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Truncated Complex Moment Problem with Data in a Circle

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Abstract. Let $\gamma \equiv \{\gamma_{ij}\}$ $(0 \le i+j \le 2n)$ be a collection of complex numbers with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$. The truncated complex moment problem for γ entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that $\gamma_{ij} = \int \bar{z}^i z^j d\mu(z)$ $(0 \le i+j \le 2n)$. We solve this truncated moment problem with data in a circle and discuss the behavior of data in an extended moment matrix.

1. Introduction and preliminaries

Given a doubly indexed finite sequence of complex numbers

$$\gamma : \gamma_{00},\ \gamma_{01},\ \gamma_{10},\ \gamma_{02},\ \gamma_{11},\ \gamma_{20},\ \cdots,\ \gamma_{0,2n},\ \gamma_{1,2n-1},\ \cdots,\ \gamma_{2n-1,1},\ \gamma_{2n,0}$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, *the truncated complex moment problem* entails finding a positive Borel measure μ supported in the complex plane $\mathbb{C}$ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \le i+j \le 2n);$$

μ is called *a representing measure* for γ, and γ is called *a truncated moment sequence*. This truncated complex moment problem has been well-developed in several articles ([4], [5], [6], [7], [8], [10], [9]). Also, given a closed subset $K \subset \mathbb{C}$ and a doubly indexed infinite sequence of complex numbers $\gamma = \{\gamma_{ij}\}_{i,j=0}^{\infty}$:

$$\gamma_{00},\ \gamma_{01},\ \gamma_{10},\ \gamma_{02},\ \gamma_{11},\ \gamma_{20},\ \cdots,\ \gamma_{0,2n},\ \gamma_{1,2n-1},\ \cdots,\ \gamma_{2n-1,1},\ \gamma_{2n,0},\ \cdots$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, *the (full) complex moment problem* entails finding a positive Borel measure μ such that

$$\gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (\ i,\ j \ge 0)$$

and supp $\mu \subset K$ (cf. [2]). In [4] and [5], one studied the truncated complex moment problem based on positivity and extension properties of the associated moment

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matrix $M(n)$ which is defined below. For $n \geq 1$, let $m = m(n) := (n+1)(n+2)/2$. For $A \in \mathcal{M}_m(\mathbb{C})$ (the set of $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering

$$\underbrace{\mathit{1}}_{(1)}, \underbrace{Z, \bar{Z}}_{(2)}, \underbrace{Z^2, \bar{Z}Z, \bar{Z}^2}_{(3)}, \underbrace{Z^3, \bar{Z}Z^2, \bar{Z}^2Z, \bar{Z}^3}_{(4)}, \cdots, \underbrace{Z^n, \cdots, \bar{Z}^n}_{(n+1)}.$$

In particular, rows or columns indexed by $\mathit{1}, Z, \cdots, Z^n$ are said to be *analytic*. For $0 \leq i+j \leq n$, $0 \leq l+k \leq n$, we denote the entry in row $\bar{Z}^l Z^k$, column $\bar{Z}^i Z^j$ by $A_{(l,k)(i,j)}$. We define $M(n) := M(n)(\gamma) \in \mathcal{M}_{m(n)}(\mathbb{C})$ as follows: for $0 \leq k+l \leq n$, $0 \leq i+j \leq n$, the entry in row $\bar{Z}^k Z^l$ and column $\bar{Z}^i Z^j$ is $M(n)_{(k,l)(i,j)} = \gamma_{l+i,j+k}$.

For example, if $n = 1$, the quadratic moment matrix for

$$\gamma : \gamma_{00},\ \gamma_{01},\ \gamma_{10},\ \gamma_{02},\ \gamma_{11},\ \gamma_{20}$$

corresponds to

$$M(1) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{pmatrix},$$

and if $n = 2$, the quartic moment matrix for

$$\gamma : \gamma_{00},\ \gamma_{01},\ \gamma_{10},\ \gamma_{02},\ \gamma_{11},\ \gamma_{20},\ \gamma_{03},\ \gamma_{12},\ \gamma_{21},\ \gamma_{30},\ \gamma_{04},\ \gamma_{13},\ \gamma_{22},\ \gamma_{31},\ \gamma_{40}$$

corresponds to

$$M(2) = \begin{pmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{pmatrix}.$$

In the recent works, one discussed moment problem concentrated on support of the given moment measure. As a different approach we discuss moment problem concentrated on a given data. In this note we consider moment problem with a data on a circle.

2. Some known properties

In this section we recall some known properties from [3] and [4], which are used frequently in this note.

($\mathbf{P}_1$). Let A_k be the compression of A to the first $(k+1)$ rows and columns, i.e.,

$$A = \begin{pmatrix} A_k & * \\ * & * \end{pmatrix},$$

and let $\Delta_k := \det(A_k)$. Assume that $A \geq 0$ and that $\Delta_k = 0$ for some k. Then, $\Delta_l = 0$ for all $l \geq k$.

(P$_2$). If $M(1) \geq 0$ and *rank* $M(1) = 1$, then $\gamma_{00} \cdot \delta_{\gamma_{01}/\gamma_{00}}$ is the unique representing measure of γ.

For k, $l \in \mathbb{Z}_+$, let $A \in \mathcal{M}_k(\mathbb{C})$, $A = A^*$, $B \in M_{k,l}(\mathbb{C})$, $C = C^* \in M_l(\mathbb{C})$; we refer to any matrix of the form

$$\widetilde{A} \equiv \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

as an extension of A .

(P$_3$). Let A, B, C and $\widetilde{A}$ be as above, let V_1, V_2, $\cdots$, V_k be the columns of A, let V_{k+1}, $\cdots$, V_{k+l} be the columns of B, and let $\widetilde{V}_1$, $\widetilde{V}_2$, $\cdots$, $\widetilde{V}_k$, $\widetilde{V}_{k+1}$, $\widetilde{V}_{k+l}$ be the columns of $\widetilde{A}$. Assume that $\widetilde{A} \geq 0$.

i) If there exist scalars a_1, a_2, $\cdots$, a_k such that $\sum_{i=1}^{k} a_i V_i = 0$, then $\sum_{i=1}^{k} a_i \widetilde{V}_i = 0$.

ii) If $\widetilde{A}$ is a flat extension of A and $\sum_{i=1}^{k+l} a_i V_i = 0$, then $\sum_{i=1}^{k+l} a_i \widetilde{V}_i = 0$.

(P$_4$). If γ is of flat data type and $M(n) = M(n)(\gamma) \geq 0$, then $M(n)$ also admits a unique flat extension $M(\infty) \geq 0$, where $M(\infty)$ is a finite-rank positive infinite moment matrix.

3. Main results

We now begin the main section with some lemmas.

Lemma 3.1. *Let $M(1)$ be a positive moment matrix. Given*

$$\gamma = \gamma^{(2)} : \gamma_{00},\ \gamma_{01},\ \gamma_{10},\ \gamma_{02},\ \gamma_{11},\ \gamma_{20}$$

with $\gamma_{00} > 0$ and $\gamma_{ji} = \bar{\gamma}_{ij}$, if $\{\gamma_{ij}\}_{0 \leq i+j \leq 2}$ lies in $\rho\mathbb{T} := \{\rho z : |z| = 1\}$ $(\rho > 0)$, then

(i) *there exists the unique representing measure $\mu = \rho \cdot \delta_{\gamma_{01}/\rho}$ and*

(ii) $\gamma_{01}^2 = \rho \cdot \gamma_{02}$.

Proof. Since $\gamma_{00} > 0$ and $\gamma_{00} \in \rho\mathbb{T}$, we have $\gamma_{00} = \rho$. For brevity, we write $\gamma_{01} = u$, $\gamma_{02} = v$, $\gamma_{11} = r$. Then we have

$$M(1) = \begin{pmatrix} \rho & u & \overline{u} \\ \overline{u} & r & \overline{v} \\ u & v & r \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} \rho & u \\ \overline{u} & r \end{pmatrix}.$$

Since $M(1)$ is self adjoint, obviously r is a real number. Since $M(1)$ is a moment matrix, $r \geq 0$ and so $r = \rho$. Since $M(1) \geq 0$ and $\det(A) = \rho^2 - |u|^2 = 0$, by ($P_1$), $\det M(1) = 0$. On the other hand, by some computation, we have

$$M(1) \overset{R}{\sim} \begin{pmatrix} \rho & u & \overline{u} \\ \rho^2 & \rho u & u\overline{v} \\ \rho^2 & \overline{u}v & \overline{u}\rho \end{pmatrix} \overset{R}{\sim} \begin{pmatrix} \rho & u & \overline{u} \\ 0 & 0 & u\overline{v} - \overline{u}\rho \\ 0 & \overline{u}v - u\rho & 0 \end{pmatrix},$$

where $\overset{R}{\sim}$ is row equivalent. Hence

$$\det M(1) = 0 = -\rho|\overline{u}v - u\rho|^2.$$

So $\overline{u}v = u\rho$, and so

$$\gamma_{02} = v = \frac{u^2\rho}{|u|^2} = \frac{u^2}{\rho}.$$

Since $M(1) \geq 0$ and rank $M(1) = 1$, by (P_2), there exist the unique representing measure $\mu = \rho \cdot \delta_{\gamma_{01}/\rho}$ of γ. □

Lemma 3.2. *Let*

$$\gamma = \gamma^{(2n)} : \gamma_{00},\ \gamma_{01},\ \gamma_{10},\ \gamma_{02},\ \gamma_{11},\ \gamma_{20},\ \cdots,\ \gamma_{0,2n},\ \gamma_{1,2n-1},\ \cdots,\ \gamma_{2n-1,1},\ \gamma_{2n,0}$$

be a data of complex numbers. Let $M(n)$ be the positive moment matrix corresponding to γ. If the data $\gamma^{(2n)}$ lies in $\rho\mathbb{T}$, then

(i) *the extended infinite moment matrix $M(\infty)$ satisfies rank $M(\infty)$ = rank $M(n) = 1$,*

(ii) *there exists the unique associated representing measure $\widetilde{\mu} = \rho \cdot \delta_{\gamma_{01}/\rho} (= \mu)$ for $M(\infty)$,*

(iii) *$\{\gamma_{ij}\}_{i,j=0}^{\infty}$ satisfies $\gamma_{ij} = \rho^{1+i-j}\gamma_{01}^{j-i}$, and*

(iv) *the flat extended moment sequence $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ is also in $\rho\mathbb{T}$.*

Proof. Since $M(n)$ is a flat extension of $M(1)$, according to (P_3) and (P_4), the representing measure for $M(n)$ also is $\mu = \rho\delta_{\gamma_{01}/\rho}$. Since

$$\gamma_{ij} = \int \bar{z}^i z^j \, d\mu \qquad (0 \leq i + j \leq 2n),$$

we have

$$\gamma_{00} = \int \rho \, d\delta_{\gamma_{01}/\rho} = \rho \int d\delta_{\gamma_{01}/\rho} = \rho.$$

Hence $\rho = \gamma_{00} = \rho_k$, and $u_k = \gamma_{01}/\rho$. Thus $\mu_k = \mu$ for all $k = 1, 2, \cdots$. That is, $M(\infty)$ has a unique representing measure $\mu = \rho\delta_{\gamma_{01}/\rho}$. Moreover, since

$$\begin{aligned} \gamma_{ij} &= \int \bar{z}^i z^j \, d\mu_k = \int \bar{z}^i z^j \rho d\delta_{\gamma_{01}/\rho} \\ &= \rho \cdot \left(\frac{\bar{\gamma}_{01}}{\rho}\right)^i \left(\frac{\gamma_{01}}{\rho}\right)^j = \rho^{1-i-j} \cdot \bar{\gamma}_{01}^i \gamma_{01}^j \end{aligned}$$

and $\bar{\gamma}_{01} = \rho^2\gamma_{01}^{-1}$(*indeed*, $\gamma_{01}\bar{\gamma}_{01} = \rho^2$), we have

$$\gamma_{ij} = \rho^{1-i-j} \cdot \rho^{2i}\gamma_{01}^{-i} \cdot \gamma_{01}^{j} = \rho^{1+i-j}\gamma_{01}^{j-i}.$$

Finally, since

$$|\gamma_{ij}| = \left|\rho^{1+i-j}\gamma_{01}^{j-i}\right| = \rho,$$

$\{\gamma_{ij}\}_{i,j=0}^{\infty}$ lies in $\rho\mathbb{T}$. □

Recall that if $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 2}$ is in $\rho\mathbb{T}$, then by Lemma 3.2 the infinite moment matrix $M(\infty)$ is well-constructed.

Theorem 3.3. *Suppose $M(n)$ is a positive moment matrix. Let $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 2n}$ be a given data lies in $\rho\mathbb{T}$ $(\rho > 0)$ and let $\widetilde{\gamma} = \{\gamma_{ij}\}_{i,j=0}^{\infty}$ be the flat extension data corresponding to $M(\infty)$. Then,*

i) *if $\gamma_{01} = \rho \cdot e^{i2\pi\frac{n}{m}}$ with $(m,n) = 1$, then $\widetilde{\gamma}$ is exactly the set of vertices of a regular m-polygon inscribed in $\rho\mathbb{T}$,*

ii) *if $\gamma_{01} = \rho \cdot e^{i2\pi\theta}$, where θ is an irrational number, then $\widetilde{\gamma}$ is dense in $\rho\mathbb{T}$.*

Proof. i) Since $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 2n}$ lies in $\rho\mathbb{T}$, by Lemma 3.2 the flat extended moment sequences $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ lies also in $\rho\mathbb{T}$ and the associated representing measure is $\widetilde{\mu} = \rho \cdot \delta_{\gamma_{01}/\rho}$. Hence the extended flat data $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ satisfies

$$\begin{aligned} \gamma_{ij} &= \rho^{1+i-j}\gamma_{01}^{j-i} \quad (\text{since } |\gamma_{01}| = \rho) \\ &= \rho^{1+i-j}(\rho \cdot e^{i2\pi\frac{n}{m}})^{j-i} \\ &= \rho \cdot (e^{i2\pi\frac{n}{m}})^{j-i}. \end{aligned}$$

Therefore we have

$$\begin{aligned} \{\gamma_{ij}\}_{i,j=0}^{\infty} &= \{\rho \cdot (e^{i2\pi\frac{n}{m}})^{k} \mid k \in \mathbb{Z}\} \\ &= \{\rho \cdot (e^{i\frac{2\pi}{m}})^{nk} \mid k = 0, 1, \cdots, m-1\}. \end{aligned}$$

Since

$$\frac{2\pi}{m}n(k+1) - \frac{2\pi}{m}nk = \frac{2\pi}{m}n,$$

it is independent on k and the number of vertices is m. So, $\widetilde{\gamma} = \{\gamma_{ij}\}_{i,j=0}^{\infty}$ is exactly the set of vertices of a regular m-polygon inscribed in $\rho\mathbb{T}$.

ii) Since

$$\{\gamma_{ij}\}_{i,j=0}^{\infty} = \{\rho \cdot e^{i2\pi\theta k} \mid k \in \mathbb{Z}\}$$

and θ is an irrational number, it is obvious that $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ is dense in $\rho\mathbb{T}$. □

The following corollary comes immediately from Theorem 3.3.

Corollary 3.4. *Suppose $M(n)$ is a positive moment matrix. Let $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 2n}$ is exactly the set Γ_k of vertices of a regular k-polygon inscribed in $\rho\mathbb{T}$. Then $\widetilde{\gamma} =$*

$\{\gamma_{ij}\}_{i,j=0}^{\infty}$ *is the same set* Γ_k.

Remark 3.5. We can consider an arbitrary circle $\partial D(z_0,\rho) := \rho\mathbb{T}+z_0$ instead of $\rho\mathbb{T}$ in Section 3. But the extended data $\widetilde{\gamma} = \{\gamma_{ij}\}_{i,j=0}^{\infty}$ can not be contained in $\partial D(z_0,\rho)$ when $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 2n}$ is in $\partial D(z_0,\rho)$. We can give a counter examples for this concept. Consider $\gamma_{00} = 1$, $\gamma_{01} = 0$, $\gamma_{02} = 0$, $\gamma_{11} = 1$. Then $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 2} \subset \partial D(1/2, 1/2)$. By some computations, we may obtain

$$M(2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & i & -i & \sqrt{2} \\ 0 & 0 & 1 & \sqrt{2} & i & -i \\ 0 & -i & \sqrt{2} & 3 & -1+\sqrt{2}i & -2\sqrt{2}i \\ 1 & i & -i & -1-\sqrt{2}i & 3 & -1+\sqrt{2}i \\ 0 & \sqrt{2} & i & 2\sqrt{2}i & -1-\sqrt{2}i & 3 \end{pmatrix}$$

with $\text{rank}M(2) = \text{rank}M(1) = 3$, and so $M(2)$ is a flat extension of $M(1)$. Obviously we have that $\{\gamma_{ij}\}_{0\le i+j\le 4} \nsubseteq \partial D(1/2, 1/2)$. Hence $\widetilde{\gamma} = \{\gamma_{ij}\}_{i,j=0}^{\infty}$ can not be contained in $\partial D(z_0,\rho)$ (See [8] for more examples).

Remark 3.6. Like the approach in this section, when a data $\gamma = \{\gamma_{ij}\}_{0\le i+j\le 2n}$ is in $D(0,\rho) := \{z \in \mathbb{C} : |z| \le 1\}$, does the flat extension infinite data $\{\gamma_{ij}\}_{i,j=0}^{\infty}$ lie in $D(0,\rho)$? The answer is negative (indeed, consider a data $\gamma : 1, 0, 0, 0, 1, 0$, i.e.,

$$M(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then matrix $M(2)$ in Remark 3.5 is a flat extension of $M(1)$, but the data $\{\gamma_{ij}\}_{0\le i+j\le 4}$ doesn't lie in $D(0, 1)$.)

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