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TRILINEAR FORMS AND THE SPACE OF COMTRANS ALGEBRAS

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Abstract. Comtrans algebras are modules equipped with two trilinear operations: a left alternative commutator and a translator satisfying the Jacobi identity, the commutator and translator being connected by the so-called comtrans identity. These identities have analogues for trilinear forms. On a given vector space, the set of all comtrans algebra structures itself forms a vector space. In this paper, the dimension of the space of comtrans algebra structures on a finite-dimensional vector space is determined.

1. Introduction

Comtrans algebras (which arise naturally in many different contexts [2] – [8], [10]) are unital modules over a commutative ring R, equipped with two basic trilinear operations: a *commutator* $[x, y, z]$ satisfying the *left alternative identity*

$$[x, x, y] = 0, \tag{1.1}$$

and a *translator* $\langle x, y, z\rangle$ satisfying the *Jacobi identity*

$$\langle x, y, z\rangle + \langle y, z, x\rangle + \langle z, x, y\rangle = 0, \tag{1.2}$$

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such that together the commutator and translator satisfy the *comtrans identity*

$$[x, y, x] = \langle x, y, x \rangle. \tag{1.3}$$

Comtrans algebras were introduced in [9] as an answer to a problem from differential geometry, asking for a determination of the tangent bundle algebra structure corresponding to the coordinate t-ary loop of a $(t+1)$-web (compare [1]). The role played by comtrans algebras here is analogous to the role played by the Lie algebra of a Lie group.

The set of all comtrans algebras on a given module has a linear structure: If $(E, [x, y, z]_1, \langle x, y, z \rangle_1)$ and $(E, [x, y, z]_2, \langle x, y, z \rangle_2)$ are comtrans algebras on an R-module E, then so is

$$(E, r_1[x, y, z]_1 + r_2[x, y, z]_2, r_1\langle x, y, z \rangle_1 + r_2\langle x, y, z \rangle_2)$$

for any scalars r_1, r_2 from R. The main result of the paper, Corollary 3.2, determines the dimension of the space $\mathrm{CT}(n)$ of comtrans algebras on an n-dimensional vector space. Note that the set of all Lie algebras on a given space does not have a comparable linear structure, due to the nesting of brackets in the Jacobi identity for Lie algebras. Consider, say, the space $\mathbb{R}^3$ with standard ordered basis (e_1, e_2, e_3). Let $(\mathbb{R}^3, [x, y]_1)$ be the usual vector cross product algebra with $[x, y]_1 = x \times y$. Let $(\mathbb{R}^3, [x, y]_2)$ be the Lie algebra specified by $[e_1, e_2] = [e_2, e_3] = [e_3, e_1] = e_1$. The Jacobi identity holds for the bracket $[x, y]_1 + [x, y]_2$ if and only if $[x, [y, z]_2]_1 + [y, [z, x]_2]_1 + [z, [x, y]_2]_1 + [x, [y, z]_1]_2 + [y, [z, x]_1]_2 + [z, [x, y]_1]_2 = 0$. However, setting $x = e_1$, $y = e_2$ and $z = e_3$ yields the non-zero result $-e_3 + e_2$.

In computing the dimension of the space of comtrans algebras on an n-dimensional vector space, one may take advantage of the fact that the results of the ternary comtrans operations may be decomposed as linear combinations of given basis vectors. Each component in such a decomposition determines a ternary form on the space, taking values in the ground field. Thus the comtrans algebra structure is equivalent to

n so-called "commutator forms" and n "translator forms" (Section 2). A pair of such forms, satisfying the identities corresponding to those specifying a comtrans algebra, is called a "comtrans form." Theorem 3.1 determines the dimension of the linear space of all comtrans forms on an n-dimensional vector space. The dimension of the space of all comtrans algebras is then obtained as n times the dimension of the space of all comtrans forms.

2. Comtrans algebras and comtrans forms

Consider a comtrans algebra on a vector space E with ordered basis $(e_1, \ldots, e_n)$. The comtrans algebra is determined by the *structure constants* c^l_{ijk} and d^l_{ijk} (for $1 \leq i, j, k, l \leq n$) with

$$[e_i, e_j, e_k] = \sum_{l=1}^{n} c^l_{ijk} e_l$$

and

$$\langle e_i, e_j, e_k \rangle = \sum_{l=1}^{n} d^l_{ijk} e_l \,.$$

Since the defining identities (1.1) – (1.3) of a comtrans algebra do not involve any nesting of the basic operations, it is convenient to reduce the specification of a comtrans algebra of dimension n over a field to the specification of n pairs

$$(c^1_{ijk}, d^1_{ijk}), \ldots, (c^n_{ijk}, d^n_{ijk})$$

of trilinear forms corresponding to the coefficients of the products with respect to a given basis.

Definition 2.1. Let E be a module over a commutative ring R. Then a *comtrans form* on E is defined to be a pair

$$([\![x, y, z]\!], \langle\!\langle x, y, z \rangle\!\rangle) \tag{2.1}$$

of trilinear forms on E (with values in R) satisfying the analogues of (1.1) – (1.3), namely the *(formal) left alternative identity*

$$[\![x, x, y]\!] = 0\,, \tag{2.2}$$

the *(formal) Jacobi identity*

$$\langle\!\langle x, y, z\rangle\!\rangle + \langle\!\langle y, z, x\rangle\!\rangle + \langle\!\langle z, x, y\rangle\!\rangle = 0 \tag{2.3}$$

and the *(formal) comtrans identity*

$$[\![x, y, x]\!] = \langle\!\langle x, y, x\rangle\!\rangle\,. \tag{2.4}$$

The first component of (2.1) is called the *commutator form*, while the second component is called the *translator form*.

Proposition 2.2. *Let E be a vector space with ordered basis $(e_1, \ldots, e_n)$. Consider the equations*

$$[x, y, z] = \sum_{l=1}^{n} [\![x, y, z]\!]_l e_l \tag{2.5}$$

and

$$\langle x, y, z\rangle = \sum_{l=1}^{n} \langle\!\langle x, y, z\rangle\!\rangle_l e_l\,. \tag{2.6}$$

(a) *If $(E, [x, y, z], \langle x, y, z\rangle)$ is a comtrans algebra, then the equations (2.5) and (2.6) determine n comtrans forms*

$$([\![x, y, z]\!]_1, \langle\!\langle x, y, z\rangle\!\rangle_1), \ldots, ([\![x, y, z]\!]_n, \langle\!\langle x, y, z\rangle\!\rangle_n) \tag{2.7}$$

on E.

(b) *Given a system (2.7) of n comtrans forms on E, the equations (2.5) and (2.6) determine a comtrans algebra on E.*

3. The space of comtrans forms

Consider a comtrans form (c_{ijk}, d_{ijk}) on a vector space of dimension n, specified by the tensor representations of its component trilinear forms. The left alternative identity implies the tensor equations

$$c_{ijk} + c_{jik} = 0 \tag{3.1}$$

for $1 \leq i, j, k \leq n$ and

$$c_{iij} = 0 \tag{3.2}$$

for $1 \leq i, j \leq n$. The Jacobi identity implies the tensor equations

$$d_{ijk} + d_{jki} + d_{kij} = 0 \tag{3.3}$$

for $1 \leq i, j, k \leq n$. The comtrans identity implies the tensor equations

$$c_{ijk} + c_{kji} - d_{ijk} - d_{kji} = 0 \tag{3.4}$$

for $1 \leq i, j, k \leq n$ and

$$c_{iji} - d_{iji} = 0 \tag{3.5}$$

for $1 \leq i, j \leq n$.

The full space $\mathrm{TF}(n)$ of tensors c_{ijk} and d_{ijk} for $1 \leq i, j, k \leq n$ has dimension $2n^3$. Within that space, the system (3.1) – (3.5) of equations specifies the subspace $\mathrm{CTF}(n)$ of comtrans forms.

Theorem 3.1. *On an n-dimensional vector space, the space $\mathrm{CTF}(n)$ of comtrans forms has dimension $\frac{5}{6}n^3 - \frac{1}{2}n^2 - \frac{1}{3}n$.*

Proof. First, consider a suffix $1 \leq i \leq n$. On the subspace $\mathrm{TF}(n)_i$ of $\mathrm{TF}(n)$ spanned by the ordered basis (c_{iii}, d_{iii}), the equations (3.2), (3.3) and (3.5) form a system of linear equations with coefficient matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 1 & -1 \end{bmatrix}.$$

Since this matrix row-reduces to a matrix including the 2×2 identity matrix, the dimension of the solution space $\mathrm{CTF}(n)_i$ is zero.

Next, consider a pair $1 \leq i < j \leq n$ of distinct suffices. Let $\mathrm{TF}(n)_{i,j}$ denote the subspace of $\mathrm{TF}(n)$ spanned by the ordered basis

$$(c_{iij}, c_{iji}, c_{jii}, c_{jji}, c_{jij}, c_{ijj}, d_{iij}, d_{iji}, d_{jii}, d_{jji}, d_{jij}, d_{ijj})\,.$$

With respect to this basis, the equations (3.1) – (3.5) form a system of linear equations with coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Since thie matrix row-reduces to a matrix of rank 8, the solution space $\mathrm{CTF}(n)_{i,j}$ has dimension 4.

Finally, consider a triple $1 \leq i < j < k \leq n$ of distinct suffices. Let $\mathrm{TF}(n)_{i,j,k}$ denote the subspace of $\mathrm{TF}(n)$ spanned by the ordered basis

$$(c_{ijk}, c_{jki}, c_{kij}, c_{ikj}, c_{kji}, c_{jik}, d_{ijk}, d_{jki}, d_{kij}, d_{ikj}, d_{kji}, d_{jik})\,.$$

With respect to this basis, the equations (3.1), (3.3) and (3.4) form a system of linear equations with coefficient matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}.$$

Since thie matrix row-reduces to a matrix of rank 7, the solution space $\mathrm{CTF}(n)_{i,j,k}$ has dimension 5. Within $\mathrm{CTF}(n)$, there are $\binom{n}{2}$ subspaces of the form $\mathrm{CTF}(n)_{i,j}$, and $\binom{n}{3}$ subspaces of the form $\mathrm{CTF}(n)_{i,j,k}$. Thus the total dimension of $\mathrm{CTF}(n)$ is

$$5\binom{n}{3} + 4\binom{n}{2} = \frac{5}{6}n^3 - \frac{1}{2}n^2 - \frac{1}{3}n,$$

as required. □

Corollary 3.2. *On an n-dimensional vector space E, the space* $\mathrm{CT}(n)$ *of comtrans algebras has dimension* $\frac{5}{6}n^4 - \frac{1}{2}n^3 - \frac{1}{3}n^2$.

Proof. For each index $1 \leq l \leq n$, Theorem 3.1 shows that there is a $(\frac{5}{6}n^3 - \frac{1}{2}n^2 - \frac{1}{3}n)$-dimensional space of comtrans forms (c^l_{ijk}, d^l_{ijk}). By Proposition 2.2, the $(\frac{5}{6}n^4 - \frac{1}{2}n^3 - \frac{1}{3}n^2)$-dimensional space of all these forms then comprises the space of structure constants of comtrans algebras on E. □

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