal of S. We well denote the set of all left ideals [resp. right ideals and ideals] of S as $\mathrm{LI}(S)$ [resp. $\mathrm{RI}(S)$ and $\mathrm{I}(S)$].

Definition 2.1[9]. Let S be a semigroup and let $0_\sim \neq A \in \mathrm{IFS}(S)$. Then A is called an :

(1) *intuitionistic fuzzy subsemigroup* (in short, IFSG) of S if

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$$

for any $x, y \in S$,

(2) *intuitionistic fuzzy left ideal* (in short, IFLI) of S if

$$\mu_A(xy) \geq \mu_A(y) \text{ and } \nu_A(xy) \leq \nu_A(y)$$

for any $x, y \in S$,

(3) *intuitionistic fuzzy right ideal* (in short, IFSG) of S if

$$\mu_A(xy) \geq \mu_A(x) \text{ and } \nu_A(xy) \leq \nu_A(x)$$

for any $x, y \in S$,

(4) *intuitionistic fuzzy (two-sided) ideal* (in short, IFI) of S if it is both an intuitionistic fuzzy left and an intuitionistic fuzzy right ideal of S.

We well denote the set of all IFSGs [resp. IFLIs, IFRIs and IFIs] of S as IFSG(S) [resp. IFLI(S), IFRI(S) and IFI(S)]. It is clear that $A \in$ IFI(S) if and only if $\mu_A(xy) \geq \mu_A(x) \vee \mu_A(y)$ and $\nu_A(xy) \leq \nu_A \wedge \nu_A(y)$ for any $x, y \in S$, and if $A \in$ IFLI(S)[resp. IFRI(S) and IFI(S)], then $A \in$ IFSG(S).

Result 2.A[9, Proposition 3.7 and 14, Proposition 2.3]. Let S be a semigroup and let $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$. Then $A \in$ IFSG(S) [resp. IFI(S), IFLI(S) and IFRI(S)] if and only if $A^{(\lambda,\mu)}$ is a subsemigroup [resp. ideal, left ideal and right ideal] of S.

It is well-known[4] that I is complete completely distributive lattice. Thus we have the following result.

Proposition 2.2. Let S be a semigroup. Then IFI(S) is a complete completely distributive lattice with with respect to the meet "$\cap$" and the union "$\cup$".

Proof. Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset$ IFI(S), where Γ denotes the index set. Let $x, y \in S$. Then

$$\begin{aligned}&\mu_{\cup_{\alpha\in\Gamma}A_\alpha}(xy) = \textstyle\bigvee_{\alpha\in\Gamma}\mu_{A_\alpha}(xy)\\ &\geq \textstyle\bigvee_{\alpha\in\Gamma}[\mu_{A_\alpha}(x)\vee\mu_{A_\alpha}(y)] \quad (\text{Since } A_\alpha \in \text{IFI}(S) \text{ for each } \alpha\in\Gamma)\\ &= (\textstyle\bigvee_{\alpha\in\Gamma}\mu_{A_\alpha}(x))\vee(\textstyle\bigvee_{\alpha\in\Gamma}\mu_{A_\alpha}(y)) = (\mu_{\cup_{\alpha\in\Gamma}A_\alpha}(x))\vee(\mu_{\cup_{\alpha\in\Gamma}A_\alpha}(y))\end{aligned}$$

and

$$\begin{aligned}&\nu_{\cup_{\alpha\in\Gamma}A_\alpha}(xy) = \textstyle\bigwedge_{\alpha\in\Gamma}\nu_{A_\alpha}(xy) \leq \textstyle\bigwedge_{\alpha\in\Gamma}[\nu_{A_\alpha}(x)\wedge\nu_{A_\alpha}(y)]\\ &= (\textstyle\bigwedge_{\alpha\in\Gamma}\nu_{A_\alpha}(x))\wedge(\textstyle\bigwedge_{\alpha\in\Gamma}\nu_{A_\alpha}(y)) = (\nu_{\cup_{\alpha\in\Gamma}A_\alpha}(x))\wedge(\nu_{\cup_{\alpha\in\Gamma}A_\alpha}(y)).\end{aligned}$$

Also,

$$\begin{aligned}&\mu_{\cap_{\alpha\in\Gamma}A_\alpha}(xy) = \textstyle\bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(xy)\\ &= \textstyle\bigwedge_{\alpha\in\Gamma}[\mu_{A_\alpha}(x)\vee\mu_{A_\alpha}(y)] \quad (\text{Since } A_\alpha \in \text{IFI}(S) \text{ for each } \alpha\in\Gamma)\\ &\geq (\textstyle\bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(x))\vee(\textstyle\bigwedge_{\alpha\in\Gamma}\mu_{A_\alpha}(y)) = (\mu_{\cap_{\alpha\in\Gamma}A_\alpha}(x))\vee(\mu_{\cap_{\alpha\in\Gamma}A_\alpha}(y))\end{aligned}$$

and

$$\begin{aligned}&\nu_{\cap_{\alpha\in\Gamma}A_\alpha}(xy) = \textstyle\bigvee_{\alpha\in\Gamma}\nu_{A_\alpha}(xy) = \textstyle\bigvee_{\alpha\in\Gamma}[\nu_{A_\alpha}(x)\vee\nu_{A_\alpha}(y)]\\ &\leq (\textstyle\bigvee_{\alpha\in\Gamma}\nu_{A_\alpha}(x))\vee(\textstyle\bigvee_{\alpha\in\Gamma}\nu_{A_\alpha}(y)) = (\nu_{\cap_{\alpha\in\Gamma}A_\alpha}(x))\vee(\nu_{\cap_{\alpha\in\Gamma}A_\alpha}(y)).\end{aligned}$$

Hence $\cup_{\alpha\in\Gamma}A_\alpha$, $\cap_{\alpha\in\Gamma}A_\alpha \in$ IFI(S). This completes the proof. □

Definition 2.3. Let S be a semigroup and let $A \in$ IFS(S). Then the least IFLI[resp. IFRI and IFI] of S containing A is called the IFLI[resp. IFRI and IFI] of S *generated* by A and is denoted by $(A)_L$[resp. $(A)_R$ and (A)].

Lemma 2.4. Let X be a set, let $A \in$ IFS(X) and let $x \in X$. Then $A(x) = (\bigvee_{x\in A^{(\lambda,\mu)}}\lambda, \bigwedge_{x\in A^{(\lambda,\mu)}}\mu)$, where $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$.

Proof. Let $\lambda_0 = \bigvee_{x\in A^{(\lambda,\mu)}}\lambda$, let $\mu_0 = \bigwedge_{x\in A^{(\lambda,\mu)}}\mu$ and let $\varepsilon > 0$. Then $\bigvee_{x\in A^{(\lambda,\mu)}}\lambda > \lambda_0 - \varepsilon$ and $\bigwedge_{x\in A^{(\lambda,\mu)}}\mu < \mu_0 + \varepsilon$. Thus there exists $(s,t) \in \{(\lambda,\mu) : x \in A^{(\lambda,\mu)}\}$ such that $s > \lambda_0 - \varepsilon$ and $t < \mu_0 + \varepsilon$.

Since $x \in A^{(\lambda,\mu)}$, $\mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu$. Then $\mu_A(x) > \lambda_0 - \varepsilon$ and $\nu_A(x) < \mu_0 + \varepsilon$. Since ε is an arbitrary real number, $\mu_A(x) \geq \lambda_0$ and $\nu_A(x) \leq \mu_0$. On the other hand, let $A(x) = (s,t)$. Then $x \in A^{(s,t)}$. Thus $(s,t) \in \{(\lambda,\mu) : x \in A^{(\lambda,\mu)}\}$. So $s \leq \bigvee_{x \in A^{(\lambda,\mu)}} \lambda$ and $t \geq \bigwedge_{x \in A^{(\lambda,\mu)}} \mu$, i.e., $\mu_A(x) = s \leq \lambda_0$ and $\nu_A(x) = t \geq \mu_0$. Hence $A(x) = (\mu_A(x), \nu_A(x)) = (\lambda_0, \mu_0)$. □

Theorem 2.5. Let S be a semigroup, let $A \in \mathrm{IFS}(S)$ and let $(\lambda,\mu) \in I \times I$ with $\lambda + \mu \leq 1$. We define a complex mapping $A^* = (\mu_{A^*}, \nu_{A^*}) : S \to I \times I$ as follows: for each $x \in S$

$$A^*(x) = \Big(\bigvee_{x \in (A^{(\lambda,\mu)})} \lambda, \bigwedge_{x \in (A^{(\lambda,\mu)})} \mu\Big).$$

Then $A^* = (A)$, where $(A^{(\lambda,\mu)})$ denotes the ideal generated by $A^{(\lambda,\mu)}$.

Proof. For each $x \in S$, let $(s,t) \in \{(\lambda,\mu) : x \in A^{(\lambda,\mu)}\}$. Then $x \in A^{(s,t)}$. Thus $x \in (A^{(s,t)})$. So $(s,t) \in \{(\lambda,\mu) : x \in (A^{(\lambda,\mu)})\}$, i.e., $\{(\lambda,\mu) : x \in A^{(\lambda,\mu)}\} \subset \{(\lambda,\mu) : x \in (A^{(\lambda,\mu)})\}$. Then, by Lemma 2.4,

$$\mu_A(x) = \bigvee_{x \in A^{(\lambda,\mu)}} \lambda \leq \bigvee_{x \in (A^{(\lambda,\mu)})} \lambda = \mu_{A^*}(x)$$

and

$$\nu_A(x) = \bigwedge_{x \in A^{(\lambda,\nu)}} \mu \geq \bigwedge_{x \in (A^{(\lambda,\nu)})} \mu = \nu_{A^*}(x).$$

So $A \subset A^*$.

For each $(s,t) \in \mathrm{Im}\, A^*$, let $s_n = s - \frac{1}{n}$ and $t_n = t + \frac{1}{n}$ for each $n \in \mathbb{N}$. Let $x \in A^{*(s,t)}$. Then $\mu_{A^*}(x) \geq s$ and $\nu_{A^*}(x) \leq t$. Thus, for each $n \in \mathbb{N}$

$$\bigvee_{x \in (A^{(\lambda,\mu)})} \lambda \geq s > s - \tfrac{1}{n} = s_n$$

and

$$\bigwedge_{x \in (A^{(\lambda,\mu)})} \mu \leq t < t + \tfrac{1}{n} = t_n.$$

So there exists a $(\lambda_n, \mu_n) \in \{(\lambda,\mu) : x \in (A^{(\lambda,\mu)})\}$ such that $\lambda_n > s_n$ and $\mu_n < t_n$. Then $A^{(\lambda_n,\mu_n)} \subset A^{(s_n,t_n)}$. So $x \in (A^{(\lambda_n,\mu_n)}) \subset (A^{(s_n,t_n)})$. Consequently, we have $x \in \bigcap_{n \in \mathbb{N}} (A^{(s_n,t_n)})$. Now let $x \in \bigcap_{n \in \mathbb{N}} (A^{(s_n,t_n)})$. Then clearly $(s_n,t_n) \in \{(\lambda,\mu) : x \in (A^{(\lambda,\mu)})\}$ for each $n \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$,

$s - \frac{1}{n} = s_n \leq \bigvee_{x \in (A^{(\lambda,\mu)})} \lambda = \mu_{A^*}(x)$

and

$t + \frac{1}{n} = t_n \geq \bigwedge_{x \in (A^{(\lambda,\mu)})} \mu = \nu_{A^*}(x)$.

Since n is an arbitrary positive integer, $s \leq \mu_{A^*}(x)$ and $t \geq \nu_{A^*}(x)$. Thus $(s,t) \in A^{*(s,t)}$. So $A^{*(s,t)} = \bigcap_{n \in \mathbb{N}}(A^{(s_n,t_n)})$. It is clear that $\bigcap_{n \in \mathbb{N}}(A^{(s_n,t_n)})$ is an ideal of S. So, by Result 2.A, $A^* \in \text{IFI}(S)$.

Now let $B \in \text{IFI}(S)$ such that $A \subset B$ and let $x \in S$. If $A^*(x) = (0,1)$, then clearly $\mu_{A^*}(x) \leq \mu_B(x)$ and $\nu_{A^*}(x) \geq \nu_B(x)$, i.e., $A^* \subset B$. If $A^*(x)(s,t) \neq (0,1)$, then $x \in A^{*(s,t)} = \bigcap_{n \in \mathbb{N}}(A^{(s_n,t_n)})$. Thus $x \in (A^{(s_n,t_n)}) = A^{(s_n,t_n)}S \cup SA^{(s_n,t_n)}S \cup A^{(s_n,t_n)}S \cup A^{(s_n,t_n)}$ for each $n \in \mathbb{N}$. We consider the following cases:

Case (i) : Suppose $x \in A^{(s_n,t_n)}$. Then clearly for each $n \in \mathbb{N}$

$$s_n \leq \mu_A(x) \leq \mu_B(x) \text{ and } t_n \geq \nu_A(x) \geq \nu_B(x).$$

Case (ii) : Suppose $x \in A^{(s_n,t_n)}S$. Then there exist $a \in A^{(s_n,t_n)}$ and $b \in S$ such that $x = ab$. Thus for each $n \in \mathbb{N}$

$$s_n \leq \mu_A(a) \leq \mu_B(a) \leq \mu_B(ab) = \mu_B(x)$$
$$\text{and} \quad t_n \geq \nu_A(a) \geq \nu_B(a) \geq \nu_B(ab) = \nu_B(x).$$

Case (iii) : Suppose $x \in SA^{(s_n,t_n)}$. Then, by the similar arguments of Case (ii), we have $\mu_B(x) \geq s_n$ and $\nu_B(x) \leq t_n$ for each $n \in \mathbb{N}$.

Case (iv) : Suppose $x \in SA^{(s_n,t_n)}S$. Then there exist $a \in A^{(s_n,t_n)}$ and $b \in S$ such that $x = abc$. Since $B \in \text{IFI}(S)$, for each $n \in \mathbb{N}$

$$s_n \leq \mu_A(a) \leq \mu_B(a) \leq \mu_B(x) \text{ and } t_n \geq \nu_A(a) \geq \nu_B(a) \geq \nu_B(x).$$

Since n is an arbitrary number in $\mathbb{N}$, in all, $\mu_{A^*}(x) = s \leq \mu_B(x)$ and $\nu_{A^*}(x) = t \geq \nu_B(x)$. thus $A^* \subset B$. Hence $A^* = (A)$. This complete the proof. □

Corollary 2.5. Let S be a semigroup and let $x_{(\lambda,\mu)} \in \text{IF}_P(S)$. We define a complex mapping $(x_{(\lambda,\mu)}) : S \to I \times I$ as follows: for each $x \in S$,

$$(x_{(\lambda,\mu)})(y) = \begin{cases} (\lambda,\mu) & \text{if } \ y \in (x), \\ (0,1) & \text{if } \ y \notin (x), \end{cases}$$

where (x) is the principal ideal of S generated by x. Then $(x_{(\lambda,\mu)})$ is the IFI generated by $x_{(\lambda,\mu)}$. In this case, $(x_{(\lambda,\mu)})$ is called the *intuitionistic fuzzy principal ideal*(in short, *IFPI*)*of S generated by* $x_{(\lambda,\mu)}$.

Proof. By Theorem 2.5, $(x_{(\lambda,\mu)})(y) = (\bigvee_{z\in(A^{(s,t)})} s,\ \bigwedge_{z\in(A^{(s,t)})} t)$ for each $y \in S$.

Case (i) : Suppose $y \in (x)$. Let $(s,t) \in (0,\lambda] \times [\mu, 1)$. Then $A^{(s,t)} = \{z \in S : \mu_{x_{(\lambda,\mu)}}(z) \geq s,\ \nu_{x_{(\lambda,\mu)}}(z) \leq t\} = \{x\}$. Thus $y \in (x) = (A^{(s,t)})$. If $s > \lambda$ and $t < \mu$, then clearly $x_{(\lambda,\mu)} = (0,1)$. So

$$(x_{(\lambda,\mu)})(y) = (\bigvee_{z\in(A^{(s,t)})} s,\ \bigwedge_{z\in(A^{(s,t)})} t) = (\bigvee_{0<s\leq\lambda} s,\ \bigwedge_{\mu\leq t<1} t) = (\lambda,\mu).$$

Case (ii) : Suppose $y \notin (x)$. Assume that $(x_{(\lambda,\mu)})(y) \neq (0,1)$. Then there exists $(s,t) \in (0,1] \times [0,1)$ with $s+t \leq 1$ such that $y \in (A^{(s,t)})$. Since $A^{(s,t)} \neq (0,1)$, by Case (i), $s \leq \lambda$ and $t \geq \mu$. Thus $A^{(\lambda,\mu)} = \{x\}$. So $y \in (A^{(s,t)}) = (x)$. This is a contradiction. Thus $(x_{(\lambda,\mu)})(y) = (0,1)$. Hence $(x_{(\lambda,\mu)})$ is well-defined. □

The following is an easy modification of Theorem 2.5.

Theorem 2.6. Let S be a semigroup and let $A \in \text{IFS}(S)$. We define a complex mapping $A^* : S \to I \times I$ as follows: for each $x \in S$,

$$A^*(x) = (\bigvee_{x\in(A^{(\lambda,\mu)})_L} \lambda,\ \bigwedge_{x\in(A^{(\lambda,\mu)})_L} \mu).$$

Then $A^* = (A)_L$, where $(A^{(\lambda,\mu)})_L$ denotes the left ideal generated by $A^{(\lambda,\mu)}$.

Corollary 2.6. Let S be a semigroup and let $x_{(\lambda,\mu)} \in \mathrm{IF}_P(S)$. We define two complex mappings $(x_{(\lambda,\mu)})_L : S \to I \times I$ and $(x_{(\lambda,\mu)})_R : S \to I \times I$ as follows, respectively : for each $y \in S$,

$$(x_{(\lambda,\mu)})_L(y) = \begin{cases} (\lambda, \mu) & \text{if } \ y \in (x)_L, \\ (0,1) & \text{if } \ y \notin (x)_L, \end{cases}$$

and

$$(x_{(\lambda,\mu)})_R(y) = \begin{cases} (\lambda, \mu) & \text{if } \ y \in (x)_R, \\ (0,1) & \text{if } \ y \notin (x)_R. \end{cases}$$

Then $(x_{(\lambda,\mu)})_L$[resp. $(x_{(\lambda,\mu)})_R$) is the IFLI[resp. IFRI] of S generated by $x_{(\lambda,\mu)}$ in S. In this case, $(x_{(\lambda,\mu)})_L$[resp. $(x_{(\lambda,\mu)})_R$] is called the *intuitionistic fuzzy principal left* [resp. *right*] *ideal*(in short, IFPLI[resp. IFPRI]) *generated by* $x_{(\lambda,\mu)}$.

Proof. The proofs are similar to the case of Corollary 2.5. So we omit. □

Remark 2.7. As the dual of Theorem 2.6, $(A)_R$ can be characterized by $(A)_R(x) = (\bigvee_{x \in (A^{(\lambda,\mu)})_R} \lambda, \ \bigwedge_{x \in (A^{(\lambda,\mu)})_R} \mu)$ for each $x \in S$, where $(A^{(\lambda,\mu)})_R$ denotes the right ideal generated by $A^{(\lambda,\mu)}$.

A nonempty subset A of a semigroup S is called a *bi-ideal* of S if $A^2 \subset A$ and $ASA \subset A$. We will denote the set of all bi-ideals of S as $\mathrm{BI}(S)$.

Definition 2.8[14]. Let S be a semigroup and let $0_\sim \neq A \in \mathrm{IFS}(S)$. Then A is called an *intuitionistic fuzzy bi-ideal* (in short, IFBI) of S if it satisfies the following conditions : for any $x, y, z \in S$.

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$

(ii) $\mu_A(xyz) \geq \mu_A(x) \wedge \mu_A(z)$ and $\nu_A(xyz) \leq \nu_A(x) \vee \nu_A(z)$.

We will denote the set of all IFBIs of S as IFBI(S).

Result 2.B[14, Proposition 2.8]. Let S be a semigroup and let $A \in \text{IFS}(S)$. Then $A \in \text{IFBI}(S)$ if and only if $A^{(\lambda,\mu)} \in \text{BI}(S)$ for each $(\lambda, \mu) \in I \times I$ with $\lambda + \mu \leq 1$.

Let A be a subset of a semigroup S. Then it is not difficult to see that the bi-ideal $(A)_B$ generated by A in S is $A \cup A^2 \cup ASA$.

The following can be shown by the above comment, Result 2.B and a moderate modification of Theorem 2.5.

Theorem 2.9. Let S be a semigroup and let $A \in \text{IFS}(S)$. We define a complex mapping $A^* : S \to I \times I$ as follows: for each $x \in S$,

$$A^*(x) = (\bigvee_{x \in (A^{(\lambda,\mu)})_B} \lambda, \bigwedge_{x \in (A^{(\lambda,\mu)})_B} \mu).$$

Then $A^* = (A)_B$, where $(A)_B$ denotes the IFBI generated by A.

Corollary 2.9. Let S be a semigroup and let $x_{(\lambda,\mu)} \in \text{IF}_P(S)$. We define two complex mappings $(x_{(\lambda,\mu)})_B : S \to I \times I$ as follows, respectively : for each $y \in S$,

$$(x_{(\lambda,\mu)})_B(y) = \begin{cases} (\lambda, \mu) & \text{if } \ y \in (x)_B, \\ (0, 1) & \text{if } \ y \notin (x)_B. \end{cases}$$

Then $(x_{(\lambda,\mu)})_B$ is the IFBI of S generated by $x_{(\lambda,\mu)}$ in S. In this case, $(x_{(\lambda,\mu)})_B$ is called the *intuitionistic fuzzy principal bi-ideal*(in short, IF-PBI)*generated by* $x_{(\lambda,\mu)}$.

Proof. The proofs is similar to the case of Corollary 2.5. So we omit. □

It is well-known that every ideal of a semigroup S is the union of some principal ideals of S. Similarly, we have the following result.

Theorem 2.10. Let S be a semigroup. Then every IFI of S is the union of some IFPIs of S.

Proof. Let $A \in \mathrm{IFI}(S)$. Then, by Result 1.A,

$$A = \bigcup_{x_{(\lambda,\mu)} \in A} x_{(\lambda,\mu)} = \bigcup_{x \in A^{(0,1)}} x_{A(x)}.$$

Let $y \in S$.

Case (i) : Suppose $A(y) \neq (0,1)$. Then

$$\begin{aligned}(\bigcup_{x \in A^{(0,1)}} x_{A(x)})(y) &= (\bigcup_{y \in (z), z \in A^{(0,1)}} (z_{A(z)}))(y) \\ &= (\bigvee_{y \in (z), z \in A^{(0,1)}} \mu_{A(z)}, \bigwedge_{y \in (z), z \in A^{(0,1)}} \nu_{A(z)}).\end{aligned}$$

If $z \neq y$, then there exist $a_1, a_2, b_1, b_2 \in S$ such that $y = za_1$ or $y = a_2 z$ or $y = b_1 z b_2$. For any cases, since $A \in \mathrm{IFI}(S)$, $\mu_A(y) \geq \mu_a(z)$ and $\nu_A(y) \leq \nu_a(z)$. Thus

$$\begin{aligned}(\bigcup_{x \in A^{(0,1)}} x_{A(x)})(y) &= (\bigvee_{y \in (z), z \in A^{(0,1)}} \mu_{A(z)}, \bigwedge_{y \in (z), z \in A^{(0,1)}} \nu_{A(z)}) \\ &= (\mu_A(y), \nu_A(y)) = A(y).\end{aligned}$$

Case (ii) : Suppose $A(y) = (0,1)$. Assume that there exists $z \in A^{(0,1)}$ such that $y \in (z)$. Then $\mu_A(y) \geq \mu_A(z)$ and $\nu_A(y) \leq \nu_A(z)$ as above. Thus $A(y) \neq (0,1)$. This is a contradiction. Then $y \notin (z)$ for each $z \in A^{(0,1)}$. So

$$A(y) = (\bigcup_{x \in A^{(0,1)}} (x_{A(x)}))(y) = (0,1).$$

Hence, in all, $A = \bigcup_{x \in A^{(0,1)}} x_{A(x)}$. This completes the proof. □

3. Some special cases

In this case, we study intuitionistic fuzzy ideal generated by an IFS A in S^1.

Theorem 3.1. Let S be a semigroup and let $A \in \mathrm{IFS}(S^1)$. Then

$$(A)(a) = (\bigvee_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3\in S^1}} \mu_A(x_2), \bigwedge_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3\in S^1}} \nu_A(x_2)) \text{ for each } a \in S.$$

Proof. Let $a \in S$ such that $a = x_1x_2x_3$ for some$x_1, x_2, x_3 \in S^1$ and let $A(x_2) = (s,t)$. Then $x_2 \in A^{(s,t)}$. Thus $a \in (A^{(s,t)})$. So $A(x_2) \in \{(s,t) : a \in (A^{(s,t)})\}$. By theorem 2.5,

$$\mu_{(A)}(a) = \bigvee_{a\in(A^{(s,t)})} s \geq \bigvee_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3\in S^1}} \mu_A(x_2)$$

and $(*)$

$$\nu_{(A)}(a) = \bigwedge_{a\in(A^{(s,t)})} t \leq \bigwedge_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3\in S^1}} \nu_A(x_2).$$

On the other hand, let $(\lambda, \mu) \in \{(s,t) : a \in (A^{(s,t)})\}$. Then clearly $a \in (A^{(\lambda,\mu)})$. Thus there exist $x_1, x_3 \in S^1$ and $x_2 \in A^{(\lambda,\mu)}$ such that $a = x_1x_2x_3$. Since $x_2 \in A^{(\lambda,\mu)}$, $\mu_A(x_2) \geq \lambda$ and $\nu_A(x_2) \leq \mu$. Then

$$\mu_A(a) = \bigvee_{a\in(A^{(s,t)})} s \leq \bigvee_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3\in S^1}} \mu_A(x_2)$$

and $(*')$

$$\nu_A(a) = \bigwedge_{a\in(A^{(s,t)})} t \geq \bigwedge_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3\in S^1}} \nu_A(x_2).$$

Hence, by $(*)$ and $(*')$,

$$A(a) = (\bigvee_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3 \in S^1}} \mu_A(x_2), \bigwedge_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3 \in S^1}} \nu_A(x_2)).$$

This completes the proof. □

Remark 3.2. By theorem 2.5 and its dual, we can easily obtain $(A)_L$[resp. $(A)_R$] generated by A in S^1 defined by

$$A_L(a) = (\bigvee_{\substack{a=x_1x_2 \\ x_1,x_2 \in S^1}} \mu_A(x_2), \bigwedge_{\substack{a=x_1x_2 \\ x_1,x_2 \in S^1}} \nu_A(x_2))$$

[resp. $A_R(a) = (\bigvee_{\substack{a=x_1x_2 \\ x_1,x_2 \in S^1}} \mu_A(x_1), \bigwedge_{\substack{a=x_1x_2 \\ x_1,x_2 \in S^1}} \nu_A(x_1))$], for each $a \in S$.

Theorem 3.3. Let S be a regular semigroup and let $A \in \text{IFS}(S^1)$. Then

$$(A)_B(a) = (\bigvee_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3 \in S^1}} [\mu_A(x_1) \wedge \mu_A(x_3)], \bigwedge_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3 \in S^1}} [\nu_A(x_1) \vee \nu_A(x_3)])$$

for each $a \in S$.

Proof. Let $a \in S$ such that $a = x_1x_2x_3$ for some$x_1, x_2, x_3 \in S^1$ and let $(s,t) = (\mu_A(x_1) \wedge \mu_A(x_3), \nu_A(x_1) \vee \nu_A(x_3))$. Then clearly $x_1, x_3 \in A^{(s,t)}$. Thus $a \in (A^{(s,t)})_B$. So $(\mu_A(x_1) \wedge \mu_A(x_3), \nu_A(x_1) \vee \nu_A(x_3)) \in \{(s,t) : a \in (A^{(s,t)})_B\}$. By theorem 2.9,

$$\mu_{(A^{(s,t)})_B}(a) = \bigvee_{a \in (A^{(s,t)})_B} s \geq \bigvee_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3 \in S^1}} [\mu_A(x_1) \wedge \mu_A(x_3)]$$

and (**)

$$\nu_{(A^{(s,t)})_B}(a) = \bigwedge_{a \in (A^{(s,t)})_B} t \leq \bigwedge_{\substack{a=x_1x_2x_3 \\ x_1,x_2,x_3 \in S^1}} [\nu_A(x_1) \vee \nu_A(x_3)].$$

Now let $(\lambda, \mu) \in \{(s,t) : a \in (A^{(s,t)})_B\}$. Then

$a=(A^{(s,t)})_B=A^{(s,t)}\cup A^{(s,t)}A^{(s,t)}\cup A^{(s,t)}S^1A^{(s,t)}=A^{(s,t)}\cup A^{(s,t)}S^1A^{(s,t)}$. Since S^1 is regular, $A^{(s,t)}\subset A^{(s,t)}S^1A^{(s,t)}$. Then $a\in(A^{(s,t)})_B=A^{(s,t)}S^1A^{(s,t)}$. Thus there exist $x_1,x_2\in A^{(s,t)}$ and $x_2\in S^1$ such that $a=x_1x_2x_3$. Since $x_1,x_3\in A^{(s,t)}$, $\mu_A(x_1)\geq s$, $\nu_A(x_1)\leq t$ and $\mu_A(x_3)\geq s$, $\nu_A(x_3)\leq t$. Then $\mu_A(x_1)\wedge\mu_A(x_3)\geq s$, $\nu_A(x_1)\vee\nu_A(x_3)\leq t$.
Thus

$$\mu_{(A^{(s,t)})_B}(a)=\bigvee_{a\in(A^{(s,t)})_B} s\leq\bigvee_{\substack{a=x_1x_2x_3\\x_1,x_2,x_3\in S^1}}[\mu_A(x_1)\wedge\mu_A(x_3)]$$

and $(**')$

$$\nu_{(A^{(s,t)})_B}(a)=\bigwedge_{a\in(A^{(s,t)})_B} t\geq\bigwedge_{\substack{a=x_1x_2x_3\\x_1,x_2,x_3\in S^1}}[\nu_A(x_1)\vee\nu_A(x_3)].$$

Hence, by $(**)$ and $(**')$,

$$(A^{(s,t)})_B(a)=(\bigvee_{\substack{a=x_1x_2x_3\\x_1,x_2,x_3\in S^1}}[\mu_A(x_1)\wedge\mu_A(x_3)],\bigwedge_{\substack{a=x_1x_2x_3\\x_1,x_2,x_3\in S^1}}[\nu_A(x_1)\vee\nu_A(x_3)]).$$

This completes the proof. □

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