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Weighted L^p Estimates for a Rough Maximal Operator

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Abstract. This paper is concerned with studying the weighted L^p boundedness of a class of maximal operators related to homogeneous singular integrals with rough kernels. We obtain appropriate weighted L^p bounds for such maximal operators. Our results are extensions and improvements of the main theorems in [2] and [5].

1. Introduction

Let $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma\,(x')$. Let Ω be a homogeneous function of degree zero on $\mathbf{R}^n$, with $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega\,(x')\, d\sigma\,(x') = 0, \tag{1.1}$$

where $x' = x/\,|x|$ for any $x \neq 0$.

Let $\mathcal{H}$ = the set of all radial functions h satisfying

$$\left(\int_0^\infty |h(r)|^2 \,\frac{dr}{r}\right)^{1/2} \leq 1.$$

For a suitable C^1 function γ on the interval $(0,\infty)$ we define the maximal operator $\mathcal{M}_{\Omega,\gamma}$ by

$$\mathcal{M}_{\Omega,\gamma} f(x) = \sup_{h\in\mathcal{H}} \left| \int_{\mathbf{R}^n} f(x-\gamma(|y|)y')h(|y|)\Omega\,(y')\,|y|^{-n}\,dy \right|, \tag{1.2}$$

where $y' = y/\,|y| \in \mathbf{S}^{n-1}$ and $f \in \mathcal{S}(\mathbf{R}^n)$, the space of Schwartz functions.

For the sake of simplicity, we denote $\mathcal{M}_{\Omega,\gamma} = \mathcal{M}_\Omega$ if $\gamma(t) = t$.

In [2], L. K. Chen and H. Lin studied the L^p boundedness of the maximal operator $\mathcal{M}_\Omega$ under a smoothness condition on Ω. In fact, they proved the following:

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Theorem A ([2]). *Assume $n \geq 2$ and $\Omega \in C(\mathbf{S}^{n-1})$ satisfying* (1.1). *Then*

$$\|\mathcal{M}_\Omega(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)}$$

for $2n/(2n-1) < p < \infty$ and $f \in L^p$. Moreover, the range of p is the best possible.

Recently, in [5] Y. Ding and H. Qingzheng showed that the smoothness condition assumed on Ω was not necessary as described in the following theorem.

Theorem B ([5]). *Assume $n \geq 2$ and $\Omega \in L^2(\mathbf{S}^{n-1})$ satisfying* (1.1). *Then*

$$\|\mathcal{M}_\Omega(f)\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)}, \tag{1.3}$$

if p and ω satisfy one of the following conditions:

(a) $2 \leq p < \infty$ *and* $\omega \in A_{p/2}(\mathbf{R}^n)$;

(b) $2n/(2n-1) < p < 2$, $\omega(x) = |x|^\alpha$, *and* $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np-2n-p)$.

Here $A_p(\mathbf{R}^n)$ is the Muckenhoupt's weight class whose definition will be recalled in Section 2 and the weighted $L^p(\omega) = L^p(\mathbf{R}^n, \omega(x)dx)$, $\omega \geq 0$, is defined by

$$L^p(\mathbf{R}^n, \omega(x)dx) = \left\{ f : \|f\|_{L^p(\omega)} = \left(\int_{\mathbf{R}^n} |f(x)|^p \,\omega(x)dx \right)^{1/p} < \infty \right\}.$$

In light of the above results, the following natural questions arise:

Question 1. Under similar conditions on ω in Theorem B, does the $L^p(\omega)$ boundedness of the operator $\mathcal{M}_\Omega$ still hold under the condition $\Omega \in L^q(\mathbf{S}^{n-1})$, for some $q \neq 2$?

Question 2. Does the $L^p(\mathbf{R}^n)$ (or $L^p(\omega)$) boundedness of the operator $\mathcal{M}_\Omega$ would hold under a weaker condition than the condition $\Omega \in L^q(\mathbf{S}^{n-1})$, for $q > 1$?

We are able to obtain answers to these questions in the affirmative. More precisely, we have the following results:

Theorem 1.1. *Suppose that $n \geq 2$ and $\Omega \in L^q(\mathbf{S}^{n-1})$ $(q > 1)$ satisfying* (1.1). *Then*

$$\|\mathcal{M}_\Omega(f)\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)}, \tag{1.4}$$

if p and ω satisfy one of the following conditions:

(a) $\delta \leq p < \infty$ *and* $\omega \in A_{p/\delta}$;

(b) $2n\delta/(2n+n\delta-2) < p < 2$, $\omega(x) = |x|^\alpha$, $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np-2n-p)$, *where $\delta = \max\{2, q'\}$ and q' is the dual exponent of q.*

For radial weights we are able to prove the following sharper and more general result:

Theorem 1.2. *Suppose that* $n \geq 2$ *and* Ω *satisfies* (1.1). *Suppose* γ *is in* $C^2([0,\infty))$, *convex, and increasing function with* $\gamma(0) = 0$. *Then*

$$\|\mathcal{M}_{\Omega,\gamma}(f)\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)} \tag{1.5}$$

if p, Ω *and* ω *satisfy one of the following conditions*:

(a) $\Omega \in H^1(\mathbf{S}^{n-1})$, $\omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+)$, $2 \leq p < \infty$;

(b) $\Omega \in L^q(\mathbf{S}^{n-1})$ $(q > 1)$, $2n\delta/(2n + n\delta - 2) < p < 2$, $\omega(x) = |x|^\alpha$, $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np - 2n - p)$.

Here $\tilde{A}^I_p(\mathbf{R}_+)$ is a special class of radial weights introduced by Duoandikoetxea [7] and $H^1(\mathbf{S}^{n-1})$ represents the Hardy space on the unit sphere. The definitions of $\tilde{A}^I_p(\mathbf{R}_+)$ and $H^1(\mathbf{S}^{n-1})$ will be reviewed in Section 2.

It is known that for any $q > 1$,

$$C(\mathbf{S}^{n-1}) \subseteq L^q(\mathbf{S}^{n-1}) \subseteq L \log^+ L(\mathbf{S}^{n-1}) \subseteq H^1(\mathbf{S}^{n-1}) \tag{1.6}$$

and all inclusions are proper.

By the relationships in (1.6) remarked above one sees that even in the special case $\gamma(t) = t$ $(\mathcal{M}_{\Omega,\gamma} = \mathcal{M}_\Omega)$, Theorems 1.1-1.2 represent improvements of Theorems A and B.

The paper is organized as follows. A few definitions and lemmas will be recalled or proved in Section 2. Section 3 contains the proofs of the main theorems.

Throughout this paper, the letter C will stand for a positive constant that may vary at each occurrence. However, C does not depend on any of the essential variables.

2. Some definitions and lemmas

We start this section with recalling the definition of some special classes of weights and some of their important properties which relevant to our current study.

Definition 2.1. A locally integrable nonnegative function ω is said to belong to $A_p(\mathbf{R}^n)$ $(1 < p < \infty)$ if there is a positive constant C such that

$$\sup_{Q \subset \mathbf{R}^n} \left(|Q|^{-1} \int_Q \omega(x)dx\right) \left(|Q|^{-1} \int_Q \omega(x)^{-1/(p-1)}dx\right)^{p-1} \leq C,$$

and a locally integrable nonnegative function ω is said to belong to $A_1(\mathbf{R}^n)$ if there is a positive constant C such that

$$|Q|^{-1} \int_Q \omega(y)dy \leq C\omega(x), \text{ a.e. } x \in Q,$$

or equivalently $M^*\omega(x) \leq C\omega(x)$ a.e. $x \in \mathbf{R}^n$, where Q denotes a cube in $\mathbf{R}^n$ with its sides parallel to the coordinate axes and M^*f denotes the usual Hardy-Littlewood maximal function.

Definition 2.2. Let $1 \leq p < \infty$. If $\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p}$,where either $\nu_i \in A_1(\mathbf{R}_+)$ is decreasing or $\nu_i^2 \in A_1(\mathbf{R}_+)$, $i = 1, 2$, thenwe say that $\omega \in \tilde{A}_p(\mathbf{R}_+)$.

Let $A_p^I(\mathbf{R}^n)$ be the weight class defined by exchanging the cubes in the definitions of A_p for all n-dimensional intervals with sides parallel to coordinate axes (see [11]). Let $\tilde{A}_p^I = \tilde{A}_p \cap A_p^I$. If $\omega \in \tilde{A}_p$, it follows from [7] that M^*f is bounded on $L^p(\mathbf{R}^n, \omega(|x|)dx)$. Therefore, if $\omega(t) \in \tilde{A}_p(\mathbf{R}_+)$, then $\omega(|x|) \in A_p(\mathbf{R}^n)$.

By following the same argument as in the proof of the elementary properties of A_p weight class (see for example [10]) we get the following:

Lemma 2.3. *If $1 \leq p < \infty$, then the weight class $\tilde{A}_p^I(\mathbf{R}_+)$ has the following properties*:

(i) $\tilde{A}_{p_1}^I \subset \tilde{A}_{p_1}^I$, *if* $1 \leq p_1 < p_2 < \infty$;

(ii) *For any $\omega \in \tilde{A}_p^I$, there exists an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}_p^I$*;

(iii) *For any $\omega \in \tilde{A}_p^I$ and $p > 1$, there exists an $\varepsilon > 0$ such that $p - \varepsilon > 1$ and $\omega \in \tilde{A}_{p-\varepsilon}^I$.*

Now, let us recall the definition of the Hardy space $\mathbf{H}^1(\mathbf{S}^{n-1})$ and some of its important properties. The Hardy space has many equivalent definitions, one of which is given in terms of the following radial maximal operator on $\mathbf{S}^{n-1}$:

$$P^+ : f \to \sup_{0 \leq r < 1} \left| \int_{\mathbf{S}^{n-1}} P_{rx}(y)f(y)d\sigma(y) \right|,$$

where $P_x(y) = \left(1 - |x|^2\right) / |x - y|^n$.

Definition 2.4. An integrable function $f$on $\mathbf{S}^{n-1}$ is in the space $\mathbf{H}^1(\mathbf{S}^{n-1})$ if $\|f\|_{\mathbf{H}^1(\mathbf{S}^{n-1})} = \|P^+f\|_{L^1(\mathbf{S}^{n-1})} < \infty$.

Now let us recall the atomic decomposition of $\mathbf{H}^1(\mathbf{S}^{n-1})$. For $x_0 \in \mathbf{S}^{n-1}$ and $\rho > 0$ we let $\mathbf{B}(x_0, \rho) = \mathbf{S}^{n-1} \cap \{y \in \mathbf{R}^n : |y - x_0| < \rho\}$.

Definition 2.5. A function $a(\cdot)$ on $\mathbf{S}^{n-1}$ is called an ∞-regular atom if there exist $x_0 \in \mathbf{S}^{n-1}$ and $\rho \in (0, 2]$ such that

$$\operatorname{supp}(a) \subset \mathbf{B}(x_0, \rho); \tag{2.1}$$

$$\|a\|_\infty \leq \rho^{-n+1}; \tag{2.2}$$

$$\int_{\mathbf{S}^{n-1}} a(y')d\sigma(y') = 0. \tag{2.3}$$

A very useful characterization of the space $\mathbf{H}^1(\mathbf{S}^{n-1})$ is its atomic decomposition. The atomic decomposition of $\mathbf{H}^1(\mathbf{S}^{n-1})$ is given by the following lemma:

Lemma 2.6. *If $\Omega \in \mathbf{H}^1(\mathbf{S}^{n-1})$ and satisfies the mean value zero property* (1.1)*, then there exist $\{c_j\}_{j\in\mathbf{N}} \subseteq \mathbf{C}$ and ∞-regular atoms $\{a_j\}_{j\in\mathbf{N}}$ such that $\Omega = \sum_{j=1}^{\infty} c_j a_j$ and $\|\Omega\|_{\mathbf{H}^1(\mathbf{S}^{n-1})} \approx \sum_{j=1}^{\infty} |c_j|$.*

For any non-zero $\xi = (\xi_1, \cdots, \xi_n) \in \mathbf{R}^n$, we write $\xi' = \xi/|\xi| = (\xi_1', \cdots, \xi_n') = (\xi_1', \xi_*')$. For a fixed $\rho > 0$, we let $L_\rho(\xi) = (\rho^2\xi_1, \rho\xi_2, \cdots, \rho\xi_n) = (\rho^2\xi_1, \rho\xi_*)$ and let $r \equiv r(\xi') = |\xi|^{-1}|L_\rho(\xi)|$.

In proving our main results we shall need the following two results proved by Fan and Pan in [9]:

Lemma 2.7. *Let a be an ∞-regular atom on $\mathbf{S}^{n-1}$ $(n \geq 3)$ with supp$(a) \subseteq \mathbf{B}(\xi', \rho)$ $(0 < \rho \leq 1)$. Let*

$$F_a(s, \xi') = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{\mathbf{S}^{n-2}} a(s, (1-s^2)^{1/2}\tilde{y})d\sigma(\tilde{y}).$$

Then up to a constant multiplier independent of a, $F_a(s, \xi')$ is an ∞-regular atom on $\mathbf{R}$. More precisely, there is a constant C independent of a such that

$$\text{supp}(F_a) \subseteq (\xi_1' - 3r, \xi_1' + 3r); \tag{2.4}$$

$$\|F_a\|_\infty \leq Cr^{-1}; \tag{2.5}$$

$$\int_{\mathbf{R}} F_a(s)ds = 0, \tag{2.6}$$

where $r \equiv r(\xi') = \left|(\rho^2\xi_1', \rho\xi_*')\right|$.

Lemma 2.8. *Suppose that $n = 2$ and a is an ∞-regular atom satisfying* (2.1)-(2.3)*. Let $\xi' = (\xi_1', \xi_2') \in \mathbf{S}^1$ be the center of the support of $a(\cdot)$. Let*

$$f_a(s, \xi') = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)\left\{a\left(s, \sqrt{1-s^2}\right) + a\left(s, -\sqrt{1-s^2}\right)\right\}.$$

Then up to a constant multiplier independent of a, $f_a(s, \xi')$ is an ∞-regular atom on $\mathbf{R}$. The radius of their support is $r \equiv r(\xi') = \rho\sqrt{\rho\left(\xi_1'\right)^2 + \left(\xi_2'\right)^2}$, and the center of their support is ξ_1'.

Lemma 2.9. *Suppose that $a(\cdot)$ is an ∞-regular atom on $\mathbf{S}^{n-1}$ with supp$(a) \subseteq \mathbf{B}(\mathbf{e}, \rho)$, where $\mathbf{e} = (1, \cdots, 0) \in \mathbf{S}^{n-1}$. Let γ be in $C^2([0, \infty))$, convex, and increasing function with $\gamma(0) = 0$ and let*

$$I_{a,\gamma,k}(\xi) = \left(\int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{S}^{n-1}} a(y')e^{-i\gamma(t)<\xi,y'>}d\sigma(y')\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Then there exists a positive constant C independent of k, ξ and ρ such that

$$|I_{a,\gamma,k}(\xi)| \leq C\left|\gamma(2^{k+1})L_\rho(\xi)\right|^{\frac{1}{4}}; \tag{2.7}$$

$$|I_{a,\gamma,k}(\xi)| \leq C\left|\gamma(2^{k})L_\rho(\xi)\right|^{-\frac{1}{4}}. \tag{2.8}$$

Proof. We shall only prove (2.7)-(2.8) for the case $n > 2$, since the proof for $n = 2$ is essentially the same (we use Lemma 2.5 instead of Lemma 2.4). For any $\xi \neq 0$, we choose a rotation θ such that $\theta(\xi) = |\xi|\,\mathbf{e}$. Let $y' = (s, y_2', \cdots, y_n')$. Then

$$|I_{a,\gamma,k}(\xi)|^2 = \int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{S}^{n-1}} a(\theta^{-1}y')e^{-i\gamma(t)|\xi|<\mathbf{e},y'>}d\sigma(y')\right|^2 \frac{dt}{t},$$

where θ^{-1} is the inverse of θ. Now, $a(\theta^{-1}y')$ is again a regular $\infty-$atom with support in $\mathbf{B}(\xi',\rho)$, where $\xi' = \xi/|\xi|$. For simplicity, we still denote it by $a(y')$. Thus we have

$$|I_{a,\gamma,k}(\xi)|^2 = \int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{R}} F_a(s,\xi')e^{-i\gamma(t)|\xi|s}ds\right|^2 \frac{dt}{t},$$

where $F_a(s,\xi')$ has support in $(\xi_1' - 3r, \xi_1' + 3r)$ (see Lemma 2.4). By the cancelation property of $F_a(s,\xi')$ and the conditions on γ we have

$$\begin{aligned}
|I_{a,\gamma,k}(\xi)|^2 &= \int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{R}} F_a(s,\xi')\left(e^{-i\gamma(t)|\xi|s} - 1\right)ds\right|^2 \frac{dt}{t} \\
&\leq Cr^{-2}\left(|\xi|\,\gamma(2^{k+1})\right)^2 \left(\int_{\xi_1'-3r}^{\xi_1'+3r} |s|\,ds\right)^2 \\
&\leq C\left|\gamma(2^{k+1})L_\rho(\xi)\right|^2.
\end{aligned}$$

Thus by combining the last estimate with the trivial estimate $|I_{a,\gamma,k}(\xi)| \leq C$ we get (2.7). To prove (2.8), we notice that

$$|I_{a,\gamma,k}(\xi)|^2 = \int_{\mathbf{R}\times\mathbf{R}} \left(\int_{2^k}^{2^{k+1}} e^{-i\gamma(t)|\xi|(s-u)}\frac{dt}{t}\right) F_a(s,\xi')\overline{F_a(u,\xi')}dsdu.$$

Denote

$$J_k(\xi,s,u) = \left(\int_{2^k}^{2^{k+1}} e^{-i\gamma(t)|\xi|(s-u)}\frac{dt}{t}\right).$$

By a simple change of variable, we have

$$J_k(\xi,s,u) = \int_1^2 e^{-i\gamma(2^k t)|\xi|(s-u)}\frac{dt}{t} \equiv \int_1^2 G'(t)\frac{dt}{t},$$

where

$$G(t)=\int_1^t e^{-i\gamma(2^k w)|\xi|(s-u)}dw,\ 1\le t\le 2.$$

By the assumptions on γ, we obtain

$$\frac{d}{dw}\gamma(2^k w)=2^k\gamma'(2^k w)\ge\frac{\gamma(2^k w)}{w}\ge\frac{\gamma(2^k)}{t}\text{ for }1\le w\le t\le 2.$$

Thus by van der Corput's lemma, $|G(t)|\le\left|\xi(s-u)\gamma(2^k)\right|^{-1}t$, for $1\le t\le 2$. Hence by integration by parts,

$$|J_k(\xi,s,u)|\le C\left|\xi(s-u)\gamma(2^k)\right|^{-1}.$$

This estimate when combined with the trivial estimate $J_k(\xi,s,u)\le\ln 2$ gives

$$|J_k(\xi,s,u)|\le C\left|\xi(s-u)\gamma(2^k)\right|^{-\frac{1}{2}}.$$

Thus

$$\begin{aligned}|I_{a,\gamma,k}(\xi)|^2 &\le C\left|\xi\gamma(2^k)\right|^{-\frac{1}{2}}\int_{\mathbf{R}\times\mathbf{R}}|s-u|^{-\frac{1}{2}}F_a(s,\xi')\overline{F_a(u,\xi')}dsdu\\ &\le C\left|\xi\gamma(2^k)\right|^{-\frac{1}{2}}r^{-1}\int_{\mathbf{R}}\left(\int_{\xi_1'-3r}^{\xi_1'+3r}|s-u|^{-\frac{1}{2}}ds\right)\left|\overline{F_a(u,\xi')}\right|du.\end{aligned}$$

By a simple change of variable we get

$$\begin{aligned}\int_{\xi_1'-3r}^{\xi_1'+3r}|s-u|^{-\frac{1}{2}}ds &\le \int_{\xi_1'-3r-u}^{\xi_1'+3r-u}|s|^{-\frac{1}{2}}ds\\ &\le \int_{-6r}^{6r}|s|^{-\frac{1}{2}}ds\le Cr^{\frac{1}{2}}.\end{aligned}$$

So,

$$|I_{a,\gamma,k}(\xi)|\le C\left|\xi r\gamma(2^k)\right|^{-\frac{1}{4}}\|F_a\|_1^{\frac{1}{2}}\le C\left|\xi r\gamma(2^k)\right|^{-\frac{1}{4}}=C\left|\gamma(2^k)L_\rho(\xi)\right|^{-\frac{1}{4}}.$$

This completes the proof of Lemma 2.9. □

For any $\Omega\in L^1(\mathbf{S}^{n-1})$, we define the maximal operator

$$M^*_{\Omega,\gamma}f(x)=\sup_{k\in\mathbf{Z}}|\mu_{k,\gamma,\Omega}*f(x)|,$$

where

$$\mu_{k,\gamma,\Omega}*f(x)=\int_{2^k\le|y|<2^{k+1}}f(x-\gamma(|y|)y')\frac{|\Omega(y')|}{|y|^n}dy.$$

If $\gamma(t)=t$, we denote $M^*_{\Omega,\gamma}$ by M^*_Ω. Then we have the following results related to the maximal operators $M^*_{\Omega,\gamma}$ and M^*_Ω.

Lemma 2.10. *Let $\Omega \in L^1(\mathbf{S}^{n-1})$ and $\omega \in \tilde{A}_p(\mathbf{R}^+)$, $1<p<\infty$. Let γ be in $C^2([0,\infty))$, convex, and increasing function with $\gamma(0)=0$. Then*

$$\left\| M^*_{\Omega,\gamma}(f) \right\|_{L^p(\omega)} \le C_p \left\| \Omega \right\|_{L^1(\mathbf{S}^{n-1})} \left\| f \right\|_{L^p(\omega)}, \tag{2.9}$$

where C_p is a constant independent of Ω and $f \in L^p(\omega)$.

Proof. Using the spherical coordinate we have

$$|\mu_{k,\gamma,\Omega} * f(x)| \le \int_{2^k}^{2^{k+1}} \int_{\mathbf{S}^{n-1}} |f(x-\gamma(t)y')| \, |\Omega(y')| \, d\sigma(y') \frac{dt}{t}.$$

Let $s=\gamma(t)$. By the assumptions on γ, we have

$$\begin{aligned} M^*_{\Omega,\gamma} f(x) &\le \int_{\gamma(2^k)}^{\gamma(2^{k+1})} \int_{\mathbf{S}^{n-1}} |f(x-sy')| \, |\Omega(y')| \, d\sigma(y') \frac{ds}{s} \\ &\le \int_{\mathbf{S}^{n-1}} |\Omega(y')| \, M_{y'} f(x) d\sigma(y') \qquad (2.10) \end{aligned}$$

where

$$M_{y'} f(x) = \sup_{R \in \mathbf{R}} \frac{1}{R} \int_0^R |f(x-sy')| \, ds$$

is the Hardy-Littlewood maximal function of f in the direction of y'. By equation (8) in [7] and since $\omega \in \tilde{A}_p(\mathbf{R}^+)$ we have

$$\| M_{y'} f \|_{L^p(\omega)} \le C \| f \|_{L^p(\omega)} \text{ with } C \text{ independent of } y'. \tag{2.11}$$

By (2.10) and Minkowski's inequality for integrals we have

$$\left\| M^*_{\Omega,\gamma} f \right\|_{L^p(\omega)} \le \int_{\mathbf{S}^{n-1}} |\Omega(y')| \, \| M_{y'} f \|_{L^p(\omega)} \, d\sigma(y')$$

and hence by (2.11) we get (2.9). This completes the proof of the lemma. □

We shall need the following lemma from [7, p. 873].

Lemma 2.11. *Let $\Omega \in L^d(\mathbf{S}^{n-1})$ for some $d>1$. Then the maximal operator M^*_Ω is bounded from $L^p(\omega)$ to itself, when p and ω satisfy one of the following conditions*:

(a) $d' \le p < \infty$, $p \ne 1$ *and* $\omega \in A_{p/d'}$;

(b) $1 < p \le d$, $p \ne \infty$ *and* $\omega^{1-p'} \in A_{p'/d'}$.

Let $\mathcal{M}_S$ be the spherical maximal operator defined by

$$\mathcal{M}_S f(x) = \sup_{r>0} \int_{\mathbf{S}^{n-1}} |f(x - r\theta)|\, d\sigma(\theta).$$

We shall need the following result concerning the weighted L^p boundedness of $\mathcal{M}_S$ with power weights.

Lemma 2.12 ([8]). *Suppose that $n \geq 2$, $p > n/(n-1)$ and $1 - n < \alpha < (n-1)(p-1) - 1$. Then $\mathcal{M}_S(f)$ is bounded on $L^p(\mathbf{R}^n, |x|^{\alpha})$.*

3. Proof of theorems

In the proof of our results we will apply the machinery developed by Duoandikoetxea and Rubio de Francia in [6]. We shall start by presenting a proof of Theorem 1.2.

The proof of Theorem 1.2 under condition (a).

In view of the atomic decomposition of Ω, it suffices to show that

$$\|\mathcal{M}_{a,\gamma}(f)\|_{L^p(\omega)} \leq C_p \|f\|_{L^p(\omega)} \tag{3.1}$$

holds for $\omega \in \tilde{A}^I_{p/2}(\mathbf{R}_+)$, $2 \leq p < \infty$ and $f \in L^p(\omega)$ when a satisfies

(i) supp$(a) \subseteq \mathbf{B}(z_0, \rho)$ for some $z_0 \in \mathbf{S}^{n-1}$ and $\rho \in (0, 2]$;

(ii) $\|a\|_\infty \leq \rho^{-(n-1)}$;

(iii) $\int_{\mathbf{S}^{n-1}} a(y) d\sigma(y) = 0$.

Let us first prove (3.1) for the case $2 < p < \infty$. For each $k \in \mathbf{Z}$, let $\eta_k = \gamma(2^k)$. Since γ is convex and increasing in $(0, \infty)$, we have $\gamma(t)/t$ is also increasing for $t > 0$. Therefore, the sequence $\{\eta_k : k \in \mathbf{Z}\}$ is a lacunary sequence with $\eta_{k+1}/\eta_k \geq 2$. Since the weight function ω is radial, by using an appropriate rotation on $\mathbf{S}^{n-1}$, we may assume w.l.o.g. that $z_0 = (0, \cdots, 0, 1)$. Let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the intervals $E_j = [\eta_{j+1}^{-1}, \eta_{j-1}^{-1}]$. More precisely, we require the following:

$$\begin{aligned} \Phi_j &\in C^\infty,\ 0 \leq \Phi_j \leq 1,\ \sum_j \Phi_j(t) = 1, \\ \text{supp } \Phi_j &\subseteq E_j,\ \left| \frac{d^s \Phi_j(t)}{dt^s} \right| \leq \frac{C}{t^s}. \end{aligned}$$

By duality,

$$\begin{aligned} \mathcal{M}_{a,\gamma} f(x) &= \left(\int_0^\infty \left| \int_{\mathbf{S}^{n-1}} a(\xi) f(x - \gamma(r)\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2} \\ &= \left(\sum_{k \in \mathbf{Z}} \int_{2^k}^{2^{k+1}} \left| \int_{\mathbf{S}^{n-1}} a(\xi) f(x - \gamma(r)\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2}. \end{aligned}$$

Now if we let $\widehat{\Psi_j}(\xi) = \Phi_j(|L_\rho(\xi)|)$, then we have $f = \sum_j \Psi_{j+k} * f$ for $f \in \mathcal{S}(\mathbf{R}^n)$ and for any $k \in \mathbf{Z}$. Thus

$$\mathcal{M}_{a,\gamma}f(x) = \left(\sum_{k\in\mathbf{Z}} \int_{2^k}^{2^{k+1}} \left| \sum_j \int_{\mathbf{S}^{n-1}} a(\xi)\left(\Psi_{j+k} * f\right)(x - \gamma(r)\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2}.$$

By Minkowski's inequality, we have

$$\mathcal{M}_{a,\gamma}f(x) \le \sum_j \mathcal{M}_{a,\gamma,j}f(x), \tag{3.2}$$

where

$$\mathcal{M}_{a,\gamma,j}f(x) = \left(\sum_{k\in\mathbf{Z}} \int_{2^k}^{2^{k+1}} \left| \int_{\mathbf{S}^{n-1}} a(\xi)\left(\Psi_{j+k} * f\right)(x - \gamma(r)\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \right)^{1/2}.$$

By Plancherel's theorem and Lemma 2.9 we get

$$\begin{aligned}
\|\mathcal{M}_{a,\gamma,j}(f)\|_2^2 &= \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{2^k}^{2^{k+1}} \left| \int_{\mathbf{S}^{n-1}} a(\xi)(\Psi_{j+k} * f)(x - \gamma(r)\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} dx \\
&\le \sum_{k\in\mathbf{Z}} \int_{\Delta_{j+k}} \left(\int_{2^k}^{2^{k+1}} \left| \int_{\mathbf{S}^{n-1}} a(\xi) e^{-i\gamma(r)<\xi,x>} d\sigma(\xi) \right|^2 \frac{dr}{r} \right) \left| \hat{f}(x) \right|^2 dx \\
&\le C 2^{-2\alpha|j|} \sum_{k\in\mathbf{Z}} \int_{\Delta_{j+k}} \left| \hat{f}(x) \right|^2 dx \\
&\le C 2^{-2\alpha|j|} \|f\|_2^2,
\end{aligned}$$

where

$$\Delta_j = \{x \in \mathbf{R}^n : \eta_{j+1}^{-1} \le |L_\rho(x)| < \eta_{j-1}^{-1}\}.$$

Therefore,

$$\|\mathcal{M}_{a,\gamma,j}(f)\|_2 \le C 2^{-\alpha|j|} \|f\|_2 . \tag{3.3}$$

Now, we need to compute the $L^p(\omega)$-norm of $\mathcal{M}_{a,j}(f)$ for $p > 2$. By duality, there is a function g in $L^{(p/2)'}(\omega^{1-(p/2)'})$ with $\|g\|_{(p/2)',\omega^{1-(p/2)'}} \le 1$ such that

$$\begin{aligned}
&\left\|\mathcal{M}_{a,\gamma,j}(f)\right\|_{p,\omega}^{2} \\
= \; & \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{2^k}^{2^{k+1}}\left|\int_{\mathbf{S}^{n-1}} a(\xi)\left(\Psi_{j+k}*f\right)(x-\gamma(r)\xi)d\sigma(\xi)\right|^2\frac{dr}{r}\left|g(x)\right|dx \\
\leq \; & \|a\|_1\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\left|\Psi_{k+j}*f(x)\right|^2\int_{2^k}^{2^{k+1}}\int_{\mathbf{S}^{n-1}}|a(\xi)|\left|g(x+\gamma(r)\xi)\right|d\sigma(\xi)\frac{dr}{r}dx \\
\leq \; & C\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\left|\Psi_{k+j}*f(x)\right|^2 M^*_{a,\gamma}(\tilde{g})(-x)dx, \text{ with } \tilde{g}(x)=g(-x), \\
\leq \; & C\left\|\sum_{k\in\mathbf{Z}}\left|\Psi_{k+j}*f\right|^2\right\|_{p/2,\omega}\left\|M^*_{a,\gamma}(\tilde{g})\right\|_{(p/2)',\omega^{1-(p/2)'}}.
\end{aligned}$$

By Lemma 2.3 and since $\omega\in\tilde{A}_r(\mathbf{R}_+)$ if and only if $\omega^{1-r'}\in\tilde{A}_{r'}(\mathbf{R}_+)$ we get $\omega\in\tilde{A}^I_{p/2}(\mathbf{R}_+)\subset\tilde{A}^I_p(\mathbf{R}_+)\subset\tilde{A}_p(\mathbf{R}_+)$ and $\omega^{1-(p/2)'}\in\tilde{A}_{(p/2)'}(\mathbf{R}_+)$. Therefore, by the weighted Littlewood-Paley theory [11] and Lemma 2.10, we have

$$\left\|\mathcal{M}_{a,\gamma,j}(f)\right\|_{p,\omega}\leq C\left\|f\right\|_{p,\omega} \text{ for } 2<p<\infty \text{ and } \omega\in\tilde{A}^I_{p/2}(\mathbf{R}_+). \tag{3.4}$$

By interpolating between (3.3) and (3.4) with $\omega=1$, we get

$$\left\|\mathcal{M}_{a,\gamma,j}(f)\right\|_{p}\leq C_p2^{-\theta|j|}\left\|f\right\|_{p} \tag{3.5}$$

for $2\leq p<\infty$ and for some $\theta>0$.

Now, by Lemma 2.3, for any $\omega\in\tilde{A}^I_{p/2}(\mathbf{R}_+)$, there is an $\varepsilon>0$ such that $\omega^{1+\varepsilon}\in\tilde{A}^I_{p/2}(\mathbf{R}_+)$. Thus by (3.4) we have

$$\left\|\mathcal{M}_{a,\gamma,j}(f)\right\|_{p,\omega^{1+\varepsilon}}\leq C\left\|f\right\|_{p,\omega^{1+\varepsilon}} \text{ for } 2<p<\infty \text{ and } \omega\in\tilde{A}^I_{p/2}(\mathbf{R}_+). \tag{3.6}$$

Therefore, using the Stein-Weiss interpolation theorem with change of measure [15], we may interpolate between (3.5) and (3.6) to obtain a positive number τ such that

$$\left\|\mathcal{M}_{a,\gamma,j}(f)\right\|_{p,\omega}\leq C_p2^{-\tau|j|}\left\|f\right\|_{p,\omega} \text{ for } 2<p<\infty \text{ and } \omega\in\tilde{A}^I_{p/2}(\mathbf{R}_+) \tag{3.7}$$

which in turn implies

$$\left\|\mathcal{M}_{a,\gamma}(f)\right\|_{p,\omega}\leq C_p\sum_{j}\left\|\mathcal{M}_{a,\gamma,j}(f)\right\|_{p,\omega}\leq C_p\left\|f\right\|_{p,\omega} \tag{3.8}$$

for $2<p<\infty$ and $\omega\in\tilde{A}^I_{p/2}(\mathbf{R}_+)$. In the endpoint case $p=2$ and $\omega\in\tilde{A}^I_1(\mathbf{R}_+)$, by Schwarz inequality, and (2.10) we have

$$\begin{aligned}
&\|\mathcal{M}_{a,\gamma,j}(f)\|_{2,\omega}^2\\
=\ &\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_{2^k}^{2^{k+1}}\left|\int_{\mathbf{S}^{n-1}}a(\xi)\left(\Psi_{j+k}*f\right)(x-\gamma(r)\xi)d\sigma(\xi)\right|^2\frac{dr}{r}\omega(x)dx\\
\leq\ &\|a\|_1\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}|\Psi_{k+j}*f(x)|^2\int_{2^k}^{2^{k+1}}\int_{\mathbf{S}^{n-1}}|a(\xi)|\,|\omega(x+\gamma(r)\xi)|\,d\sigma(\xi)\frac{dr}{r}dx\\
\leq\ &C\,\|a\|_1\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}|\Psi_{k+j}*f(x)|^2\,M_{a,\gamma}^*(\tilde{\omega})(-x)dx,\ \text{with}\ \tilde{\omega}(x)=\omega(-x)\\
\leq\ &C\,\|a\|_1\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}|\Psi_{k+j}*f(x)|^2\int_{\mathbf{S}^{n-1}}|a(y')|\,M_{y'}\tilde{\omega}(-x)d\sigma(y')dx.
\end{aligned}$$

By the arguments in the proof of Theorem 7 in [7, p. 875] we infer that

$$M_{y'}(\tilde{\omega})(-x)\leq C\omega(x)\ \text{with}\ C\ \text{independent of}\ y'.$$

Thus,

$$\begin{aligned}
\|\mathcal{M}_{a,\gamma,j}(f)\|_{2,\omega}^2\ &\leq\ C\,\|a\|_1^2\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}|\Psi_{k+j}*f(x)|^2\,\omega(x)dx\\
&\leq\ C\left\|\left(\sum_{k\in\mathbf{Z}}|\Psi_{k+j}*f|^2\right)^{1/2}\right\|_{2,\omega}^2.
\end{aligned}$$

Since $\omega\in\tilde{A}_1^I(\mathbf{R}_+)\subset A_1(\mathbf{R}_+)$, by the weighted Littlewood-Paley theory we get

$$\|\mathcal{M}_{a,\gamma,j}(f)\|_{2,\omega}\leq\|f\|_{2,\omega}\ \text{for}\ \omega\in\tilde{A}_1^I(\mathbf{R}_+). \tag{3.9}$$

As above, by using the interpolation theorem with change of measure between (3.3) and (3.9), Lemma 2.3 and using (3.2) we get (3.1) for $p=2$. Thus the proof Theorem 1.2 under condition (a) is complete. □

The proof of Theorem 1.2 under condition (*b*).

Let $\{\Phi_j\}_{-\infty}^{\infty}$ be as before and $\widehat{\Lambda_j}(\xi)=\Phi_j(|\xi|)$ for $\xi\in\mathbf{R}^n$. Then by the above arguments and changing variables we get

$$\mathcal{M}_{\Omega,\gamma}f(x)\leq\sum_j S_{\Omega,\gamma,j}f(x), \tag{3.10}$$

where

$$S_{\Omega,\gamma,j}f(x)=\left(\sum_{k\in\mathbf{Z}}\int_{\gamma(1)}^{\gamma(2)}\left|\int_{\mathbf{S}^{n-1}}\Omega(\xi)\left(\Lambda_{j+k}*f\right)(x-r2^k\xi)d\sigma(\xi)\right|^2\frac{dr}{r}\right)^{1/2}.$$

By the same proof as that of (3.3) we have

$$\left\|S_{\Omega,\gamma,j}(f)\right\|_2 \le C2^{-\alpha|j|}\left\|f\right\|_2. \tag{3.11}$$

By (3.10)-(3.11), interpolation theorem with change of measure and following the same argument as in the proof of (3.7), we notice that the proof of Theorem 1.2 for condition (b) is completed if we can show that

$$\left\|S_{\Omega,\gamma,j}(f)\right\|_{p,|x|^\alpha} \le \left\|f\right\|_{p,|x|^\alpha} \tag{3.12}$$

for $2n\delta/(2n+n\delta-2) < p < 2$, $\frac{1}{2}(1-n)(2-p) < \alpha < \frac{1}{2}(2np-2n-p)$. To this end, we use the duality argument. In fact, by duality there is a function $g = g_{k,j}(x,r)$ satisfying $\|g\| \le 1$ and $g_{k,j}(x,r) \in L^{p'}\left(l^2\left[L^2\left(\left[\gamma(1),\gamma(2)\right], \frac{dt}{t}\right), k\right], |x|^{-\alpha p'/p}\,dx\right)$ such that

$$\begin{aligned}
&\left\|S_{\Omega,\gamma,j}f\right\|_{p,|x|^\alpha}\\
= &\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\gamma(1)}^{\gamma(2)}\int_{\mathbf{S}^{n-1}}\Omega(\xi)\left(\Lambda_{j+k}*f\right)(x-2^kr\xi)g_{k,j}(x,r)d\sigma(\xi)\frac{dr}{r}dx\\
= &\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\gamma(1)}^{\gamma(2)}\int_{\mathbf{S}^{n-1}}\Omega(\xi)\left(\Lambda_{j+k}*f\right)(x)g_{k,j}(x+2^kr\xi,r)d\sigma(\xi)\frac{dr}{r}dx\\
\le &\left\|\left(\sum_{k\in\mathbf{Z}}\left(\int_{\gamma(1)}^{\gamma(2)}\int_{\mathbf{S}^{n-1}}\Omega(\xi)g_{k,j}(\cdot+2^kr\xi,r)d\sigma(\xi)\frac{dr}{r}\right)^2\right)^{1/2}\right\|_{p',|x|^{-\alpha p'/p}}\\
&\times\left\|\left(\sum_{k\in\mathbf{Z}}\left|\Lambda_{j+k}*f\right|^2\right)^{1/2}\right\|_{p,|x|^\alpha}.
\end{aligned}$$

Now set

$$F(g) = \sum_{k\in\mathbf{Z}}\left(\int_{\gamma(1)}^{\gamma(2)}\int_{\mathbf{S}^{n-1}}\left|\Omega(\xi)\right|\left|g_{k,j}(\cdot+2^kr\xi,r)\right|d\sigma(\xi)\frac{dr}{r}\right)^2.$$

Since $|x|^\alpha \in A_p(\mathbf{R}^n)$ if and only if $-n < \alpha < n(p-1)$, by the weighted Littlewood-Paley theory we have

$$\left\|S_{\Omega,\gamma,j}f\right\|_{p,|x|^\alpha} \le C_p\left\|f\right\|_{p,|x|^\alpha}\left\|(F(g))^{1/2}\right\|_{p',|x|^{-\alpha p'/p}}. \tag{3.13}$$

Since $\left\|(F(g))^{1/2}\right\|_{p',|x|^{-\alpha p'/p}} = \left\|F(g)\right\|^{1/2}_{p'/2,|x|^{-\alpha p'/p}}$ and $p' > 2$, there is a function $b \in L^{(p'/2)'}(\mathbf{R}^n, |x|^{-\alpha p'/p})$ such that $\|b\|_{(p'/2)',|x|^{-\alpha p'/p}} \le 1$ and

$$\begin{aligned}
&\|F(g)\|_{p'/2,|x|^{-\alpha p'/p}} \\
= \quad & \int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\left(\int_{\gamma(1)}^{\gamma(2)}\int_{\mathbf{S}^{n-1}}|\Omega(\xi)|\left|g_{k,j}(x+2^k r\xi,r)\right|d\sigma(\xi)\frac{dr}{r}\right)^2|b(x)|\,dx.
\end{aligned}$$

Now, we need to consider two cases:

Case 1. $2n\delta/(2n+n\delta-2)<p<2$, $\frac{1}{2}(1-n)(2-p)<\alpha<\frac{1}{2}(2np-2n-p)$ and $q\geq 2$.

By Hölder's inequality, we have

$$\begin{aligned}
&\left(\int_{\gamma(1)}^{\gamma(2)}\int_{\mathbf{S}^{n-1}}|\Omega(\xi)|\left|g_{k,j}(x+2^k r\xi,r)\right|d\sigma(\xi)\frac{dr}{r}\right)^2 \\
\leq \quad & \|\Omega\|_q^2\int_{\gamma(1)}^{\gamma(2)}\left(\int_{\mathbf{S}^{n-1}}\left|g_{k,j}(x+2^k r\xi,r)\right|^{q'}d\sigma(\xi)\right)^{2/q'}\frac{dr}{r} \\
\leq \quad & \|\Omega\|_q^2\int_{\gamma(1)}^{\gamma(2)}\int_{\mathbf{S}^{n-1}}\left|g_{k,j}(x+2^k r\xi,r)\right|^2 d\sigma(\xi)\frac{dr}{r}.
\end{aligned}\tag{3.14}$$

Thus, by (3.14) and a simple change of variable we get

$$\begin{aligned}
&\|F(g)\|_{p'/2,|x|^{-\alpha p'/p}} \\
\leq \quad & C\int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\gamma(1)}^{\gamma(2)}|g_{k,j}(x,r)|^2\left(\int_{\mathbf{S}^{n-1}}\left|b(x-2^k r\xi)\right|d\sigma(\xi)\right)\frac{dr}{r}dx \\
\leq \quad & C\int_{\mathbf{R}^n}\left(\sum_{k\in\mathbf{Z}}\int_{\gamma(1)}^{\gamma(2)}|g_{k,j}(x,r)|^2\frac{dr}{r}\right)\mathcal{M}_S(|b|)(x)dx \\
\leq \quad & \left\|\left(\sum_{k\in\mathbf{Z}}\int_{\gamma(1)}^{\gamma(2)}|g_{k,j}(\cdot,r)|^2\frac{dr}{r}\right)^{1/2}\right\|_{p',|x|^{-\alpha p'/p}}^2 \|\mathcal{M}_S(|b|)\|_{(p'/2)',|x|^{2\alpha/(2-p)}}.
\end{aligned}$$

By the conditions on $p,\ q$ and α we have $(p'/2)'>n/(n-1)$ and $1-n<\frac{2\alpha}{2-p}<(n-1)\left((p'/2)'-1\right)-1$. Thus by Lemma 2.12 and the choices of g and b we obtain

$$\|F(g)\|_{p'/2,|x|^{-\alpha p'/p}}\leq C$$

which proves (3.12) for $q\geq 2$.

Case 2. $2n\delta/(2n+n\delta-2)<p<2$, $\frac{1}{2}(1-n)(2-p)<\alpha<\frac{1}{2}(2np-2n-p)$ and $1<q<2$.

By Hölder's inequality, Fubini's theorem and a change of variable, we have

$$\begin{aligned}
&\|F(g)\|_{p'/2,|x|^{-\alpha p'/p}} \\
\leq\ & \|\Omega\|_q^q \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\gamma(1)}^{\gamma(2)} \int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{2-q} \left|g_{k,j}(x+2^k r\xi, r)\right|^2 d\sigma(\xi) \frac{dr}{r}\, |b(x)|\, dx \\
\leq\ & C \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\gamma(1)}^{\gamma(2)} |g_{k,j}(x,r)|^2 \left(\int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{2-q} \left|b(x-2^k r\xi)\right| d\sigma(\xi) \right) \frac{dr}{r} dx \\
\leq\ & C \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\gamma(1)}^{\gamma(2)} |g_{k,j}(x,r)|^2 \left(\int_{\mathbf{S}^{n-1}} \left|b(x-2^k r\xi)\right|^{q'/2} d\sigma(\xi) \right)^{2/q'} \frac{dr}{r} dx \\
\leq\ & C \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\gamma(1)}^{\gamma(2)} |g_{k,j}(x,r)|^2 \left(\int_{\mathbf{S}^{n-1}} \left|b(x-2^k r\xi)\right|^{q'/2} d\sigma(\xi) \right)^{2/q'} \frac{dr}{r} dx \\
\leq\ & C \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} \int_{\gamma(1)}^{\gamma(2)} |g_{k,j}(x,r)|^2 \frac{dr}{r} \right) \left(\mathcal{M}_S(|b|^{q'/2})(x) \right)^{2/q'} dx \\
\leq\ & C \left\| \left(\sum_{k\in\mathbf{Z}} \int_{\gamma(1)}^{\gamma(2)} |g_{k,j}(\cdot,r)|^2 \frac{dr}{r} \right)^{1/2} \right\|^2_{p',|x|^{-\alpha p'/p}} \times \\
& \left\| \left(\mathcal{M}_S(|b|^{q'/2}) \right)^{2/q'} \right\|_{(p'/2)',|x|^{2\alpha/(2-p)}} .
\end{aligned}$$

Since $(2/q')(p'/2)' > n/(n-1)$, by Lemma 2.12 we get (3.12) in the case $1 < q < 2$. This completes the proof of Theorem 1.2 under condition (b). $\square$

Proof of Theorem 1.1.

Let $\varphi(r)$ be a smooth function supported on $\left\{r : \frac{1}{2} < r < 2\right\}$ and $\sum_j \varphi(2^j r) = 1$. Let $\widehat{\Upsilon_j}(\xi) = \varphi(2^j|\xi|)$ and

$$H_{r,k,j}f(x) = \int_{\mathbf{S}^{n-1}} \Omega(\xi) \left(\Upsilon_{j+k} * f\right)(x - 2^k r\xi) d\sigma(\xi).$$

Since $f = \sum_j \Upsilon_{j+k} * f$ for $f \in \mathcal{S}(\mathbf{R}^n)$ and for any $k \in \mathbf{Z}$, applying Minkowski's inequality to get

$$\mathcal{M}_\Omega f(x) \leq \sum_j \mathcal{M}_{\Omega,j} f(x), \tag{3.15}$$

where

$$\mathcal{M}_{\Omega,j} f(x) = \left(\sum_{k\in\mathbf{Z}} \int_1^2 |H_{r,k,j} f(x)|^2 \frac{dr}{r} \right)^{1/2} .$$

It is clear that by Theorem 1.2, we only need to prove Theorem 1.1 under condition (a).

By the same arguments as in the proof of (3.3) we have

$$\|\mathcal{M}_{\Omega,j}(f)\|_2 \le C2^{-\alpha|j|}\|f\|_2 . \tag{3.16}$$

As in the proof of Theorem 1.2 under condition (b) and (3.15)-(3.16), we only need to show that

$$\|\mathcal{M}_{\Omega,j}f\|_{p,\omega} \le C_p \|f\|_{p,\omega} \text{ for } \delta \le p < \infty,\ \omega \in A_{p/\delta}. \tag{3.17}$$

The proof of (3.17) will be divided into two steps.

Case (1): $\delta \le p < \infty$, $\omega \in A_{p/\delta}$ and $q \ge 2$. In this case $2 \le p < \infty$ and $\omega \in A_{p/2}$. First we consider the case $p > 2$. By duality, there is a function $g \in L^{(p/2)'}(\omega^{1-(p/2)'})$ with $\|g\|_{(p/2)',\omega^{1-(p/2)'}} \le 1$ such that

$$\|\mathcal{M}_{\Omega,j}(f)\|_{p,\omega}^2 = \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}\int_1^2 \left|\int_{\mathbf{S}^{n-1}}\Omega(\xi)\left(\Upsilon_{j+k}*f\right)(x-r2^k\xi)d\sigma(\xi)\right|^2 \frac{dr}{r}|g(x)|\,dx.$$

By Hölder's inequality, we have

$$\begin{aligned}
&\left|\int_{\mathbf{S}^{n-1}}\Omega(\xi)\left(\Upsilon_{j+k}*f\right)(x-r2^k\xi)d\sigma(\xi)\right|^2 \\
\le\ & \|\Omega\|_q^2\left(\int_{\mathbf{S}^{n-1}}\left|\left(\Upsilon_{k+j}*f\right)(x-r2^k\xi)\right|^{q'}d\sigma(\xi)\right)^{2/q'} \\
\le\ & \|\Omega\|_q^2\int_{\mathbf{S}^{n-1}}\left|\left(\Upsilon_{k+j}*f\right)(x-r2^k\xi)\right|^2 d\sigma(\xi).
\end{aligned} \tag{3.18}$$

Thus, by Fubini's theorem and a simple change of variable we get

$$\begin{aligned}
&\|\mathcal{M}_{\Omega,j}(f)\|_{p,\omega}^2 \\
\le\ & C\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}|\Upsilon_{k+j}*f(x)|^2\int_1^2\int_{\mathbf{S}^{n-1}}\left|g(x+r2^k\xi)\right|d\sigma(\xi)\frac{dr}{r}dx \\
\le\ & C\sum_{k\in\mathbf{Z}}\int_{\mathbf{R}^n}|\Upsilon_{k+j}*f(x)|^2 M^*(\tilde{g})(-x)dx, \text{ with } \tilde{g}(x)=g(-x) \\
\le\ & C\left\|\sum_{k\in\mathbf{Z}}|\Upsilon_{k+j}*f|^2\right\|_{p/2,\omega}\|M^*(\tilde{g})\|_{(p/2)',\omega^{1-(p/2)'}} .
\end{aligned}$$

Therefore, by the weighted L^p $(1<p<\infty)$ boundedness of the Hardy-Littlewood maximal operator M^* and the weighted Littlewood-Paley theory [11] we get

$$\|\mathcal{M}_{\Omega,j}(f)\|_{p,\omega} \le C_p\|f\|_{p,\omega} \text{ for } 2<p<\infty \text{ and } \omega\in A_{p/2}(\mathbf{R}^n). \tag{3.19}$$

In the endpoint case $p = 2$ and $\omega \in A_1(\mathbf{R}^n)$, by (3.18) and the definition of A_1 weight we have

$$\begin{aligned}
&\|\mathcal{M}_{\Omega,j}(f)\|_{2,\omega}^2 \\
= \;& \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_1^2 \left| \int_{\mathbf{S}^{n-1}} \Omega(\xi) \left(\Upsilon_{j+k} * f\right)(x - r2^k\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} \omega(x) dx \\
\le \;& \|\Omega\|_q^2 \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} |\Upsilon_{k+j} * f(x)|^2 \left(\int_1^2 \int_{\mathbf{S}^{n-1}} \omega(x + r2^k\xi) d\sigma(\xi) \frac{dr}{r} \right) dx \\
\le \;& C \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} |\Upsilon_{k+j} * f(x)|^2 M^*(\tilde{\omega})(-x) dx, \text{ with } \tilde{\omega}(x) = \omega(-x) \\
\le \;& C \left\| \left(\sum_{k\in\mathbf{Z}} |\Upsilon_{k+j} * f|^2 \right)^{1/2} \right\|_{2,\omega}^2 .
\end{aligned}$$

Thus, by the weighted Littlewood-Paley theory we get

$$\|\mathcal{M}_{\Omega,j}(f)\|_{2,\omega} \le \|f\|_{2,\omega} \text{ for } \omega \in A_1(\mathbf{R}^n). \tag{3.20}$$

Case (2): $\delta \le p < \infty$, $\omega \in A_{p/\delta}$ and $1 < q < 2$. In this case we have $q' \le p < \infty$, $\omega \in A_{p/q'}$ and $p > 2$.

As above, by duality, there is a function $g \in L^{(p/2)'}(\omega^{1-(p/2)'})$ and satisfies $\|g\|_{(p/2)',\omega^{1-(p/2)'}} \le 1$ such that

$$\|\mathcal{M}_{\Omega,j}(f)\|_{p,\omega}^2 = \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_1^2 \left| \int_{\mathbf{S}^{n-1}} \Omega(\xi) \left(\Upsilon_{j+k} * f\right)(x - r2^k\xi) d\sigma(\xi) \right|^2 \frac{dr}{r} |g(x)| \, dx.$$

By Hölder's inequality, Fubini's theorem and a change of variable, we have

$$\begin{aligned}
&\|\mathcal{M}_{\Omega,j}(f)\|_{p,\omega}^2 \\
\le \;& \|\Omega\|_1 \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} \int_1^2 \int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{2-q} \left| \left(\Upsilon_{k+j} * f\right)(x - r2^k\xi) \right|^2 d\sigma(\xi) \frac{dr}{r} |g(x)| \, dx \\
\le \;& C \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} |\Upsilon_{k+j} * f(x)|^2 M^*_{\Omega^{(2-q)}}(\tilde{g})(-x) dx \\
\le \;& C \left\| \left(\sum_{k\in\mathbf{Z}} |\Upsilon_{k+j} * f|^2 \right)^{1/2} \right\|_{p,\omega}^2 \left\| M^*_{\Omega^{(2-q)}}(\tilde{g}) \right\|_{(p/2)',\omega^{1-(p/2)'}} .
\end{aligned}$$

By the above arguments, the proof of (3.17) will be completed in Case (2) if we can show that

$$\left\| M^*_{\Omega^{(2-q)}}(\tilde{g}) \right\|_{(p/2)',\omega^{1-(p/2)'}} \le C \|g\|_{(p/2)',\omega^{1-(p/2)'}} . \tag{3.21}$$

To this end, we invoke Lemma 2.11. In fact, if we let $d = q/(2-q)$, then we notice that $|\Omega|^{2-q} \in L^d(\mathbf{S}^{n-1})$, $d' = q'/2$, $\left(\omega^{1-(p/2)'}\right)^{1-(p/2)} = \omega \in A_{p/q'} = A_{(p/2)/d'}$ and $(p/2)' < d$. Therefore, d, $(p/2)'$ and $\omega^{1-(p/2)'}$ satisfy condition (b) in Lemma 2.11. This finishes the proof of Theorem 1.1 for condition (a). This ends the proofs of Theorems 1.1 and 1.2. $\square$

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