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On Prime Near-rings with Generalized (σ,τ)-derivations

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Abstract. Let N be a prime left near-ring with multiplicative center Z and f be a generalized (σ,τ)-derivation associated with d. We prove commutativity theorems in prime near-rings with generalized (σ,τ)-derivation.

1. Introduction

H. E. Bell and G. Mason have proved some results on commutativity of prime near-rings with derivations in [1] and [2]. Ö. Gölbaşı and N. Aydın have generalized these theorems for a prime near-rings with (σ,τ)-derivation in [5]. On the other hand, the notion of generalized derivation of a prime ring was introduced by M. Bresar [3] and B. Hvala [4]. An additive map f of an associative ring R is called a generalized derivation if there is a derivation d of R such that

$$f(xy) = f(x)y + xd(y), \quad \text{for all } x,\ y \in R.$$

Many authors have investigated the properties of prime and semi-prime rings with derivations or generalized derivations. It is our aim in this note to study the commutativity prime near-rings with generalized (σ,τ)-derivations.

Throughout this paper, N will denote a zero-symetric left near-ring with multiplicative center Z. N is called a prime near-ring if $aNb = \{0\}$ implies that $a = 0$ or $b = 0$. An additive mapping $f : N \to N$ is generalized derivation of N associated with nonzero (σ,τ)-derivation d such that $\tau d = d\tau$, $\sigma d = d\sigma$. For $x, y \in N$, the symbol $[x, y]$ will denote $xy - yx$ while the symbol (x, y) will denote $x + y - x - y$. As for terminologies used here without mention, we refer to G. Pilz [6].

Definition 1. Let N be a near-ring, d a (σ,τ)-derivation of N. An additive mapping $f : N \to N$ is said to be left generalized (σ,τ)-derivation of associated with d if

$$f(xy) = d(x)\tau(y) + \sigma(x)f(y) \quad \text{for all } x,\ y \in R. \tag{1.1}$$

and f is said to be right generalized (σ,τ)-derivation of associated with d if

$$f(xy) = f(x)\tau(y) + \sigma(x)d(y) \quad \text{for all } x,\ y \in R. \tag{1.2}$$

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$f$is said to be a generalized (σ,τ)-derivation of associated with d if it is both a left and right generalized (σ,τ)-derivation of associated with d.

Lemma 1.

(i) *Let f be a right generalized (σ,τ)-derivation of near ring N with associated d. Then $f(xy)=\sigma(x)d(y)+f(x)\tau(y)$ for all $x,\ y\in N$.*

(ii) *Let f be a left generalized (σ,τ)-derivation of near ring N with associated d. Then $f(xy)=\sigma(x)f(y)+d(x)\tau(y)$ for all $x,\ y\in N$.*

Proof. (i) For any $x,\ y\in N$, we get

$$\begin{aligned}f(x(y+y)) &= f(x)\tau(y+y)+\sigma(x)d(y+y)\\ &= f(x)\tau(y)+f(x)\tau(y)+\sigma(x)d(y)+\sigma(x)d(y)\end{aligned}$$

and

$$f(xy+xy)=f(x)\tau(y)+\sigma(x)d(y)+f(x)\tau(y)+\sigma(x)d(y).$$

Comparing these two expressions, one can obtain

$$f(x)\tau(y)+\sigma(x)d(y)=\sigma(x)d(y)+f(x)\tau(y)$$

and so,

$$f(xy)=\sigma(x)d(y)+f(x)\tau(y),\ \text{for all } x,\ y\in N.$$

(ii) Similarly. □

Lemma 2.

(i) *Let f be a right generalized (σ,τ)-derivation of near ring N with associated d. Then $(f(x)\tau(y)+\sigma(x)d(y))\tau(z)=f(x)\tau(y)\tau(z)+\sigma(x)d(y)\tau(z)$, for all $x,\ y\in N$.*

(ii) *Let f be a generalized (σ,τ)-derivation of near ring N with associated d. Then $(d(x)\tau(y)+\sigma(x)f(y))\tau(z)=d(x)\tau(y)\tau(z)+\sigma(x)f(y)\tau(z)$ for all $x,\ y\in N$.*

(iii) *Let f be a left generalized (σ,τ)-derivation of near ring N with associated d. Then $(d(x)\tau(y)+\sigma(x)d(y))\tau(z)=d(x)\tau(y)\tau(z)+\sigma(x)d(y)\tau(z)$ for all $x,\ y\in N$.*

Proof. (i) For all $x,\ y,\ z\in N$, we get

$$f((xy)z)=f(xy)\tau(z)+\sigma(xy)d(z).$$

On the other hand,

$$\begin{aligned}f(x(yz)) &= f(x)\tau(yz)+\sigma(x)d(yz)\\ &= f(x)\tau(yz)+\sigma(x)d(y)\tau(z)+\sigma(xy)d(z).\end{aligned}$$

For these two expressions of $f(xyz)$, we obtain that, for all $x,\ y,\ z\in N$,

$$(f(x)\tau(y)+\sigma(x)d(y))\tau(z)=f(x)\tau(y)\tau(z)+\sigma(x)d(y)\tau(z).$$

(ii) For all $x,\ y,\ z\in N$, we have

$$f((xy)z)=f(xy)\tau(z)+\sigma(xy)d(z).$$

On the other hand,

$$\begin{aligned}f(x(yz))&=d(x)\tau(yz)+\sigma(x)f(yz)\\&=d(x)\tau(yz)+\sigma(x)f(y)\tau(z)+\sigma(xy)d(z).\end{aligned}$$

Comparing these two expressions of $f(xyz)$, we obtain that, for all $x,\ y,\ z\in N$,

$$(d(x)\tau(y)+\sigma(x)f(y))\tau(z)=d(x)\tau(y)\tau(z)+\sigma(x)f(y)\tau(z).$$

(iii) For all $x,\ y,\ z\in N$, we get

$$f((xy)z)=d(xy)\tau(z)+\sigma(xy)f(z).$$

On the other hand,

$$\begin{aligned}f(x(yz))&=d(x)\tau(yz)+\sigma(x)f(yz)\\&=d(x)\tau(yz)+\sigma(x)d(y)\tau(z)+\sigma(xy)f(z).\end{aligned}$$

Comparing these two expressions of $f(xyz)$, we obtain that, for all $x,\ y,\ z\in N$,

$$(d(x)\tau(y)+\sigma(x)f(y))\tau(z)=d(x)\tau(y)\tau(z)+\sigma(x)f(y)\tau(z).$$

□

Lemma 3. *Let N be a prime near-ring, f a generalized (σ,τ)-derivation of N with associated d and $a\in N$.*

(i) *If $af(N)=0$ then $a=0$.*

(ii) *If $f(N)\tau(a)=0$ then $a=0$.*

Proof. (i) For all $x,\ y\in N$, we get

$$0=af(xy)=af(x)\tau(y)+a\sigma(x)d(y).$$

and so,

$$aNd(N)=0.$$

Since N is a prime near-ring and $d\neq 0$ (σ,τ)-derivation of N, we obtain $a=0$.

ii) A similar argument works if $f(N)\tau(a)=0$. □

Theorem 1. *Let f be a generalized (σ,τ)-derivation of N with associated d such that $f\sigma=\sigma f$. If N is a 2-torsion free near-ring and $f^2=0$ then $f=0$.*

Proof. (i) For arbitrary $x,\ y\in N$, we have

$$\begin{aligned}0&=f^2(xy)=f(f(xy))=f(f(x)\tau(y)+\sigma(x)d(y))\\&=f^2(x)\tau^2(y)+\sigma(f(x))d(\tau(y))+f(\sigma(x))\tau(d(y))+\sigma^2(x)d^2(y).\end{aligned}$$

By the hypothesis,

$$\sigma(f(x))d(\tau(y))+f(\sigma(x))\tau(d(y))+\sigma^2(x)d^2(y)=0 \ \text{ for all } x,\ y\in N. \tag{1.3}$$

Writing $\sigma^{-1}(\ f(x))$ by x in (1.3) and using $f\sigma=\sigma f$, we get

$$f(\sigma(x))d^2(y)=0 \ \text{ for all } x,\ y\in N.$$

By Lemma 3(ii), we obtain that $d^2(N)=0$. That is $d=0$ from [5, Lemma 4].
Suppose now that $d=0$. Then for all $x,\ y\in N$, we have $f(xy)=f(x)\tau(y)=\sigma(x)f(y)$ and so, $0=f^2(xy)=f(\sigma(x)f(y))=f(\sigma(x))\tau(f(y))+\sigma^2(x)d(f(y))$.
That is

$$f(\sigma(x))\tau(f(y))=0, \quad \text{for all } x,\ y\in N.$$

It gives $f=0$ from Lemma 3(ii). □

Theorem 2. *Let N be a prime near-ring with a nonzero generalized (σ,τ)-derivation f associated with d. If $f(N)\subset Z$ then $(N,+)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof. Suppose that $a\in N$ such that $f(a)\neq 0$. So, $f(a)\in Z\backslash\{0\}$ and $f(a)+f(a)\in Z\backslash\{0\}$. For all $x,\ y\in N$, we have

$$(x+y)(f(a)+f(a))=(f(a)+f(a))(x+y),$$

that is, $xf(a)+xf(a)+yf(a)+yf(a)=f(a)x+f(a)y+f(a)x+f(a)y$.
Since $f(a)\in Z$, we get

$$f(a)x+f(a)y=f(a)y+f(a)x,$$

and so,

$$f(a)(x,y)=0 \ \text{ for all } x,\ y\in N.$$

Since $f(a)\in Z\backslash\{0\}$ and N is a prime near-ring, it follows that $(x,y)=0$, for all $x,\ y\in N$. Thus $(N,+)$ is abelian.
Using the hypothesis, for any $x,\ y,\ z\in N$,

$$\tau(z)f(xy)=f(xy)\tau(z)$$

By Lemma 2(ii), we have

$$\tau(z)d(x)\tau(y)+\tau(z)\sigma(x)f(y)=d(x)\tau(y)\tau(z)+\sigma(x)f(y)\tau(z).$$

Using $f(N) \subset Z$ and $(N,+)$ is abelian, we obtain that

$$\tau(z)d(x)\tau(y) - d(x)\tau(y)\tau(z) = [\sigma(x),\tau(z)]f(y), \quad \text{for all } x,\ y,\ z \in N. \tag{1.4}$$

Substituting $\tau^{-1}(f(z))$ for z in (1.4), we get

$$f(z)[d(x),\tau(y)] = 0, \quad \text{for all } x,\ y,\ z \in N.$$

Since $f(z) \in Z$ and f a nonzero (σ,τ)-generalized derivation with associated d, we get $d(N) \subset Z$. So, N is commutative ring by [5, Theorem 2]. □

Theorem 3. *Let N be a prime near-ring with a nonzero generalized (σ,τ)-derivation f associated with d such that $\tau f = f\tau, \sigma f = f\sigma$. If $[f(N), f(N)] = 0$ then $(N,+)$ is abelian. Moreover, if N is 2-torsion free, then N is a commutative ring.*

Proof. The argument used in the proof of Theorem 2 shows that if both z and $z+z$ commute elementwise with $f(N)$, then we have

$$zf(x,y) = 0 \quad \text{for all } x,\ y \in N. \tag{1.5}$$

Substituting $f(t), t \in N$ for z in (1.5), we get $f(t)f(x,y) = 0$. By Lemma **??**(i), we obtain that $f(x,y) = 0$ for all $x,\ y \in N$. For any $w \in N$, we have $0 = f(wx,wy) = f(w(x,y)) = d(w)\tau(x,y) + \sigma(w)f(x,y)$ and so,

$$d(w)\tau(x,y) = 0, \quad \text{for all } x,\ y \in N.$$

Replacing w by wr and using Lemma 2(iii), we get $0 = d(w)\tau(r)\tau(x,y) + \sigma(w)d(r)\tau(x,y)$ and so,

$$d(w)N\tau(x,y) = 0, \quad \text{for all } x,y,w \in N.$$

Since $d \neq 0$, we obtain that $(N,+)$ is abelian.
Now, assume that N is 2-torsion free. By the assumption $[f(N), f(N)] = 0$, we have

$$f(\tau(z))f(f(x)y) = f(f(x)y)f(\tau(z)) \quad \text{for all } x,\ y,\ z \in N.$$

Using $\tau f = f\tau$ and $\sigma f = f\sigma$, we get

$$f(\tau(z))d(f(x))\tau(y)+f(\tau(z))\sigma(f(x))f(y) = d(f(x))\tau(y)\tau(f(z))+\sigma(f(x))f(y)\tau(f(z))$$

$$f(\tau(z))d(f(x))\tau(y)+f(\sigma(x))f(\tau(z))f(y) = d(f(x))\tau(y)f(\tau(z))+f(\sigma(x))f(\tau(z))f(y)$$

and so,

$$f(\tau(z))d(f(x))\tau(y) = d(f(x))\tau(y)f(\tau(z)), \ \text{for all } x,\ y,\ z \in N. \tag{1.6}$$

If we take yw instead of y in (1.6), then

$$f(\tau(z))d(f(x))\tau(y)\tau(w) = d(f(x))\tau(y)\tau(w)f(\tau(z))$$

and so,

$$d(f(x))\tau(y)f(\tau(z))\tau(w) = d(f(x))\tau(y)\tau(w)f(\tau(z))$$

Hence, we get

$$d(f(x))N[f(\tau(z)), \tau(w)] = 0, \quad \text{for all } x, z, w \in N.$$

Since N is a prime near-ring, we have

$$d(f(N)) = 0 \quad \text{or} \quad f(N) \subset Z$$

Let assume that $d(f(N)) = 0$. Then $0 = d(f(xy)) = d(d(x)\tau(y) + \sigma(x)f(y))$ and so,

$$(1.7) \qquad d^2(x)\tau^2(y) + \sigma(d(x))d(\tau(y)) + d(\sigma(x))\tau(f(y)) = 0, \quad \text{for all } x,\ y \in N.$$

Replacing y by$\tau^{-1}(y)$ in this equation, we get

$$d^2(x)\tau(y) + \sigma(d(x))d(y) + d(\sigma(x))f(y) = 0, \quad \text{for all } x,\ y \in N.$$

Writing yz by y in this equation and using this equation, we have

$$\begin{aligned} 0 &= d^2(x)\tau(yz) + \sigma(d(x))d(yz) + d(\sigma(x))f(yz) \\ &= d^2(x)\tau(yz) + \sigma(d(x))d(y)\tau(z) + \sigma(d(x))\sigma(y)d(z) + d(\sigma(x))f(y)\tau(z) \\ &\quad + d(\sigma(x))\sigma(y)d(z) \\ &= \Big\{d^2(x)\tau(y) + \sigma(d(x))d(y) + d(\sigma(x))f(y)\Big\}\tau(z) + \sigma(d(x))\sigma(y)d(z) \\ &\quad + d(\sigma(x))\sigma(y)d(z). \end{aligned}$$

That is

$$2\sigma(d(x))\sigma(y)d(z) = 0, \text{ for all } x,\ y,\ z \in N$$

Since N is a 2-torsion free near-ring, we get

$$d(N)Nd(N) = 0.$$

Thus, we obtain that $d = 0$ from primenessly of N. It contradicts by $d \neq 0$. □

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