Remarks on Fixed Point Theorems of Non-Lipschitzian Self-mappings

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Abstract. In 1994, Lim-Xu asked whether the Maluta’s constant $D(X) < 1$ implies the fixed point property for asymptotically nonexpansive mappings and gave a partial solution for this question under an additional assumption for $T$, i.e., weakly asymptotic regularity of $T$. In this paper, we shall prove that the result due to Lim-Xu is also satisfied for more general non-Lipschitzian mappings in reflexive Banach spaces with weak uniform normal structure. Some applications of this result are also added.

1. Introduction

Let $C$ be a nonempty subset of a real Banach space $X$ and let $\mathbb{N}$ be the set of natural numbers. Let $T : C \to C$ be a mapping. $T$ is said to be Lipschitzian if for each $n \in \mathbb{N}$, there exists a real number $k_n$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad x, y \in C.$$ 

In particular, $T$ is said to be asymptotically nonexpansive [8] if $\lim_{n \to \infty} k_n = 1$, and it is said to be nonexpansive if $k_n = 1$ for all $n \in \mathbb{N}$. A set $K$ satisfying $T(K) \subset K$ is said to be invariant under $T$ or $T$-invariant. Let $K$ be a nonempty subset of $C$. For each $x \in K$, we set

$$c_n(x; K) = \sup_{y \in K} (\|T^n x - T^n y\| - \|x - y\|) \lor 0.$$ 

We say that $T$ is of partly asymptotically nonexpansive type if there exists a nonempty bounded closed convex and $T$-invariant subset $K$ of $C$ such that $c_n(x; K) \to 0$ for each $x \in K$. Recall that if $c_n(x) := c_n(x; C) \to 0$ for each $x \in C$, then $T$ is said to be of asymptotically nonexpansive type (see [16]). A point $x \in C$ is a fixed point of $T$ provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of $T$; that is, $Fix(T) = \{x \in C : Tx = x\}$.

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In 1965, Kirk [15] proved that if $C$ is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping $T$ of $C$ has a fixed point, where a nonempty convex subset $C$ of a normed linear space is said to have normal structure if each bounded convex subset $K$ of $C$ consisting of more than one point contains a nondiametral point; that is, a point $z \in K$ such that $\sup\{\|z - x\| : x \in K\} < \text{diam}(K)$. Seven years later, in 1972, Goebel-Kirk [8] proved that if the space $X$ is assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping $T$ of $C$ has a fixed point. This was immediately extended to mappings of asymptotically nonexpansive type in a space with its characteristic of convexity, $\epsilon_o(X) < 1$, by Kirk [16] in 1974. More recently these results have been extended to wider classes of spaces, see for example [4], [6], [7], [14], [19], [18], [22]. In particular, Lim-Xu [19] and Kim-Xu [14] have demonstrated the existence of fixed points for asymptotically nonexpansive mappings in Banach spaces with uniform normal structure, see also [6] for some related results. Very recently, the result due to Kim-Xu [14] was extended to mappings of asymptotically nonexpansive type by Li-Sims [17] and Kim [10] independently.

On the other hand, fixed point theorems due to Lim-Xu [19] for asymptotically nonexpansive mappings defined on a weakly compact convex subset $C$ in a Banach space $X$ with either a weakly continuous duality mapping or for which $D(X) < 1$ having an additional condition, i.e., weak asymptotic regularity on $C$ for $T$, where $D(X)$ is Maluta’s constant (see [20]), were carried over continuous mappings of asymptotically nonexpansive type by Kim-Kim [13].

In this paper, we modify some results in [13] and carry over these to a wider class of continuous mappings of partly asymptotically nonexpansive type in a Banach space with weak uniform normal structure (see Theorem 3.2). Some applications and examples of non-Lipschitzian mappings of partly asymptotically nonexpansive type which are not of asymptotically nonexpansive type are also added.

2. Preliminaries

Let $X$ be a real Banach space. First, let us introduce normal structure coefficient of $X$ introduced by Bynum [5]. For $A \subset X$, $\text{diam}(A)$ and $r_A(A)$ denote the diameter and the self-Chebyshev radius of $A$, respectively, i.e.,

$$
\text{diam}(A) = \sup_{x,y \in A} \|x - y\|,
$$

$$
\quad r_A(A) = \inf_{x \in A} (\sup_{y \in A} \|x - y\|)
$$

Recall that $X$ has uniform normal structure (simply $\text{UNS}$) if $N(X) > 1$, where

$$
N(X) = \inf \left\{ \frac{\text{diam}(A)}{r_A(A)} : A \subset X \text{ bounded closed convex with } \text{diam}(A) > 0 \right\}.
$$

Obviously, if $N(X) > 1$, then $X$ has normal structure.
Recall that if $X$ is a non-Schur Banach space, then the weakly convergent sequence coefficient of $X$, denoted by $WCS(X)$, is defined by

$$WCS(X) = \sup \{ M > 0 : \text{for each weakly convergent sequence } \{x_n\}, $$

$$\exists y \in \overline{co}(\{x_n\}) \text{ such that } M \cdot \limsup_{n \to \infty} \|x_n - y\| \leq A(\{x_n\}) \},$$

where $\overline{co}(K)$ denotes the closed convex hull of a set $K$ and $A(\{x_n\})$ denotes the asymptotic diameter of $\{x_n\}$, i.e.,

$$A(\{x_n\}) = \lim_{n \to \infty} \sup \{\|x_i - x_j\| : i, j \geq n\}.$$

It is easy to give a sharp expression $WCS(X)$ as follows;

$$WCS(X) = \sup \{ M : x_n \rightharpoonup u \Rightarrow M \cdot \limsup_{n \to \infty} \|x_n - u\| \leq D(\{x_n\}) \},$$

where $D(\{x_n\}) := \limsup_{n \to \infty} \limsup_{n \to \infty} \|x_n - x_m\|$ and “$\rightharpoonup$” means the weak convergence. For more details, see [5] and [12].

Note that if $X$ is reflexive, then $1 \leq N(X) \leq BS(X) \leq WCS(X) \leq 2$ (cf., [5]), where $BS(X)$ means the bounded sequence coefficient of $X$, i.e.,

$$BS(X) = \sup \{ M : \text{for any bounded sequence } \{x_n\} \text{ in } X, $$

$$\exists y \in \overline{co}(\{x_n\}) \text{ such that } M \cdot \limsup_{n \to \infty} \|x_n - y\| \leq A(\{x_n\}) \}.$$

While $N(X)$ and $BS(X)$ can be defined in every Banach space, $WCS(X)$ is well defined only in infinite dimensional reflexive spaces, where, by Eberlein-Šmulian theorem, we can assure the existence of weakly convergent sequences which do not converge.

The coefficient $WCS(X)$ plays important roles in fixed point theory. A space $X$ such that $WCS(X) > 1$ is said to have weak uniform normal structure. It is well-known [5] that if $WCS(X) > 1$, then $X$ has weak normal structure; that is, any weakly compact convex subset $C$ of $X$ with $\text{diam}(C) > 0$ has a nondiametral point.

Let $X$ be a Banach space. Recall that Maluta’s constant $D(X)$ [20] of $X$ is defined by

$$D(X) = \sup \left\{ \limsup_{n \to \infty} d(x_{n+1}, \overline{co}(\{x_1, x_2, \ldots, x_n\})) / \text{diam}(\{x_n\}) \right\},$$

where the supremum is taken over all bounded nonconstant sequences $\{x_n\}$ in $X$.

We remark the following properties for Maluta’s constant given in [20].

**Lemma 2.1.** Let $X$ be a Banach space. Then

(a) $D(X) \leq N(X) := 1/N(X)$,

(b) $D(X) = \sup \{ D(Y) : Y \subset X \text{ separable} \}$,

(c) $D(X) = 0$ if and only if $X$ is finite-dimensional.
(d) If $X$ is reflexive, then $D(X) \leq 1/WCS(X)$.

(e) If $D(X) < 1$, then the Banach space $X$ is reflexive and has normal structure.

**Remark 2.1.** (i) The property (a) says that if $X$ has uniform normal structure, then $D(X) < 1$. However, the converse does not hold (see Example 5.1 and Corollary 5.2 in [20]).

(ii) In view of (d), Maluta asked whether $D(X) = 1/WCS(X)$ holds true for every infinite dimensional reflexive space $X$. In 1985, Amir [2] gave a partial solution for this question. In other words, the converse inequality $D(X) \geq 1/WCS(X)$ holds if $X$ satisfies Opial’s property, i.e., for any sequence $\{x_n\}$ converging weakly to $x$, there holds the inequality

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad y (\neq x) \in X.$$  

Five years later, this question was completely solved by Prus [21].

(iii) The converse of (e) also does not hold (see Example 4.1 in [20], $X = (\sum \oplus \ell_n)_2$ is reflexive and has normal structure although $D(X) = 1$).

Note that, by (e) of Lemma 2.1, if $D(X) < 1$, $X$ has normal structure and hence the fixed point property for nonexpansive mappings; that is, for every weakly compact convex subset $C$ of $X$, every nonexpansive map $T : C \to C$ has a fixed point. However, it is still open whether $D(X) < 1$ implies the fixed point property for asymptotically nonexpansive mappings. In 1994, Lim-Xu [19] gave a partial answer for this question as follows:

**Theorem LX [19].** Suppose that $X$ is a Banach space such that $D(X) < 1$, that $C$ is a closed bounded convex subset of $X$, and that $T : C \to C$ is an asymptotically nonexpansive mapping. Suppose, in addition, that $T$ is weakly asymptotically regular on $C$, i.e., $T^{n+1}x - T^n x \to 0$ for all $x \in C$. Then $T$ has a fixed point.

Immediately, Theorem LX was extended to all mappings of asymptotically nonexpansive type by Kim-Kim (see Corollary 3.3 in [13]). In fact, under the assumption of weakly asymptotic regularity of $T$, the conditions for $X$ and $T$ can be weakened, in other words, Theorem LX can be extended to mappings of partly asymptotically nonexpansive type with $WCS(X) > 1$. Finally we need the following two well known properties for ultrafilters (for example, see [1]).

**Lemma 2.2.** Let $X$ be a Hausdorff topological linear space and let $\mathcal{U}$ be an ultrafilter on a set $I$. Then, the following properties hold.

(i) if $\{x_i\}_{i \in I}$ and $\{y_i\}_{i \in I}$ are two subsets of $X$ and $\lim_{\mathcal{U}} x_i = x$ and $\lim_{\mathcal{U}} y_i = y$ both exist, then $\lim_{\mathcal{U}} (x_i + y_i) = x + y$ and $\lim_{\mathcal{U}} (\alpha x_i) = \alpha x$ for any scalar $\alpha$.

(ii) $K$ is a compact subset of $X$ if and only if any set $\{x_i\}_{i \in I} \subset K$ is convergent over any ultrafilter $\mathcal{U}$ on $I$.

3. Fixed point theorems

Let $C$ be a nonempty subset of a Banach space $X$, and let $T : C \to C$ be
a mapping. Suppose there exists a nonempty subset $K$ of $C$ and the weak limit $w\text{-}\lim_{\mathcal{U}} T^nx$ exists in $K$ for each $x \in K$, where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. We then can define a mapping $S : K \to K$ by

$$Sx = w\text{-}\lim_{\mathcal{U}} T^nx, \quad x \in K. \quad (1)$$

Note first that if $K$ is weakly compact and $T$-invariant, by (ii) of Lemma 2.2, the weak limit $w\text{-}\lim_{\mathcal{U}} T^nx$ always exists in $K$ for each $x \in K$. Furthermore, we can see that $Fix(T) \cap K \subset Fix(S)$. What are conditions on $X$ and $T$ for which the converse inclusion remains true? Our purpose is to find some conditions on $X$ and $T$ to answer the above question.

First, we exhibit the following easy lemma for our argument.

**Lemma 3.1.** Let $C$ be a nonempty subset of a reflexive Banach space $X$. If $T : C \to C$ is a continuous mapping of partly asymptotically nonexpansive type, then there exist a nonempty weakly compact convex and $T$-invariant subset $K$ of $C$ such that $c_n(x; K) \to 0$ for each $x \in K$, and a nonexpansive mapping $S : K \to K$.

**Proof.** Since $T$ is of partly asymptotically nonexpansive type and $X$ is reflexive, there exists a nonempty weakly compact convex and $T$-invariant subset $K$ of $C$ such that $c_n(x; K) \to 0$ for each $x \in K$. Now defining $S : K \to K$ as in (1), $S$ is nonexpansive. In fact, for $x, y \in K$, $Sx = w\text{-}\lim_{\mathcal{U}} T^nx$ and $Sy = w\text{-}\lim_{\mathcal{U}} T^ny$. By (i) of Lemma 2.2, we have $Sx - Sy = w\text{-}\lim_{\mathcal{U}} (T^nx - T^ny)$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $T^{n_k}x - T^{n_k}y \to Sx - Sy$ as $k \to \infty$. Since the norm $\| \cdot \|$ is weakly lower semicontinuous and $c_n(x; K) \to 0$ as $n \to \infty$ for each $x \in K$, we have

$$\|Sx - Sy\| \leq \liminf_{k \to \infty} \|T^{n_k}x - T^{n_k}y\|$$

$$\leq \limsup_{k \to \infty} (\|T^{n_k}x - T^{n_k}y\| - \|x - y\| + \|x - y\|$$

$$\leq \lim_{k \to \infty} c_{n_k}(x; K) + \|x - y\| = \|x - y\|$$

for all $x, y \in K$. \hfill $\Box$

Now we will present a partial answer of the above question; that is, a sufficient condition for $Fix(S) \subset Fix(T) \cap K$, with a slight modification of the proof in Lemma 3.1 of [13]. Here we shall give the detailed proof for convenience sake.

**Theorem 3.2.** Let $C$ be a nonempty subset of a reflexive Banach space $X$ with $WCS(X) > 1$. If $T : C \to C$ is a continuous mapping of partly asymptotically nonexpansive type and weakly asymptotically regular on $C$, then there exist a nonempty weakly compact convex and $T$-invariant subset $K$ of $C$ and a nonexpansive mapping $S : K \to K$ such that $Fix(T) \cap K = Fix(S) \neq \emptyset$.

**Proof.** Let $K$ and $S : K \to K$ be as in Lemma 3.1. Clearly, $Fix(S) \neq \emptyset$ by Kirk [15]. Now to complete the proof, it suffices to show that $Fix(S) \subset Fix(T) \cap K$. 

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To this end, let \( x \in \text{Fix}(S) \); that is, \( w \)-\( \lim \)\( T^nx = x \in K \). Then there exists a subsequence \( \{T^{n_k}x\} \) of the sequence \( \{T^n x\} \) in \( K \) such that \( T^{n_k}x \to x \) as \( k \to \infty \). By the well known property of \( WCS(X) \),

\[
\limsup_{k \to \infty} \|T^{n_k}x - x\| \leq \frac{1}{WCS(X)} D(\{T^{n_k}x\}). \tag{2}
\]

By weakly asymptotic regularity of \( T \), it follows that \( T^{n_k+m}x \to x \) as \( k \to \infty \) for any \( m \geq 0 \). On the other hand, for each \( i, j \in \mathbb{N} \) with \( i > j \), the weak lower semicontinuity of the norm \( \| \cdot \| \) immediately yields that

\[
\|T^{n_j}x - T^{n_i}x\| \\
\leq (\|T^{n_j}x - T^{n_j}(T^{n_i-x_j}x)\| - \|x - T^{n_i-x_j}x\|) + \|x - T^{n_i-x_j}x\| \\
\leq c_{n_j}(x; K) + \|x - T^{n_i-x_j}x\| \quad (T^{n_k+m}x \to x \text{ as } k \to \infty, \text{ with } m = n_i - n_j) \\
\leq c_{n_j}(x; K) + \liminf_{k \to \infty} \|T^{n_k+m}x - T^{n_i-x_j}x\| \\
\leq c_{n_j}(x; K) + c_{n_i-x_j}(x; K) + \limsup_{k \to \infty} \|x - T^{n_k}x\|.
\]

Taking \( \limsup_{k \to \infty} \) first and next \( \limsup_{j \to \infty} \) on both sides, since \( c_n(x; K) \to 0 \) for each \( x \in K \), this yields

\[
D(\{T^{n_k}x\}) \leq \limsup_{k \to \infty} \|x - T^{n_k}x\|,
\]

and this together with (2) gives \( (WCS(X) - 1) \cdot \limsup_{k \to \infty} \|T^{n_k}x - x\| \leq 0 \), which in turn implies that \( x = \lim_{k \to \infty} T^{n_k}x \). By the continuity and weak asymptotic regularity of \( T \), we have \( Tx = x \), i.e., \( x \in \text{Fix}(T) \). \( \square \)

**Remark 3.1.** (i) Note that if \( C \) is weakly compact convex, the reflexivity of \( X \) can be removed in Theorem 3.2.

(ii) Following (ii) of Remark 2.1, \( D(X) = 1/WCS(X) \) for every infinite dimensional reflexive space \( X \). Therefore, the assumption in Theorem 3.2 which \( X \) is a reflexive Banach space with \( WCS(X) > 1 \) can be replaced by \( D(X) < 1 \).

(iii) As a direct consequence of the proof of Theorem 3.2, we notice that, under the same assumptions of \( C \), \( X \) and \( T \), if \( \{T^{n_k}x\} \) is a subsequence of \( \{T^n x\} \) converging weakly to \( x \in K \), then \( \lim_{k \to \infty} T^{n_k}x = x \). However, if the whole sequence \( \{T^n x\} \) converges weakly, the weakly asymptotic regularity on \( C \) for \( T \) is abundant.

**Lemma 3.3.** Let \( C \) be a nonempty subset of a reflexive Banach space \( X \) with \( WCS(X) > 1 \). If \( T : C \to C \) is a continuous mapping of partly asymptotically nonexpansive type, then \( \lim_{n \to \infty} T^n x = x \in K \Rightarrow \lim_{n \to \infty} T^n x = x \in \text{Fix}(T) \).

With the similar method of the proof as in Theorem 3.2, we observe the following

**Theorem 3.4.** Let \( C \) be a nonempty bounded subset of a Banach space \( X \) with
\[ D(X) < 1. \] Let \( T : C \to C \) be a continuous mapping of asymptotically nonexpansive type which is weakly asymptotically regular on \( C \). Suppose there exists a nonempty closed convex subset \( K \) of \( C \) with the following property
\[
x \in K \implies \omega_{w}(x) \subseteq K, \tag{\omega}
\]
where \( \omega_{w}(x) \) is the weak \( \omega \)-limit set of \( T \) at \( x \); namely, \( \omega_{w}(x) = \{ y \in X : y = \text{w-lim}_{k \to \infty} T^{n_{k}}x \text{ for some } n_{k} \uparrow \infty \} \). Then there exists a nonexpansive mapping \( S : K \to K \) such that \( \text{Fix}(T) \cap K = \text{Fix}(S) \neq \emptyset \).

**Proof.** Since \( X \) is reflexive, \( K \) is weakly compact and \( WSC(X) > 1 \). Since the sequence \( \{ T^{n}x \} \) belongs to \( C \), and \( \text{co}(C) \) is weakly compact, the weak limit \( \text{w-lim}_{n} T^{n}x \) always exists in \( \text{co}(C) \) for each \( x \in K \) by (ii) of Lemma 2.2. Define \( Sx = \text{w-lim}_{n} T^{n}x \) for each \( x \in K \). Then, there exists a subsequence \( \{ n_{k} \} \) of \( n \) such that \( T^{n_{k}}x \to Sx \) as \( k \to \infty \). By property of \( (\omega) \), it follows that \( Sx \in \omega_{w}(x) \subseteq K \). Therefore, \( S : K \to K \) is well defined, and also nonexpansive. Thus, repeating the method of proof in Theorem 3.2, we can easily obtain the conclusion. \( \square \)

It is clear that if \( C \) is a nonempty bounded subset of a Banach space \( X \), and if \( T : C \to C \) is an asymptotically nonexpansive mapping with its Lipschitz constant of \( T^{n} \), \( k_{n} \geq 1 \), then \( T \) is a uniformly Lipschitzian mapping of asymptotically nonexpansive type. Therefore, we have the following easy result.

**Corollary 3.5.** Let \( C \) be a nonempty bounded subset of a Banach space \( X \) with \( D(X) < 1 \). Let \( T : C \to C \) be an asymptotically nonexpansive mapping which is weakly asymptotically regular on \( C \). Suppose there exists a nonempty closed convex subset \( K \) of \( C \) with the property \((\omega)\). Then there exists a nonexpansive mapping \( S : K \to K \) such that \( \text{Fix}(T) \cap K = \text{Fix}(S) \neq \emptyset \).

Let \( C \) be a weakly compact convex subset of a Banach space \( X \). Consider a family \( \mathcal{F} \) of subsets \( K \) of \( C \) which are nonempty, closed, convex, and satisfy the following property \((\omega)\). The weak compactness of \( C \) now allows one to use Zorn’s lemma to obtain a minimal element (say) \( K \in \mathcal{F} \). Therefore, as a direct consequence of Theorem 3.2 or 3.4, we have the following result due to Kim-Kim [13].

**Corollary 3.6.** Let \( C \) be a nonempty weakly compact convex subset of a Banach space \( X \) with \( \text{WCS}(X) > 1 \). If \( T : C \to C \) is a continuous mapping of asymptotically nonexpansive type and weakly asymptotically regular on \( C \), then \( \text{Fix}(T) \) is a nonempty nonexpansive retract of \( C \).

**Proof.** Note first that \( T \) is of partly asymptotically nonexpansive type with \( K = C \). Since \( C \) is weakly compact and convex, in view of (i) of Remark 3.1, we can apply for Theorem 3.2 or 3.4, and hence \( \text{Fix}(T) = \text{Fix}(S) \neq \emptyset \). Since \( S \) is nonexpansive, it follows from [3] that \( \text{Fix}(S) \) is a nonempty nonexpansive retract of \( C \). \( \square \)

Recall that a Banach space \( X \) is said to be uniformly convex in every direction \([9]\) if \( \delta_{x}(\epsilon) > 0 \) for all \( \epsilon > 0 \) and all \( z \in X \) with \( \|z\| = 1 \), where \( \delta_{x}(\cdot) \) means the
The modulus of convexity of $X$ in the direction $z$, that is,

$$\delta_z(\epsilon) = \{1 - \|x + y\|/2 : \|x\| \leq 1, \|y\| \leq 1, x - y = \epsilon z\}.$$  

There is clearly a space $X$ which may be uniformly convex in every direction while failing to be uniformly convex. Obviously, such spaces are always strictly convex.

**Theorem 3.7.** Suppose that $X$ is a reflexive Banach space which is uniformly convex in every direction and for which $\text{WCS}(X) > 1$ and that $C$ is a nonempty subset of $X$. Then, if $T : C \to C$ is a continuous mapping of partly asymptotically nonexpansive type, $T$ has a fixed point.

**Proof.** Use the same argument presented in the proof of Theorem 5 in [19] and Lemma 3.3. \hfill \Box

Finally, we shall give examples of non-Lipschitzian mappings of partly asymptotically nonexpansive type which are not of asymptotically nonexpansive type, inspired by the example 4.3 and 4.4 in [11]. These examples also satisfy all assumptions of Theorem 3.2.

**Example A.** Let $X = C = \mathbb{R}$, the set of real numbers, and let $|k| < 1$. For each $x \in C$, we define

$$Tx = \begin{cases} 
kx \sin \frac{1}{2^n}, & x \neq 0, |x| \leq 1/\pi; \\
0, & x = 0; \\
\pi|x| - 1, & |x| > 1/\pi.\end{cases}$$

Then, clearly $c_n(1) = c_n(1; C) \geq T^n 1 - 1 \to \infty$, and so $T$ is not of asymptotically nonexpansive type. Note further that $c_n(x) = c_n(x, C) \to \infty$ for all fixed $x \in C$. But if we take $K = [-1/\pi, 1/\pi]$, then $K$ is $T$-invariant and also $T$ is of partly asymptotically nonexpansive type. Indeed, it suffices to show that $c_n(x; K) \to 0$ for each $x \in K$. For fixed $x \in K$ and $n \in \mathbb{N}$, set

$$H_n(y) = |T^n x - T^n y| - |x - y|, \quad y \in K.$$  

Then $H_n(\cdot)$ is continuous on $K$, and so it achieves its maximum in $K$, i.e., there exists a $y_n \in K$ such that $c_n(x; K) = H_n(y_n) \vee 0$. Since $T^n z \to 0$ uniformly on $K$, we have $c_n(x; K) \to 0$ for each $x \in K$.

**Example B.** Let $X = \mathbb{R}$ and $C = (-\infty, 1]$. First consider a continuous non-Lipschitzian mapping $f : [0, 1/2] \to [0, 1/4]$ defined by

$$f(x) = \begin{cases} 
n(2n+1)x \frac{1}{2n+1}, & 0 \leq x \leq \frac{1}{2n}, n \geq 1; \\
(n+1)(2n+1)x \frac{1}{n+2}, & \frac{1}{2(n+1)} \leq x \leq \frac{1}{2n+1}, n \geq 1; \\
0, & x = 0.
\end{cases}$$

Then $\|f(x) - f(y)\| \leq \epsilon \|x - y\|/2$, and so $f$ is asymptotically nonexpansive. But if we take $x = (1/2)^n$ and $y = (1/2)^{n+1}$, then $f(x) - f(y) = \epsilon (1/2)^{n+1}$, and so $f$ is not Lipschitzian.
Note first that for each \( n \in \mathbb{N} \), the graph of \( f \) on each subinterval \([1/2(n+1), 1/2n]\) consists of two segments connecting three points \((1/2(n+1), 1/2(n+2)), (1/2n+1, 0)\) and \((1/2n, 1/2(n+1))\). For each \( x \in C = (-\infty, 1] \), we now define

\[
T_x = \begin{cases} \frac{x}{1-2x}, & \text{if } x \leq -\frac{1}{2}; \\ f(x), & \text{if } x \in [0, 1/2]; \\ -f(-x), & \text{if } x \in [-1/2, 0]; \\ x^2, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}
\]

Obviously, \(|T^n z| \leq \frac{1}{2(n+1)}\) for \(|z| \leq \frac{1}{2} \), and so \(T^n z \to 0\) uniformly on \([-1/2, 1/2]\).

Also, since \(|Tz| \leq 1/2\) for \(z \leq -1/2\), we also have \(T^n z \to 0\) uniformly on \((-\infty, -1/2]\). We thus obtain \(T^n z \to 0\) uniformly on \((-\infty, 1/2]\). It is obvious that \(T\) is not of asymptotically nonexpansive type because \(c_n(1) = 1\) for each \(n\).

However, if we take \(K := [-1/2, 1/2]\), it is easy to see that \(K\) is \(T\)-invariant and \(T\) is of partly asymptotically nonexpansive type, i.e., \(c_n(x; K) \to 0\) for each \(x \in K\).

**Remark 3.2.** If we take \(K := [-1/2, 0]\) in Example B, for this \(T\)-invariant closed interval \(K\) of \(C\), we can further prove that \(c_n(x) \to 0\) for each \(x \in K\). Indeed, for \(x \in K\), we set

\[
c_n(x) = \sup_{y \in C}(|T^n x - T^n y| - |x - y|) \vee 0
\]

\[
= \sup_{y \in (-\infty, 1/2]}(|T^n x - T^n y| - |x - y|) \vee \sup_{y \in [1/2, 1]}(|T^n x - T^n y| - |x - y|) \vee 0
\]

\[
:= A_n(x) \vee B_n(x) \vee 0.
\]

Since \(T^n z \to 0\) uniformly on \((-\infty, 1/2]\), \(A_n(x) \to 0\) as \(n \to \infty\). Now it suffices to show that \(\limsup_{n \to \infty} B_n(x) \leq 0\). For each \(n \in \mathbb{N}\), there exists \(y_n \in [1/2, 1]\) such that \(B_n(x) = |T^n x - T^n y_n| - |x - y_n|\). If \(y_n = 1\), since \(-\frac{1}{2(n+1)} \leq T^n x \leq 0\), we have \(|T^n x - 1| = 1 - T^n x \leq 1 - x = |x - 1|\) for sufficiently large \(n\), and so \(\limsup_{n \to \infty}(|T^n x - 1| - |x - 1|) \leq 0\). Also if \(y_n \in [1/2, 1]\), we easily have

\[
\limsup_{n \to \infty}(|T^n x - T^n y_n| - |x - y_n|) = -\liminf_{n \to \infty} |x - y_n| \leq 0.
\]

Thus, \(\limsup_{n \to \infty} B_n(x) \leq 0\) is obtained, and therefore \(c_n(x) \to 0\) for each \(x \in K\).

Finally, note that every sequence \(\{T^n x\}\) converges uniformly to \(0 \in Fix(T) \cap K = \{0\}\) for each \(x \in K\).

**References**


