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Ramanujan's Continued Fraction, a Generalization and Partitions

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Abstract. We generalize a continued fraction of Ramanujan by introducing a free parameter. We give the closed form for the continued fraction. We also consider the finite form giving n^{th} convergent using partition theory.

1. Introduction

Ramanujan, in his first letter to Hardy [7, p. xxviii] stated the continued fraction

$$1+\cfrac{x}{1+\cfrac{x^2}{1+\cfrac{x^3}{\cdot}}} \tag{1}$$

now known as Rogers-Ramanujan continued fraction and gave some identities involving it. Ramanujan continued to write "the above theorem is a particular case of a theorem on the continued fraction

$$1+\cfrac{ax}{1+\cfrac{ax^2}{1+\cfrac{ax^3}{\cdot}}}, \tag{2}$$

which is a particular case of the continued fraction

$$1+\cfrac{ax}{(1+bx)+\cfrac{ax^2}{(1+bx^2)+\cfrac{ax^3}{(1+bx^3)+\cdot}}}, \tag{3}$$

which is a particular case of a general theorem on continued fractions". Andrews [1, Theorem 6] gave the continued fraction

$$1+bxq+\cfrac{xq(1+axq^2)}{(1+bxq^2)+\cfrac{xq^2(1+axq^3)}{(1+bxq^3)+\cdot}} \tag{4}$$

and thought this might be the general theorem about which Ramanujan referred.

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In this paper we give a mild generalization of the continued fraction (2), considered by Hirschhorn [4], by introducing a free parameter and then generate the generalized continued fraction, giving a closed form for the infinite continued fraction. We then consider the finite form of the generalization and give the closed form of the sum using two methods. In the first method we consider the expansion $P = \sum_{n=0}^{\infty} P_n z^{n+1}$ and from this get the n^{th} convergent. In the second method we use partition theory. Later we give an alternate expansion of the n^{th} convergent using Watson ${}_8\varphi_7$ transformation.

2. Notations

We shall use the following usual basic hypergeometric notations: For $|x| < 1$,

$(a)_0 = 1,$

$(a)_n = (1-a)(1-ax)\cdots(1-ax^{n-1})$ for $1 \le n < \infty$

$(a)_\infty = \Pi_{r=0}^{\infty}(1-ax^r)$

$\left[{n \atop r}\right] = \frac{(x)_n}{(x)_r (x)_{n-r}}$

$\sum_k p(k,n,r)x^k = \frac{(x)_{r+n}}{(x)_r (x)_n}.$

3. The continued fraction

Let

$$F(a,c,x) = 1 + \cfrac{(1-1/c)ax}{1+\cfrac{ax^2}{1+\cfrac{(1-1/cx)ax^3}{1+\cfrac{ax^4}{\cdot}}}} = \frac{P(a,c,x)}{P(ax,c,x)}. \tag{5}$$

We shall prove that

$$P(a,c,x) = \sum_{n=0}^{\infty} \frac{(x)^{\frac{n^2+n}{2}}(c)_n(-a/c)^n}{(x)_n} \tag{6}$$

Proof. We define for non-negative integer i

$$F_i = \sum_{n=0}^{\infty} \frac{(-ax^i/\lambda)_n (cx^i)_n (-\lambda x/c)^n}{(x)_n (-bx)_{n+i}} \tag{7}$$

and

$$H_i = \sum_{n=0}^{\infty} \frac{(-ax^i/\lambda)_n (cx^i)_n (-\lambda x^2/c)^n}{(x)_n (-bx)_{n+i}}. \tag{8}$$

This gives

$$
\begin{aligned}
(9)\qquad & F_i - H_i \\
= \quad & \sum_{n=0}^{\infty} \frac{(-ax^i/\lambda)_n (cx^i)_n (-\lambda x/c)^n}{(x)_n (-bx)_{n+i}} (1-x^n) \\
= \quad & (-\lambda x/c)(1+ax^i/\lambda)(1-cx^i) \sum_{n=0}^{\infty} \frac{(-ax^{i+1}/\lambda)_n (cx^{i+1})_n (-\lambda x/c)^n}{(x)_n (-bx)_{n+i+1}} \\
= \quad & (1-1/cx^i)(\lambda x^{i+1} + ax^{2i+1}) F_{i+1}.
\end{aligned}
$$

Now we transform F_i and H_i by Heine's fundamental transformation

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n \tau^n}{(x)_n (\gamma)_n} = \frac{(\alpha\beta\tau/\gamma)_\infty}{(\tau)_\infty} \sum_{n=0}^{\infty} \frac{(\gamma/\alpha)_n (\gamma/\beta)_n (\alpha\beta\tau/\gamma)^n}{(x)_n (\gamma)_n}.$$

Taking $\alpha = -ax^i/\lambda,\ \ \beta = cx^i,\ \ \tau = -\lambda x/c,\ \ \gamma = -bx^{i+1}$, we have

$$F_i = \frac{(-ax^i/\lambda)_\infty}{(-\lambda x/c)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda bx/a)_n (-bx/c)_n (-ax^i/b)^n}{(x)_n (-bx)_{n+i}}. \tag{10}$$

Then taking $\alpha = -ax^i/\lambda,\ \ \beta = cx^i,\ \ \tau = -\lambda x^2/c,\ \ \gamma = -bx^{i+1}$, we have

$$H_i = \frac{(-ax^{i+1}/\lambda)_\infty}{(-\lambda q^2/c)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda bx/a)_n (-bx/c)_n (-ax^{i+1}/b)^n}{(x)_n (-bx)_{n+i}}. \tag{11}$$

From (10) and (11), we get in the same way

$$H_i - (1+\lambda x/c) F_{i+1} = (bx^{i+1} + ax^{2i+2}) H_{i+1}. \tag{12}$$

Putting $\lambda = 0,\ b = 0$ in (9) and (12), we have

$$
\begin{aligned}
F_i - H_i &= (1-1/cx^i) ax^{2i+1} F_{i+1}, && (13)\\
H_i - F_{i+1} &= ax^{2i+2} H_{i+1}. && (14)
\end{aligned}
$$

So

$$\frac{F_0}{H_0} = \frac{\sum_{n=0}^{\infty} \frac{x^{\frac{n^2+n}{2}} (c)_n (-a/c)^n}{(x)_n}}{\sum_{n=0}^{\infty} \frac{x^{\frac{n^2+3n}{2}} (c)_n (-a/c)^n}{(x)_n}} = F(a,c,x). \tag{15}$$

Now we iterate (13) and (14), to get

$$
\begin{aligned}
\frac{F_0}{H_0} &= 1 + \frac{(1-1/c)ax}{\frac{H_0}{F_1}} = 1 + \frac{(1-1/c)ax}{1 + \frac{ax^2}{\frac{F_1}{H_1}}} \\
&= 1 + \frac{(1-1/c)ax}{1 + \frac{ax^2}{1 + \frac{(1-1/cx)ax^3}{\frac{H_1}{F_2}}}}.
\end{aligned}
$$

Hence (15) is the closed form of the infinite continued fraction (5).

Let $c \longrightarrow \infty$ and $a = 1$ in (15)

$$\frac{\sum_{n=0}^{\infty} \frac{x^{n^2}}{(x)_n}}{\sum_{n=0}^{\infty} \frac{x^{n^2+n}}{(x)_n}} = 1 + \frac{x}{1 + \frac{x^2}{1+\frac{x^3}{1}}},$$

which is the celebrated Roger's Ramanujan continued fraction. □

4. Interesting cases

(i) Taking $c = x$ in (15), we have

$$1 + \frac{(1-1/x)ax}{1 + \frac{ax^2}{1+\frac{(1-1/x^2)ax^3}{\cdot}}} = \frac{\sum_{n=0}^{\infty}(-1)^n a^n x^{\frac{n^2-n}{2}}}{\sum_{n=0}^{\infty}(-1)^n a^n x^{\frac{n^2+n}{2}}}.$$

(ii) Taking $c = -x,\ a = 1$ in (15), we have

$$\begin{aligned} 1 + \frac{(1+1/x)x}{1 + \frac{x^2}{1+\frac{(1+1/x^2)x^3}{\cdot}}} &= \frac{\sum_{n=0}^{\infty} \frac{x^{\frac{n^2-n}{2}}(-x)_n}{(x)_n}}{\sum_{n=0}^{\infty} \frac{x^{\frac{n^2+n}{2}}(-x)_n}{(x)_n}} \\ &= \frac{[(q^2;q^4)_\infty^2(q^4;q^4)_\infty + (q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty]}{(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty} \text{ by Slater [8, eq.(8) and eq.(13)]} \\ &= 1 + \frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty}. \end{aligned}$$

(iii) Writing x^2 for x and then putting $c = -x,\ a = 1$ in (15), we have the identity due to Ramanujan [5]

$$1 + \frac{x + x^2}{1 + \frac{x^4}{1+\frac{x^3+x^6}{1+\frac{x^8}{\cdot}}}} = \sum_{n=0}^{\infty} \frac{(1-x^{8n+3})(1-x^{8n+5})}{(1-x^{8n+1})(1-x^{8n+7})}.$$

5. Expression for P_n and Q_n

Let

$$(16)\quad P(z) = 1 + \sum_{n=0}^{\infty} P_n z^{n+1} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{\frac{r^2+r}{2}}(cz)_r(a/c)^r}{(z)_{r+1}},\ |a/c| < 1 \text{ and}$$

$$Q(z) = \sum_{n=0}^{\infty} Q_n z^n = \sum_{r=0}^{\infty} \frac{(-1)^r x^{\frac{r^2+3r}{2}}(cz)_r(a/c)^r}{(z)_{r+1}},\ |a/c| < 1.$$

Applying Abel's Lemma

$$\begin{aligned} P_\infty &= \lim_{n\to\infty} P_n = \lim_{z\to 1^-}(1-z)P(z) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r x^{\frac{r^2+r}{2}}(c)_r(a/c)^r}{(x)_r}, \quad |a/c|<1 \end{aligned}$$

which is (6). Similarly

$$Q_\infty = \sum_{r=0}^{\infty} \frac{(-1)^r x^{\frac{r^2+3r}{2}}(c)_r(a/c)^r}{(x)_r}, \quad |a/c|<1.$$

Explicit expressions for P_n and Q_n

To obtain explicit expressions for P_n and Q_n, we shall use (16). We have

$$\begin{aligned} P_\infty &= 1+\sum_{n=0}^{\infty} P_n z^{n+1} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{\frac{r^2+r}{2}}(cz)_r(az/c)^r}{(z)_{r+1}} \\ &= \sum_{r=0}^{\infty}(-1)^r x^{\frac{r^2+r}{2}}(a/c)^r z^r \sum_{s=0}^{r}(-1)^s x^{\binom{s}{2}}(cz)^s {\textstyle\begin{bmatrix} r \\ s\end{bmatrix}} \sum_{t=0}^{\infty} z^t {\textstyle\begin{bmatrix} t+r \\ r\end{bmatrix}}. \end{aligned}$$

Hence

$$\begin{aligned} (17)\quad P_n(a,b,c,x) &= \sum_{r+s+t=n+1}(-1)^r x^{\frac{r^2+r}{2}}(a/c)^r(-1)^s c^s x^{\frac{s^2-s}{2}} {\textstyle\begin{bmatrix} r \\ s\end{bmatrix}\begin{bmatrix} r+t \\ r\end{bmatrix}} \\ &= \sum_{r=0}^{n+1}(-1)^r x^{\frac{r^2+r}{2}}(a/c)^r \sum_{s=0}^{\min(r,n-r+1)}(-1)^s c^s x^{\frac{s^2-s}{2}} {\textstyle\begin{bmatrix} r \\ s\end{bmatrix}\begin{bmatrix} n-s+1 \\ r\end{bmatrix}}. \end{aligned}$$

Similarly

$$(18)\qquad Q_n(a,b,c,x) = \sum_{r=0}^{n}(-1)^r x^{\frac{r^2+3r}{2}}(a/c)^r \sum_{s=0}^{\min(r,n-r)}(-1)^s c^s x^{\frac{s^2-s}{2}} {\textstyle\begin{bmatrix} r \\ s\end{bmatrix}\begin{bmatrix} n-s \\ r\end{bmatrix}}.$$

(17) and (18) are the explicit expression for P_n and Q_n.

6. Explicit expression for P_n and Q_n using partitions

We shall now find explicit expressions for P_n and Q_n by using a result in partitions.

$$P_\infty = 1+\sum_{n=0}^{\infty} P_n z^{n+1} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{\frac{r^2+r}{2}}(cz)_r(az/c)^r}{(z)_{r+1}}.$$

Therefore

$$\begin{aligned}P_n(a,b,c,x) &= \sum_{r=0}^{n+1}(-1)^r x^{\frac{r^2+r}{2}}(a/c)^r \sum_{s=0}^{\min(r,n-r+1)}(-1)^s c^s \\ &\quad \sum_{0<\alpha_1\cdots\alpha_s\le r} x^{(\alpha_1-1)+(\alpha_2-1)+\cdots+(\alpha_s-1)} \\ &\quad \times \sum_{\beta_0+\beta_1+\beta_2+\cdots+\beta_s<n-r-s+1} 1^{\beta_0}x^{\beta_1}\cdots x^{\beta_r}.\end{aligned}$$

Putting $v_i=\alpha_i-t_i,\;\; t_i=i,\;\; i=1,2,\cdots,s.$

$$\begin{aligned}P_n(a,b,c,x) &= \sum_{r=0}^{n+1}(-1)^r x^{\frac{r^2+r}{2}}(a/c)^r \sum_{s=0}^{\min(r,n-r+1)}(-1)^s c^s x^{\frac{s^2-s}{2}} \\ &\quad \sum_{0<v_1\cdots<v_s\le r-s} x^{v_1+v_2+\cdots+v_s} \\ &\quad \times \sum_{\beta_1+\cdots+\beta_r<n-r-s+1} x^{\beta_1+\cdots+\beta_r}.\end{aligned}$$

Now using a result on partition [5, Art. 241], viz.,

$$\sum_k p(k,r,n)x^k = \frac{(x)_{r+n}}{(x)_r(x)_n},$$

where $p(k,r,n)$ is the number of partition of k into at most r parts not exceeding n.

Hence

$$\begin{aligned}&P_n(a,b,c,x) \\ =&\sum_{r=0}^{n+1}(-1)^r x^{\frac{r^2+r}{2}}(a/c)^r \sum_{s=0}^{\min(r,n-r+1)}(-1)^s x^{\frac{s^2-s}{2}}c^s\sum_k p(k,s,r-s)x^k \\ &\times\sum_k p(k,n-r-s+1,r)x^k \\ =&\sum_{r=0}^{n+1}(-1)^r x^{\frac{r^2+r}{2}}(a/c)^r \sum_{s=0}^{\min(r,n-r+1)}(-1)^s x^{\frac{s^2-s}{2}}c^s\frac{(x)_r}{(x)_s(x)_{r-s}}\frac{(x)_{n-s+1}}{(x)_r(x)_{n-r-s+1}} \\ =&\sum_{r=0}^{n+1}(-1)^r x^{\frac{r^2+r}{2}}(a/c)^r \sum_{s=0}^{\min(r,n-r+1)}(-1)^s x^{\frac{s^2-s}{2}}c^s{\textstyle\left[{r\atop s}\right]\left[{n-s+1\atop r}\right]}.\end{aligned}$$

Similarly for $Q_n(a,b,c,x)$.

7. Alternative forms for P_n and Q_n

We shall show that

$$(19) \quad P(a,b,c,x) = \frac{(ax/c)_\infty}{(ax)_\infty}\sum_{r=0}^{\infty}\frac{(1-ax^{2r})}{(1-ax^r)}\frac{(ax)_r(c)_r(a^2/c)^r x^{2r^2}}{(x)_r(ax/c)_r} \quad \text{and}$$

$$(20) \quad Q(a,b,c,x) = \frac{(ax^2/c)_\infty}{(ax^2)_\infty}\sum_{r=0}^{\infty}\frac{(1-ax^{2r+1})}{(1-ax^{r+1})}\frac{(ax^2)_r(c)_r(a^2x^2/c)^r x^{2r^2}}{(x)_r(ax^2/c)_r}.$$

Proof. We shall use Watson's theorem [3] to prove (19) and (20). Watson's theorem is

$$(21) \quad {}_4\phi_3\left[\begin{matrix} ax/BC, D, E, x^{-N} \\ ax/B, Ax/C, DEx^{-N}/A \end{matrix}; x\right]$$

$$= \frac{(Ax/D)_N(Ax/E)_N}{(Ax)_N(Ax/DE)_N}$$

$${}_8\phi_7\left[\begin{matrix} A, \sqrt{A}x, -\sqrt{A}x, B, C, D, E, x^{-N} \\ \sqrt{A}, -\sqrt{A}, Ax/B, Ax/C, Ax/D, Ax/E, Ax/x^{-N} \end{matrix}; ^{;a^2x^2/BCDEx^{-N}}\right].$$

Making $B,\ C,\ D,\ N\to\infty$ and putting $A=a,\ E=C$ in (21), we have

$$\begin{aligned} P &= \sum_{r=0}^{\infty} x^{\frac{r^2+r}{2}}\frac{(c)_r(a/c)^r}{(x)_r} \\ &= \frac{(ax/c)_\infty}{(ax)_\infty}\sum_{r=0}^{\infty}\frac{(a)_r(\sqrt{a}x)_r(-\sqrt{a}x)_r(c)_r}{(a)_r(-a)_r(ax/c)_r} \\ &= \frac{(ax/c)_\infty}{(ax)_\infty}\sum_{r=0}^{\infty}\frac{(1-ax^{2r})}{(1-ax^r)}\frac{(c)_r(ax)_r(a^2/c)^r x^{2r^2}}{(x)_r(ax/c)_r}. \end{aligned}$$

Similarly making $B,\ C,\ D,\ N\to\infty$ and putting $A=ax,\ \ E=C$ in (21), we have

$$Q = \frac{(ax^2/c)_\infty}{(ax^2)_\infty}\sum_{r=0}^{\infty}\frac{(1-ax^{2r+1})}{(1-ax^{r+1})}\frac{(c)_r(ax^2)_r(a^2x^2/c)^r x^{2r^2}}{(x)_r(ax^2/c)_r},$$

which proves (15) and (16). □

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