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A Length Function and Admissible Diagrams for Complex Reflection Groups $G(m, 1, n)$

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Abstract. In this paper, we introduce a length function for elements of the imprimitive complex reflection group $G(m, 1, n)$ and study its properties. Furthermore, we show that every conjugacy class of $G(m, 1, n)$ can be represented by an admissible diagram. The corresponding results for Weyl groups are well known.

1. Introduction

The imprimitive complex reflection group $G(m, 1, n)$ can be viewed as the generalized symmetric group. Its conjugacy classes have been determined by Kerber [9] and its irreducible representations can, for example, be obtained from the works of Can [1], [2]. In this paper, we introduce a length function for elements of $G(m, 1, n)$ and study its properties. Furthermore, in an analogous way to Carter [6], we show that every conjugacy class of $G(m, 1, n)$ can be represented by an admissible diagram. We refer the reader to [3] and [7] for much of the undefined terminology and quoted results.

Let V be a complex vector space of dimension n. A *reflection* in V is a linear transformation of V of finite order with exactly $(n-1)$ eigenvalues equal to 1. A *reflection group* G in V is a finite group generated by reflections in V. The dimension n of V is called the *rank* of G. For each non-zero vector $\alpha \in V$, let w_α be a reflection in V of order $m > 1$. Then there is a primitive m-th root of unity ξ such that $w_\alpha(v) = v - (1-\xi)\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$ for all $v \in V$. Thus $w_\alpha(\alpha) = \xi\alpha$ and $w_\alpha(v) = v$ if $v \in \langle\alpha\rangle^\perp$, where $\langle\alpha\rangle^\perp$ is the orthogonal complement of $\langle\alpha\rangle$ with respect to the given unitary inner product. As a convention, throughout this paper, we assume that ξ is a primitive m-th root of unity. Define $o_G : V \to \mathbf{N}$ by $o_G(v) = |G_{\langle v\rangle^\perp}|$ $(v \in V)$. Then $o_G(v) > 1$ if and only if v is a root of G. In this case, $o_G(v)$ is the order of the cyclic group generated by the reflections in G with root v. If α is a root of G then the number $o_G(\alpha)$ is called the *order* of α. Let $\mathcal{S}_n$ be the group of all $n \times n$ permutation matrices, and let $A(m, 1, n)$ be the group of all diagonal $n \times n$

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matrices with $\xi^{s_i},\ s_i \in \mathbf{Z}$ in the (i,i) position. We let $G(m,1,n) = A(m,1,n) \times \mathcal{S}_n$ (semi-direct product). $G(m,1,n)$ is an *imprimitive complex reflection group* in V generated by unitary reflections, and $G(1,1,n) = W(A_{n-1})$ (Weyl group of type A_{n-1}) and $G(2,1,n) = W(C_n)$ (Weyl group of type C_n). The group $G(m,1,n)$ has the following presentation (see [8]):

$$\begin{aligned} G(m,1,n) \quad = \quad & \langle r_1,\ \cdots,\ r_{n-1},\ w_1,\ \cdots, w_n \mid r_i^2 = (r_i r_{i+1})^3 = (r_i r_j)^2 = 1, \\ & |i-j| \geq 2,\ w_i^m = 1,\ w_i w_j = w_j w_i,\ r_i w_i = w_{i+1} r_i,\ r_i w_j = w_j r_i, \\ & j \neq i,\ i+1\rangle. \end{aligned}$$

2. The length function

Let $\Phi(m,p,n)$ $(p = 1,\ m)$ be an imprimitive root system with simple system $\pi(m,p,n) = (B,\theta)$, where

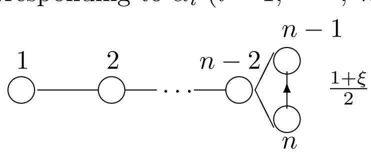
$$B = \begin{cases} \{\alpha_i = e_i - e_{i+1}\ (i = 1,\ \cdots,\ n-1),\ \alpha_n = e_n\} & \text{if } p = 1 \\ \{\beta_i = e_i - e_{i+1}\ (i = 1,\ \cdots,\ n-1),\ \beta_n = e_{n-1} - \xi e_n\} & \text{if } p = m. \end{cases}$$

Then the Cohen diagrams for $\Phi(m,1,n)$ and $\Phi(m,m,n)$ are respectively

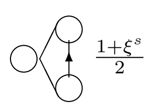
B_n^m :

where the node corresponding to α_i $(i = 1,\ \cdots,\ n)$ is denoted by i and

D_n^m :



where the node corresponding to β_i $(i = 1,\ \cdots,\ n)$ is denoted by i.

A *web* is a graph of the form 

where $s \in \{1,\ \cdots,\ m-1\}$.

Let $W = G(m,1,n)$ denote the imprimitive reflection group corresponding to $\Phi = \Phi(m,1,n)$. Now each element w in W can be expressed as a product of reflections $w = w_{a_1}^{s_1} \cdots w_{a_k}^{s_k}$, where $a_i \in \Phi$ and $s_i \in \{1,\ \cdots,\ m-1\}$. The *length* of w, denoted by $l(w)$ is the smallest value of $\sum_{i=1}^k s_i$ in any such expression for w. (Here, if $o_W(a_i) = 2$ then $s_i = 1$, and if $o_W(a_i) = m$ then $s_i \in \{1,\ \cdots,\ m-1\}$.) By convention, $l(1) = 0$. Clearly $l(w) = 1$ if and only if $w = w_a$ where $a \in \Phi$. It is also clear that if $w = w_a^s$ with $o_W(a) = m$ and $s \in \{1,\ \cdots,\ m-1\}$, then $l(w) = s$. We say that w is a product of k reflections if $l(w) = \sum_{i=1}^k s_i$. Any element $\sigma \in W$ may be written uniquely (up to reordering) as the product of disjoint cycles $\sigma = \theta_1 \cdots \theta_t$, where

$$\theta_i = \begin{pmatrix} b_{i1} & b_{i2} & \cdots & b_{ik_i} \\ \xi^{s_{i1}} b_{i2} & \xi^{s_{i2}} b_{i3} & \cdots & \xi^{s_{ik_i}} b_{i1} \end{pmatrix},$$

$b_{ij} \in \{1, \cdots, n\}$, $s_{ij} \in \{1, \cdots, m\}$, k_i is the length of the cycle θ_i, $i = 1, \cdots, t$. Let $f(\theta_i) = \sum_{j=1}^{k_i} s_{ij}$, and put $f(\sigma) = \sum_{i=1}^{t} f(\theta_i)$. Define $a_{pq}(\sigma)$ to be the number of cycles θ_i of σ of length q such that $f(\theta_i) \equiv p \ (mod\ m)$ for $1 \le p \le m$, $1 \le q \le n$. The $m \times n$ matrix $(a_{pq}(\sigma))$ is called the *type* of σ, denoted by $Ty(\sigma)$ (see [10]). Then it is well known that $\sigma, \pi \in W$ are conjugate in W if and only if $Ty(\sigma) = Ty(\pi)$ (see [9]).

Lemma 2.1. *If $\sigma, \pi \in W$ are conjugate in W, then $l(\sigma) = l(\pi)$.*

Proof. Let $\sigma = w_{a_1}^{s_1} \cdots w_{a_k}^{s_k}$, where $a_i \in \Phi$ and $s_i \in \{1, \cdots, m-1\}$. Since σ is conjugate in W to π, $\pi = w\sigma w^{-1}$ for some $w \in W$. But $w\sigma w^{-1} = w_{b_1}^{s_1} \cdots w_{b_k}^{s_k}$ where $b_i = w(a_i)$ implies that $l(\sigma) = \sum_{i=1}^{k} s_i = l(\pi)$. □

The above lemma says that two conjugate elements in W have the same length and are also product of the same number of reflections. The lemma below is a well-known property of reflection groups (see [11]).

Lemma 2.2. *Let G be a reflection group in an n-dimensional complex vector space V. If $g \in G$ and U is the subspace of V composed of all vectors fixed by g, then g is a product of reflections corresponding to roots in the orthogonal complement $U^{\perp}$ of U.*

Lemma 2.3. *Let $w \in W$. Then $l(w)$ is the sum of the powers of eigenvalues of w on V which are not equal to 1. In particular, w is a product of at most n reflections.*

Proof. Suppose that $l(w) = \sum_{i=1}^{k} s_i$. Then w is a product of k reflections and has an expression of the form $w = w_{a_1}^{s_1} \cdots w_{a_k}^{s_k}$, where $a_i \in \Phi$ and $s_i \in \{1, \cdots, m-1\}$. Now, let H_{a_i} be the reflecting hyperplane of a_i in V and let

$$H = H_{a_1} \cap H_{a_2} \cap \cdots \cap H_{a_k}.$$

Then w acts trivially on H and $\dim H \ge n - k$. Thus w has at least $(n-k)$ eigenvalues equal to 1, and so at most k eigenvalues $\xi^{s_1}, \xi^{s_2}, \cdots, \xi^{s_k}$ which are not equal to 1, by definition of the reflection. Therefore, the sum of the powers of these eigenvalues $\le l(w)$.

Conversely, suppose w has k eigenvalues $\xi^{s_1}, \xi^{s_2}, \cdots, \xi^{s_k}$ which are not equal to 1, where $s_i \in \{1, \cdots, m-1\}$. Let U be the subspace of V composed of all vectors fixed by w, and $U^{\perp}$ be the orthogonal subspace. Then at once $\dim U = n - k$ and $\dim U^{\perp} = k$, and by Lemma 2.2 w is a product of reflections corresponding to roots in $U^{\perp}$. Suppose that w fixes some vector in V. Then $k < n$ and so $\dim U^{\perp} < \dim V$. The roots in $U^{\perp}$ form a root system in the subspace they generate which has dimension less than n, and w is an element of the reflection group associated with this root system. Thus by induction w is a product of at most k reflections, i.e., $w = w_{a_1}^{s_1} \cdots w_{a_k}^{s_k}$, and so $l(w) \leq \sum_{i=1}^{k} s_i$.

Therefore, it suffices to show that if w fixes no non-zero vector in V then w can be expressed as a product of at most n reflections. Now, suppose that $w(v) \neq v$ for all $v \in V \setminus \{0\}$. Then $(w-1)v \neq 0$ for all $v \in V \setminus \{0\}$ and $ker(w-1) = \{0\}$, and so $w - 1$ is invertible.

Let $-(1-\xi)\frac{(v,a)}{(a,a)}a \in V$, where $a \in \Phi$, then there exists $v \in V$ such that

$$(w-1)v = -(1-\xi)\frac{(v,a)}{(a,a)}a.$$

Thus $w(v) = v - (1-\xi)\frac{(v,a)}{(a,a)}a = w_a(v)$, and so $w_a^s w(v) = v$,
where $s = \begin{cases} 1 & \text{if } o_W(a) = 2 \\ m-1 & \text{if } o_W(a) = m. \end{cases}$

By Lemma 2.2 $w_a^s w$ is a product of reflections corresponding to roots in $\langle v \rangle^{\perp}$. Then $w_a^s w$ is contained in a reflection group of smaller rank, and so by induction $w_a^s w$ is a product of at most $n-1$ reflections. Hence, w is a product of at most n reflections, and the proof is complete. ☐

An expression $w_{a_1}^{s_1} \cdots w_{a_k}^{s_k} \in W$ will be called *reduced* if $l(w_{a_1}^{s_1} \cdots w_{a_k}^{s_k}) = \sum_{i=1}^{k} s_i$.

Lemma 2.4. *Let $a_1, \cdots, a_k \in \Phi$ and $s_i \in \{1, \cdots, m-1\}$ for $i = 1, \cdots, k$. Then $w_{a_1}^{s_1} \cdots w_{a_k}^{s_k}$ is reduced if and only if $a_1, \cdots, a_k$ are linearly independent.*

Proof. Let $w = w_{a_1}^{s_1} \cdots w_{a_k}^{s_k}$. Suppose that the expression is reduced. Then by Lemma 2.3 w has k eigenvalues not equal to 1, and so

$$\dim(H_{a_1} \cap H_{a_2} \cap \cdots \cap H_{a_k}) = n - k.$$

(Here, the dimension cannot be larger since w acts as the identity on this subspace.) Thus, it follows that the roots $a_1, \cdots, a_k$ are linearly independent.

Conversely, suppose that the roots $a_1, \cdots, a_k$ are linearly independent. Now, consider the subspace $\text{im}(w-1)$, and select a vector $v_1 \in V$ such that

$$v_1 \in H_{a_2} \cap \cdots \cap H_{a_k} \text{ but } v_1 \notin H_{a_1}.$$

Then $w(v_1) - v_1$ is a non-zero multiple of a_1. Thus $a_1 \in \text{im}(w-1)$. Now, select once again a vector $v_2 \in V$ with

$$v_2 \in H_{a_3} \cap \cdots \cap H_{a_k} \text{ but } v_2 \notin H_{a_2}.$$

Then $w(v_2) - v_2 = \alpha a_1 + \beta a_2$, where $\alpha, \beta \in \mathbf{C}$ and $\beta \neq 0$. Hence $a_2 \in \text{im}(w-1)$. Repeating this argument will eventually show that $a_1, \cdots, a_k \in \text{im}(w-1)$, and so $\dim \text{im}(w-1) = k$. Then w is reduced, for if w has a shorter expression $w =$

$w_{b_1}^{\rho_1}\cdots w_{b_l}^{\rho_l}$ with $l<k$ and $\rho_i\in\{1,\ \cdots,\ m-1\}$, then every element of $\mathrm{im}(w-1)$ can be written as a linear combination of $b_1,\ \cdots,\ b_l$ and so $\dim\ \mathrm{im}(w-1)<k$, which is a contradiction. Furthermore, if w has an expression $w=w_{a_1}^{r_1}\cdots w_{a_k}^{r_k}$ with $r_i\le s_i$, then $w_{a_k}^{\varrho_k}\cdots w_{a_1}^{\varrho_1}w=1$ and $w_{a_k}^{\varrho_k}\cdots w_{a_1}^{\varrho_1}w_{a_1}^{r_1}\cdots w_{a_k}^{r_k}\neq 1$ where $\varrho_i=o_W(a_i)-s_i$, $(1\le i\le k)$, a contradiction. □

3. Admissible diagrams

Any element $w=w_{a_1}^{s_1}\cdots w_{a_k}^{s_k}\in W$ with $l(w)=\sum_{i=1}^{k}s_i$ can be decomposed as follows (see [1]):

$$w=\tau w_{a_{i+1}}^{s_{i+1}}\cdots w_{a_k}^{s_k},\ \text{ where }\ \tau=w_{a_1}\cdots w_{a_i}\in W(A_{n-1}).$$

Corresponding to each such decomposition of w, we define a graph Γ. Γ has k nodes, one corresponding to each root $a_1,\ \cdots,\ a_k$ with the value $o_W(a_i)$. The nodes corresponding to distinct roots $a_i,\ a_j$ are joined by a bond of weight $(a_i,\ a_j)$. If $o_W(a_i)=2$, then the number 2 in the node corresponding to the root a_i is omitted, as in Cohen [7]. If $w\in W$ has a decomposition with graph Γ, then any conjugate of w also has a decomposition with graph Γ. For if $w=w_{a_1}\cdots w_{a_i}w_{a_{i+1}}^{s_{i+1}}\cdots w_{a_k}^{s_k}$, where $w_{a_1}\cdots w_{a_i}\in W(A_{n-1})$, then we have $w'ww'^{-1}=w_{b_1}\cdots w_{b_i}w_{b_{i+1}}^{s_{i+1}}\cdots w_{b_k}^{s_k}$, where $b_j=w'(a_j)$ for $j=1,\ \cdots,\ k$.

Therefore, we say that the graph Γ is associated with this conjugacy class. (Here, we assume that the conjugacy class containing the identity element is represented by the empty graph.) By Lemma 2.4 the nodes of Γ correspond to a set of linearly independent roots.

Now we can give our principal definition.

Definition 3.1 Let Γ be a graph, then Γ is called an *admissible diagram* if

(i) the nodes of Γ correspond to a set of linearly independent roots of Φ,

(ii) each subgraph of Γ which is a cycle is equivalent to a web.

(A subgraph of Γ in this context is a subset of the nodes, together with the bonds joining the nodes in the subset. A cycle is a graph in which each node is connected to only two other nodes.)

Lemma 3.2. *Every admissible diagram associated with a conjugacy class of W is the Cohen (Dynkin) diagram of some reflection subgroup of W.*

Proof. Let Γ be such a graph. Let J be a set of roots corresponding to the nodes of Γ. Denote by $W(J)$ the group generated by all reflections $w_{a,o_W(a)}$ with $a\in J$, then $W(J)$ is a subgroup of W, so is a finite reflection group. Furthermore, J is linearly independent by definition of the admissible diagram. Thus, by (4.2) of Cohen [7] Γ is a root graph. Now, put $S=W(J)J$, define a map $g:\ S\to\mathbf{N}\backslash\{1\}$ by $g(a)=o_{W(J)}(a)$ for all $a\in S$, then the pair $\Psi=(S,g)$ is the pre-root system corresponding to J with $W(\Psi)=W(J)$ by 1.2 (ii) of Can [3]. By 1.2 (iii) of Can

[3], the pair $\Psi = (S, g)$ is a root system and so is a subsystem of Φ. Thus, $W(\Psi)$ is the reflection group of Ψ, and so Γ is the Cohen (Dynkin) diagram of the reflection subgroup $W(\Psi)$ of W, as desired. □

Here, recall that Γ may be a union of disconnected graphs Γ_i say, which satisfy the following: if Γ_i contains no web, then Γ_i is either of type A_n or B_n^m, and if Γ_i does contain a web, then Γ_i must be of type D_n^m.

The present author [3] has presented an algorithm for obtaining the graphs which are Cohen (Dynkin) diagrams of reflection subgroups of W without any reference to extended diagrams. In [4], we also interpreted it as a computer program written using the symbolic computation system Maple. The Cohen (Dynkin) diagrams of all possible reflection subgroups of W are either of the form

$$\sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}} + \sum_{j=1}^{s} D^m_{\mu_j} \quad \text{with} \quad \sum_{j=1}^{s_1}(\lambda_j^{(1)}+1) + \sum_{i=2}^{m}\sum_{j=1}^{s_i}\lambda_j^{(i)} + \sum_{j=1}^{s}\mu_j = n \quad \text{or}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{s_i} B^{m_i}_{\lambda_j^{(i)}} \quad \text{with} \quad \sum_{j=1}^{s_1}(\lambda_j^{(1)}+1) + \sum_{i=2}^{m}\sum_{j=1}^{s_i}\lambda_j^{(i)} = n,$$

where $m_i = \begin{cases} 1 & \text{if } i = 1 \\ m & \text{if } i = 2, \ \cdots, \ m \end{cases}$ (see [5]).

We now show that the admissible diagrams can be used to parametrise the conjugacy classes of W.

Theorem 3.3. *There is a one-to-one correspondence between conjugacy classes in W and admissible diagrams of the form*

$$\sum_{i=1}^{m} \ (B^{m_i}_{\lambda_1^{(i)}} + B^{m_i}_{\lambda_2^{(i)}} + \ \cdots \ + B^{m_i}_{\lambda_{s_i}^{(i)}})$$

where $\sum_{j=1}^{s_1}(\lambda_j^{(1)}+1) + \sum_{i=2}^{m}\sum_{j=1}^{s_i}\lambda_j^{(i)} = n$ and $m_i = \begin{cases} 1 & \text{if } i = 1 \\ m & \text{if } i = 2, \ \cdots, \ m. \end{cases}$

Proof. The elements of W operate on the orthonormal basis $e_1, \ \cdots, \ e_n$ of V by permuting the basis vectors and multiplying arbitrary subsets of them by a power of ξ. By ignoring these multiples, each element w of W determines a permutation of $\{1, \ \cdots, \ n\}$ which can be expressed in the usual way as a product of disjoint cycles. Let $(k_1k_2 \ \cdots \ k_r)$ be such a cycle which has the following shape

$$e_{k_1} \to \xi^{p_1}e_{k_2} \to \xi^{p_1+p_2}e_{k_3} \to \ \cdots \ \to \xi^{p_1+\cdots+p_{r-1}}e_{k_r} \to \xi^{p_1+\cdots+p_r}e_{k_1}$$

where $p_i \in \{1, \ \cdots, \ m\}$. The cycle $(k_1k_2 \ \cdots \ k_r)$ is said to be a (ξ^p, r)-cycle if $w^r(e_{k_1}) = \xi^p e_{k_1}$, where $\sum_{i=1}^{r} p_i \equiv p \pmod m$. Then the lengths of the cycles together with their values $\sum \ p_i$ determine the type of w, and two elements of W

are conjugate if and only if they have the same type, as in Kerber [9]. Thus there is a one-to-one correspondence between conjugacy classes and types. Now, consider the (ξ^p, r)-cycle

$$e_1 \to e_2 \to \cdots \to e_{r-1} \to e_r \to \xi^p e_1,$$

where $p \in \{1, \cdots, m\}$. If $p = m$, then this can be expressed as the product of elements $(12)(23) \cdots (r-1\ r)$. These factors form a complete set of simple reflections of the Weyl subgroup of type A_{r-1}, and so this $(1, r)$-cycle, denoted by $[r]$, is represented by an admissible diagram A_{r-1}, as in type A_n - see Carter [6]. If $p \in \{1, \cdots, m-1\}$, then this can be expressed as the product of elements $(12)(23) \cdots (r-1\ r)w_r^p$, where w_r^p changes e_r into $\xi^p e_r$ and fixes all other e_i. Thus these factors form a complete set of simple reflections of the reflection subgroup of type B_r^m, and so this (ξ^p, r)-cycle is represented by an admissible diagram B_r^m.

Now consider an arbitrary element of W, expressed as a product of disjoint (ξ^p, r)-cycles. Since disjoint cycles operate on orthogonal subspaces of V, the admissible diagram splits into connected components corresponding to the cycle decomposition, and so has form

$$\sum_{i=1}^{m} \left(B^{m_i}_{\lambda_1^{(i)}} + B^{m_i}_{\lambda_2^{(i)}} + \cdots + B^{m_i}_{\lambda_{s_i}^{(i)}}\right)$$

where $\sum_{j=1}^{s_1}(\lambda_j^{(1)}+1) + \sum_{i=2}^{m}\sum_{j=1}^{s_i}\lambda_j^{(i)} = n$ and $m_i = \begin{cases} 1 & \text{if } i = 1 \\ m & \text{if } i = 2, \cdots, m, \end{cases}$
as desired. □

Remark 3.4. Now, define m partitions $\lambda^{(1)}, \cdots, \lambda^{(m)}$ by

$$\lambda^{(1)} = (\lambda_1^{(1)}+1, \cdots, \lambda_{s_1}^{(1)}+1),\ \lambda^{(i)} = (\lambda_1^{(i)}, \cdots, \lambda_{s_i}^{(i)})\ \ (i = 2, \cdots, m),$$

then there is a one-to-one correspondence between conjugacy classes in W and m-sets of partitions $(\lambda^{(1)}, \cdots, \lambda^{(m)})$ of n with $\displaystyle\sum_{j=1}^{s_1}(\lambda_j^{(1)}+1) + \sum_{i=2}^{m}\sum_{j=1}^{s_i}\lambda_j^{(i)} = n$ (see Kerber [9]).

If $m = 1$, then $W = W(A_{n-1})$ (Wely group of type A_{n-1}) and if $m = 2$ then $W = W(C_n)$ (Weyl group of type C_n), and so by putting $m = 1, 2$ in Theorem 3.3, we recover the results of Carter [6]. The admissible diagrams given in Theorem 3.3 are not the only ones which could have been taken. We know that W contains a reflection subgroup $G(m, m, n) = W(D_n^m)$ (see [7]), and so D_n^m is an admissible diagram for W. However since the admissible diagrams given in Theorem 3.3 are in one-to-one correspondence with the conjugacy classes of W, we do not need to consider the remainder.

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