KYUNGPOOK Math. J. 45(2005), 167-169

Vector Bundles on Curves with Many "spread" Sections

E. Ballico
Department of Mathematics, University of Trento, 38050 Povo (TN), Italy
e-mail : ballico@science.unitn.it

Abstract. Here we introduce and study vector bundles, E, on a smooth projective curve X having many "spread" sections and for which $E^* \otimes \omega_X$ has many "spread" sections. We show that no such bundle exists on X if the gonality of X is too low.

1. Introduction

Here we are interested in vector bundles E on a smooth projective curve X having many "spread" sections and for which $E^* \otimes \omega_X$ has many "spread" sections. The corresponding notion for line bundles (primitive line bundles) was studied in [5] and [6]. Concerning the existence of stable or semistable vector bundles on X with many sections, see [1], [2], [3], [4], [7], [8] and references therein.

Definition 1. Let X be a smooth and connected projective curve and E a vector bundle on X. We will say that E is $(1,0)$-primitive if it is spanned, i.e., if it is spanned by its global sections. For any integer $a \geq 2$ we will say that E is $(a,0)$-primitive if for every vector bundle F obtained form E making a negative elementary transformations we have $h^0(X,F) = h^0(X,E) - a$, i.e., if and only if it is $(a-1,\ 0)$-primitive and every vector bundle G obtained from E making $a-1$ negative elementary transformations is $(1,0)$-primitive. For any integer $b > 0$ we will say that E is $(0,b)$-primitive if $\omega_X \otimes E^*$ is $(b,0)$-primitive. We will say that E is (a,b)-primitive if it is $(a,0)$-primitive and $(0,b)$-primitive.

Definition 2. Let X be a smooth and connected projective curve, E a vector bundle on X and a, b non-negative integers. We will say that E is strongly (a,b)-primitive if for every effective divisors A, B of X with $\deg(A) = a$ and $\deg(B) = b$ the restriction maps $\rho_{E,A} : H^0(X,E) \to H^0(A, E|A)$ and $\rho_{\omega_X^{\otimes} E^*,B} : H^0(X, \omega_X \otimes E^*) \to H^0(B, \omega_X \otimes E^*|B)$ are surjective. Here if $a = 0$ (resp. $b = 0$) we take $A = \emptyset$ (resp. $B = \emptyset$) and by convention we say that $\rho_{E,\emptyset}$ and $\rho_{\omega_X^{\otimes} E^*,\emptyset}$ are always bijective.

Remark 1. By Serre duality and Riemann-Roch E is $(0,b)$ primitive if and only

Received November 14, 2003.
2000 Mathematics Subject Classification: 14H60, 14H51.
Key words and phrases: vector bundles on curves, gonality, stable vector bundle, Brill-Noether theory for vector bundles.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

if for every vector bundle M obtained from E making b positive elementary transformations we have $h^0(X,M) = h^0(X,E)$.

Remark 2. A line bundle is $(2,0)$-primitive if and only if it is very ample. Hence a hyperellyptic curve has no $(2,1)$-primitive and no $(1,2)$-primitive line bundle.

Remark 3. A strongly (a,b)-primitive vector bundle is (a,b)-primitive. A line bundle is (a,b)-primitive if and only if it is strongly (a,b)-primitive.

Remark 4. In the definition of strongly (a,b)-primitivity we obtain the same notion if instead of A we take all effective divisors $A' \subset X$ such that $\deg(A') \le a$ and/or instead of B all effective divisors $B' \subset X$ such that $\deg(B') \le b$.

We work over an algebraically closed field $\mathbf{K}$ and prove the following results.

Theorem 1. *Fix an integer $r > 0$ and assume either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > r$. Let X be a smooth hyperelliptic curve of genus $g \ge 2$ and $R \in Pic^2(X)$ its hyperellyptic curve. Let E be a rank r strongly $(1,1)$-primitive vector bundle on X. Then there are integers $s > 0$, a_i and $b_i > 0$, $1 \le i \le s$, such that $0 \le i \le g-1$ for all i, $\sum_{i=1}^{s} b_i = r$ and $E \cong \oplus_{i=1}^{s}(R^{\otimes a_i})^{\oplus b_i}$. In particular E is stable if and only if $r = 1$ and E is semistable if and only if there is an integer a such that $0 \le a \le g-1$ and $E \cong (R^{\otimes a})^{\oplus r}$.*

Theorem 2. *Fix an integer $r > 0$ and assume either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > r$. Let X be a smooth hyperelliptic curve. Then there is no strongly $(2,1)$-primitive or strongly $(1,2)$-primitive rank r vector bundle on X.*

Theorem 3. *Fix an integer $r > 0$ and assume either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > r$. Let X be a smooth k-gonal curve of genus $g \ge 2k-3 \ge 3$. Then there is no strongly (k,k)-primitive rank r vector bundle on X.*

2. The proofs

Proof of Theorem 1. Since the "if" part is obvious, we only need to prove the "only if" part. Let $\psi : X \to \mathbf{P}^1$ the degree 2 hyperelliptic pencil and $\phi : X \to G(r,m)$, $m := h^0(X,E)$, the morphism associated to the pair $(E, H^0(X,E))$; ϕ is a morphism because E is $(1,0)$-primitive, i.e., E is spanned. Since every vector bundle on $\mathbf{P}*^1$ is a direct sum of line bundles, it is sufficient to show that ϕ factors through ψ. Fix P, $Q \in X$ such that $P \ne Q$ and $R \cong \mathcal{O}_X(P+Q)$. Since we may take as P a general point of X, it is sufficient to prove that $\phi(Q) = \phi(P)$. Assume $\phi(Q) \ne \phi(P)$. Hence there is $m \in H^0(X,E)$ such that $m(P) = 0 \in E|\{P\}$ and $m(Q) \ne 0 \in E|\{Q\}$. Notice that $E^* \otimes \omega_X \cong Hom(E,\omega_X)$. Since $E^* \otimes \omega_X$ is spanned, there is $f \in H^0(X, Hom(E,\omega_X))$ such that $(f \circ m)(Q) \ne 0 \in \omega_X|\{Q\}$. Since $m(P) = 0$, then $(f \circ m)(P) = 0$ and hence $(f \circ m)(P) \ne (f \circ m)(Q)$. Since either $\mathrm{char}(\mathbf{K}) = 0$ or $\mathrm{char}(\mathbf{K}) > r$, we have $E \otimes E^* \otimes \omega_X \cong \mathrm{ad}(E) \otimes \omega_X \oplus \omega_X$. Hence $\psi(Q) \ne \psi(P)$, contradiction. □

Proof of Theorem 2. Let $\psi : X \to \mathbf{P}^1$ the degree 2 hyperelliptic pencil. Since for every integer a with $0 \le a \le g$ the morphism induced by $R^{\otimes a}$ factors through ψ and $\omega_X \cong R^{\otimes(g-1)}$, the result follows from Theorem 1. □

Proof of Theorem 3. By assumption there are $P_1, \cdots, P_k \in X$, $P_i \neq P_j$ for $i \neq j$, such that $P_1 + \cdots + P_k$ is a g^1_k on X. Set $S := \{P_1, \cdots, P_k\}$. Assume the existence of a strongly (k,k)-primitive rank r vector bundle on X. Since either char$(\mathbf{K}) = 0$ or char$(\mathbf{K}) > r$, we have $E \otimes E^* \otimes \omega_X \cong \mathrm{ad}(E) \otimes \omega_X \oplus \omega_X$. Let $\phi_K : X \to \mathbf{P}^{g-1}$ be the canonical map. Since E is strongly $(k,0)$-primitive, the restriction map $u : H^0(X,E) \to H^0(S,E|S)$ is surjective. Since E is strongly $(0,k)$-primitive, the restriction map $v : H^0(X, E^* \otimes \omega_X) \to H^0(S, E^* \otimes \omega_X|S)$ is surjective. Since the tensor product is a right exact functor, the map $u \otimes v : H^0(X,E) \otimes H^0(X, E^* \otimes \omega_X) \to H^0(S,E|S) \otimes H^0(S, E^* \otimes \omega_X|S) \cong H^0(S, E \otimes E^* \otimes \omega_X|S)$ is surjective. Since $u \otimes v$ factors through the map $H^0(X,E) \otimes H^0(X, E^* \otimes \omega_X) \to H^0(X, E \otimes E^* \otimes \omega_X)$, we obtain the surjectivity of the restriction map $\rho : H^0(X, E \otimes E^* \otimes \omega_X) \to H^0(S), E \otimes E^* \otimes \omega_X|S))$. Since either char$(\mathbf{K}) = 0$ or char$(\mathbf{K}) > r$, we have $E \otimes E^* \otimes \omega_X \cong \mathrm{ad}(E) \otimes \omega_X \oplus \omega_X$. Hence we obtain the surjectivity of the restriction map $H^0(X, \omega_X) \to H^0(S, \omega_X|S)$ and hence $\phi_K(S)$ spans a $(k-1)$-dimensional linear subspace of $\mathbf{P}^{g-1}$, contradicting the geometric form of Riemann-Roch. □

References

[1] E. Ballico, *Brill-Noether theory for vector bundles on projective curves*, Math. Proc. Camb. Philos. Soc., **128**(1998), 483–499.

[2] L. Brambila-Paz, I. Grzegorczyk and P. E. Newstead, *Geography of Brill-Noether loci for small slopes*, J. Agebraic Geom., **6**(1997), 645–668.

[3] L. Brambila-Paz, V. Mercat, P. E. Newstead and F. Ongay, *Nonemptyness of Brill-Noether loci*, Int. J. Math., **11(6)**(2000), 737–760.

[4] J. Cilleruelo and I. Sols, *The Severi bound on sections of rank two semistable vector bundles on a Riemann surface*, Ann. Math., **154**(2001), 739–758.

[5] M. Coppens, C. Keem and G. Martens, *Primitive linear series on curves*, Manuscripta Math., **77**(1992), 237–264.

[6] M. Coppens, C. Keem and G. Martens, *The primitive length of a general k-gonal curve*, Indag. Math., N.S., **5**(1994), 145–159.

[7] H. Lange and P. E. Newstead, On Clifford's theorem for rank-3 bundles, ArXiv:math.AG/0310463.

[8] V. Mercat, *Clifford's theorem and higher rank vector bundles*, Int. J. Math., **13(7)**(2002), 785–796.