

## Some Nonlinear Alternatives in Banach Algebras with Applications II

B. C. DHAGE

*“Kasubai”, Gurukul Colony, Ahmedpur-413 515, Dist: Latur Maharashtra, India*  
e-mail : bcd20012001@yahoo.co.in

ABSTRACT. In this paper a nonlinear alternative of Leray-Schauder type is proved in a Banach algebra involving three operators and it is further applied to a functional nonlinear integral equation of mixed type

$$x(t) = k(t, x(\mu(t))) + [f(t, x(\theta(t)))] \left( q(t) + \int_0^{\sigma(t)} v(t, s) g(s, x(\eta(s))) ds \right)$$

for proving the existence results in Banach algebras under generalized Lipschitz and Carathéodory conditions.

### 1. Introduction

The topological fixed point theorems such as the Schauder fixed point principle, the Leray-Schauder nonlinear alternative and the topological transversality principle, etc., are useful in the study of nonlinear differential and integral equations for proving the existence theorems under certain compactness type conditions. An exhaustive account of this subject appears in Deimling [3], Dugundji and Granas [11], Zeidler [16] and the references therein. The existence theorems for nonlinear integral equations of mixed type are generally obtained by using the hybrid fixed point theorems of Krasnoselski [13] and Dhage [4], [5]. It has also been proved that the Leray-Schauder type hybrid fixed point theorems are also very much useful in the study of nonlinear integral equations of mixed type. Recently Dhage and O'Regan [7] proved a Leray-Schauder type hybrid fixed point theorem in Banach algebras and it is further applied to a certain nonlinear integral equation for proving the existence theorems under Lipschitz and compactness conditions. In a recent paper [10], the authors proved a similar type of fixed point theorem in a Banach algebra involving two operators under some weaker conditions than Dhage and Regan [7] and proved an existence theorem for a certain nonlinear functional integral equation. Though the main fixed point theorem of Dhage et. al. [10] is correct, the proof contains some errors and the improvement of this result is one of the motivations for this paper. In this paper we shall prove a Leray-Schauder type hybrid fixed point theorem involving three operators in a Banach algebra under more general

---

Received March 23, 2004.

2000 Mathematics Subject Classification: 47H10.

Key words and phrases: Banach algebra, fixed point theorem and functional integral equation.

conditions than that of Dhage et. al. [10] and it will be further applied to a nonlinear integral equation of mixed type for proving the existence results under the mixed Carathéodory and Lipschitz conditions.

## 2. Preliminaries

Let  $X$  be a Banach space with norm  $\|\cdot\|$ . A mapping  $A : X \rightarrow X$  is called  **$\mathcal{D}$ -Lipschitz** if there exists a continuous nondecreasing function  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfying

$$(1) \quad \|Ax - Ay\| \leq \phi_A(\|x - y\|)$$

for all  $x, y \in X$  with  $\phi_A(0) = 0$ . Sometimes we call the function  $\phi$  a  **$\mathcal{D}$ -function** of  $A$  on  $X$ . In the special case when  $\phi_A(r) = \alpha r$ ,  $\alpha > 0$ ,  $A$  is called a Lipschitz with a Lipschitz constant  $\alpha$ . In particular if  $\alpha < 1$ ,  $A$  is called a **contraction** with a contraction constant  $\alpha$ . Further if  $\phi_A(r) < r$  for  $r > 0$ , then  $A$  is called a **nonlinear contraction** on  $X$ .

The following fixed point theorem due to Boyd and Wong [1] for the nonlinear contraction is well-known and is useful for proving the existence and the uniqueness theorems for the nonlinear differential and integral equations.

**Theorem 2.1.** *Let  $A : X \rightarrow X$  be a nonlinear contraction. Then  $A$  has a unique fixed point  $x^*$  and the sequence  $\{A^n x\}$  of successive iterations of  $A$  converges to  $x^*$  for each  $x \in X$ .*

An operator  $T : X \rightarrow X$  is called **compact** if  $\overline{T(S)}$  is a compact subset of  $X$  for any  $S \subset X$ . Similarly  $T : X \rightarrow X$  is called **totally bounded** if  $T$  maps a bounded subset of  $X$  into a relatively compact subset of  $X$ . Finally  $T : X \rightarrow X$  is called a **completely continuous** operator if it is a continuous and totally bounded operator on  $X$ . It is clear that every compact operator is totally bounded, but the converse may not be true. However, the two notions are equivalent on a bounded subset of a Banach space  $X$ .

The well-known Leray-Schauder nonlinear alternative concerning the compact operators is

**Theorem 2.2.** *Let  $K$  be a convex subset of a normed linear space  $E$ ,  $U$  an open subset of  $K$  with  $0 \in U$ , and  $N : \bar{U} \rightarrow K$  a continuous and compact map. Then either*

- (i) *the equation  $x = \lambda Tx$  has a solution for  $\lambda = 1$ , or*
- (ii) *there exists an element  $u \in \partial U$  such that  $u = \lambda Tu$ , for some  $0 < \lambda < 1$ , where  $\partial U$  is a boundary of  $U$ .*

Theorem 2.2 is extensively used in the theory of nonlinear differential equations for proving existence results. The method is commonly known as *a priori bound method* for the nonlinear equations. See for example, Dugundji and Granas [11],

Zeidler [16] and the references therein. Now we combine the above two Theorems 2.1 and 2.2 in Banach algebras in a different way from that given in Dhage and O'Regan [7] and Dhage et. al. [10]. See also Dhage [8].

**Theorem 2.3.** *Let  $U$  and  $\bar{U}$  be open bounded and closed bounded subsets in a Banach algebra  $X$  such that  $0 \in U$  and let  $A, C : X \rightarrow X$  and  $B : \bar{U} \rightarrow X$  be three operators satisfying*

- (a)  *$A$  and  $C$  are  $\mathcal{D}$ -Lipschitz with  $\mathcal{D}$ -functions  $\phi_A$  and  $\phi_C$  respectively,*
- (b)  *$B$  is completely continuous, and*
- (c)  *$M\phi_A(r) + \phi_C(r) < r$  for  $r > 0$ , where  $M = \|B(\bar{U})\| = \sup \{\|B(x)\| : x \in \bar{U}\}$ .*

*Then either*

- (i) *the equation  $\lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda}) = x$  has a solution for  $\lambda = 1$ , or*
- (ii) *there is an element  $u \in \partial U$  such that  $\lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda}) = u$  for some  $0 < \lambda < 1$ , where  $\partial U$  is the boundary of  $U$ .*

*Proof.* Let  $y \in \bar{U}$  be fixed and define the mapping  $A_y : X \rightarrow X$  by

$$(2) \quad A_y(x) = AxBy + Cx$$

for  $x \in X$ . Then for any  $x_1, x_2 \in X$ , we have

$$\begin{aligned} \|A_y(x_1) - A_y(x_2)\| &= \|Ax_1By - Ax_2By\| + \|Cx_1 - Cx_2\| \\ &\leq \|Ax_1 - Ax_2\| \|By\| + \|Cx_1 - Cx_2\| \\ &\leq M\phi_A(\|x_1 - x_2\|) + \phi_C(\|x_1 - x_2\|). \end{aligned}$$

This shows that  $A_y$  is a nonlinear contraction on  $X$  in view of the hypothesis (c). Therefore an application of Theorem 2.1 yields that  $A_y$  has a unique fixed point, say  $x^*$  in  $X$ . Define the mapping  $N : \bar{U} \rightarrow X$  by

$$(3) \quad Ny = z,$$

where  $z$  is the unique solution of the equation  $z = AzBy + Cz, z \in X$ . We show that  $N$  is continuous on  $\bar{U}$ . Let  $\{y_n\}$  be a sequence in  $\bar{U}$  converging to a point  $y$ .

Now

$$\begin{aligned} \|Ny_n - Ny\| &= \|AN(y_n)By_n - AN(y)By\| + \|C(Ny_n) - C(Ny)\| \\ &\leq \|AN(y_n)By_n - AN(y)By_n\| + \|AN(y)By_n - AN(y)By\| \\ &\quad + \|C(Ny_n) - C(Ny)\| \\ &\leq \|ANy_n - AN(y)\| \|By_n\| + \|AN(y)\| \|By_n - By\| \\ &\quad + \|C(Ny_n) - C(Ny)\| \\ &\leq M\phi_A(\|Ny_n - Ny\|) + \|ANy\| \|By_n - By\| \\ &\quad + \phi_C(\|Ny_n - Ny\|). \end{aligned}$$

Now using the fact that  $\phi_A$  and  $\phi_C$  are continuous and nondecreasing functions, we obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|Ny_n - Ny\| &\leq M\phi_A\left(\limsup_{n \rightarrow \infty} \|Ny_n - Ny\|\right) \\ &\quad + \|ANy\|\left(\limsup_{n \rightarrow \infty} \|By_n - By\|\right) \\ &\quad + \phi_C\left(\limsup_{n \rightarrow \infty} \|Ny_n - Ny\|\right). \end{aligned}$$

Now from hypothesis (c) it follows that

$$\lim_{n \rightarrow \infty} \|Ny_n - Ny\| = \limsup_{n \rightarrow \infty} \|Ny_n - Ny\| = 0.$$

This shows that  $N$  is continuous on  $X$ . Next we show that  $N$  is a compact operator on  $\bar{U}$ . Now for any  $z \in \bar{U}$  we have

$$\begin{aligned} \|Az\| &\leq \|A0\| + \|Az - A0\| \\ &\leq \|A0\| + \alpha\|z - 0\| \\ &\leq c \end{aligned}$$

where  $c = \|A0\| + \alpha \operatorname{diam}(\bar{U})$ .

Let  $\epsilon > 0$  be given. Since  $B$  is completely continuous,  $B(\bar{U})$  is totally bounded. Then there is a set  $Y = \{y_1, \dots, y_n\}$  in  $\bar{U}$  such that

$$B(\bar{U}) \subset \bigcup_{i=1}^n B_\delta(w_i),$$

where  $w_i = B(y_i)$  and  $\delta = \left(\frac{1 - (\alpha M + \beta)}{c}\right)\epsilon$ . Therefore for any  $y \in U$  we have a  $y_k \in Y$  such that

$$\|By - By_k\| < \left(\frac{1 - (\alpha M + \beta)}{c}\right)\epsilon.$$

Also we have

$$\begin{aligned} \|Ny - Ny_k\| &\leq \|AzBy - Az_kBy_k\| + \|Cz - Cz_k\| \\ &\leq \|AzBy - Az_kBy\| + \|Az_kBy - Az_kBz_k\| + \|Cz - Cz_k\| \\ &\leq \|Az - Az_k\|\|By\| + \|Az_k\|\|By_k - By\| \\ &\quad + \|Cz_k - Cz\| \\ &\leq (\alpha M + \beta)\|z - z_k\| + \|Az\|\|By_k - By\| \\ &\leq \frac{c}{1 - (\alpha M + \beta)}\|By - By_k\| \\ &< \epsilon. \end{aligned}$$

This is true for every  $y \in U$  and hence

$$N(\bar{U}) \subset \bigcup_{i=1}^n B_\epsilon(w_i),$$

where  $w_i = Ny_i$ . As a result  $N(\bar{U})$  is totally bounded. Since  $N$  is continuous, it is a compact operator on  $\bar{U}$ . Now an application of Theorem 2.2 implies that either

- (i) the equation  $\lambda Nx = x$  has a solution for  $\lambda = 1$ , or
- (ii) there is an element  $u \in \partial U$  such that  $\lambda Nu = u$  for some  $0 < \lambda < 1$ , where  $\partial U$  is a boundary of  $U$ .

Assume first that  $x \in \bar{U}$  is a fixed point of the operator  $N$ . Then by the definition of  $S$ ,

$$x = Nx = A(Nx)Bx + C(Nx) = Ax Bx + Cx,$$

and so the operator equation  $x = Ax Bx + Cx$  has a solution in  $U$ . Suppose next that there is an element  $u \in \partial U$  and a real number  $\lambda \in (0, 1)$  such that  $u = \lambda Nu$ . Then

$$\frac{u}{\lambda} = Nu = A(Nu)Bu + C(Nu) = A\left(\frac{u}{\lambda}\right)Bu + \lambda C\left(\frac{u}{\lambda}\right),$$

so that

$$u = \lambda A\left(\frac{u}{\lambda}\right)Bu + \lambda C\left(\frac{u}{\lambda}\right).$$

This completes the proof. □

As a consequence of Theorem 2.3 we obtain the following corollary in its applicable form to nonlinear equations in Banach algebras.

**Corollary 2.1.** *Let  $B_r(0)$  and  $\bar{B}_r(0)$  be open and closed balls in a Banach algebra  $X$  centered at the origin  $0$  and of radius  $r$ , for some real number  $r > 0$  and let  $A, C : X \rightarrow X$  and  $B : \bar{B}_r(0) \rightarrow X$  be three operators satisfying*

- (a)  *$A$  and  $C$  are Lipschitz with Lipschitz constants  $\alpha$  and  $\beta$  respectively,*
- (b)  *$B$  is continuous and compact, and*
- (c)  *$\alpha M + \beta < 1$ , where  $M = \|B(\bar{B}_r(0))\| = \sup \{\|B(x)\| : x \in \bar{B}_r(0)\}$ .*

Then either

- (i) *the equation  $\lambda A(\frac{x}{\lambda})Bx + \lambda C(\frac{x}{\lambda}) = x$  has a solution for  $\lambda = 1$ , or*
- (ii) *there is an element  $u \in X$  such that  $\|u\| = r$  satisfying  $\lambda A(\frac{u}{\lambda})Bu + \lambda C(\frac{u}{\lambda}) = u$ , for some  $0 < \lambda < 1$ .*

When  $C \equiv 0$  in Theorem 2.3 we get the following interesting generalization of a nonlinear alternative of Dhage and O'Regan [7] and Dhage [8] under weaker conditions which seems to have numerous applications in the theory of nonlinear

differential and integral equations.

**Theorem 2.4.** *Let  $U$  and  $\bar{U}$  denote respectively an open bounded and closed bounded subset in a Banach algebra  $X$  such that  $0 \in U$  and let  $A : X \rightarrow X$  and  $B : \bar{U} \in X$  be two operators satisfying*

- (a)  $A$  is  $\mathcal{D}$ -Lipschitz with a  $\mathcal{D}$ -function  $\phi_A$ ,
- (b)  $B$  is completely continuous, and
- (c)  $M\phi_A(r) < r$  for  $r > 0$  where  $M = \|B(\bar{U})\| = \sup \{\|B(x)\| : x \in \bar{U}\}$ .

Then either

- (i) the equation  $\lambda A(\frac{x}{\lambda})Bx = x$  has a solution for  $\lambda = 1$ , or
- (ii) there is an element  $u \in \partial U$  such that  $\lambda A(\frac{u}{\lambda})Bu = u$ , for some  $0 < \lambda < 1$ , where  $\partial U$  is a boundary of  $U$ .

When  $\phi_A(r) = \alpha r$ ,  $\alpha > 0$ , we obtain the following result due to Dhage et. al. [10] with correct proof.

**Corollary 2.2.** *Let  $B_r(0)$  and  $\bar{B}_r(0)$  be the open and closed balls in a Banach algebra  $X$  centered at the origin  $0$  and of radius  $r$ , for some real number  $r > 0$  and let  $A : X \rightarrow X$  and  $B : \bar{B}_r(0) \rightarrow X$  be two operators satisfying*

- (a)  $A$  is Lipschitz with Lipschitz constant  $\alpha$ ,
- (b)  $B$  is continuous and compact, and
- (c)  $\alpha M < 1$ , where  $M = \|B(\bar{B}_r(0))\| = \sup \{\|B(\bar{B}_r(0))\| : x \in \bar{B}_r(0)\}$ .

Then either

- (i) the equation  $\lambda A(\frac{x}{\lambda})Bx = x$  has a solution for  $\lambda = 1$ , or
- (ii) there is an element  $u \in X$  with  $\|u\| = r$  such that  $\lambda A(\frac{u}{\lambda})Bu = u$ , for some  $0 < \lambda < 1$ .

### 3. Functional integral equations

Let  $\mathbb{R}$  denote the real line. Given a closed and bounded interval  $J = [0, 1]$  in  $\mathbb{R}$ , consider the nonlinear functional integral equation (in short FIE) of mixed type

$$(4) \quad x(t) = k(t, x(\mu(t))) + [f(t, x(\theta(t)))] \left( q(t) + \int_0^{\sigma(t)} v(t, s) g(s, x(\eta(s))) ds \right)$$

for all  $t \in J$ , where  $\mu, \theta, \sigma, \eta : J \rightarrow J$ ,  $q : J \rightarrow \mathbb{R}$ ,  $v : J \times J \rightarrow \mathbb{R}$  and  $f, g, k : J \times \mathbb{R} \rightarrow \mathbb{R}$ .

A special case of FIE (4) is studied in Dhage [6] via an “**a priori bound method**” for the existence theorems and the special cases of the FIE (4) occur in

some phenomena of natural, physical and social sciences, see Chandrasekhar [2], Deimling [3] and the references therein. Some special cases of FIE (4) have been discussed in Dhage [5], [6] and Dhage and O'Regan [7] for existence results. In this section we shall prove the existence theorems for the FIE (4) by an application of the abstract fixed point theorem of the previous section under some suitable conditions different from Dhage [6].

Let  $M(J, \mathbb{R})$  and  $B(J, \mathbb{R})$  denote respectively the spaces of all measurable and bounded real-valued functions on  $J$ . We shall seek the solution of FIE (4) in the space  $BM(J, \mathbb{R})$  of bounded and measurable real-valued functions on  $J$ . Define a norm

$$(5) \quad \|x\|_{BM} = \max_{t \in J} |x(t)|.$$

Clearly  $BM(J, \mathbb{R})$  is a Banach algebra with respect to this maximum norm and the multiplication “ $\cdot$ ” defined by  $(x \cdot y)(t) = x(t)y(t)$ ,  $t \in J$ . Let  $L(J, \mathbb{R})$  denote the space of Lebesgue integrable real-valued functions on  $J$  with a norm  $\|\cdot\|_{L^1}$  defined by

$$(6) \quad \|x\|_{L^1} = \int_0^1 |x(t)| dt.$$

We need the following definition in the sequel.

**Definition 3.1.** A mapping  $\beta : J \times \mathbb{R} \rightarrow \mathbb{R}$  is said to satisfy a condition of  $L^1$ -Carathéodory or simply is called  $L^1$ -Carathéodory if

- (i)  $t \mapsto \beta(t, x)$  is measurable for each  $x \in \mathbb{R}$ ,
- (ii)  $x \mapsto \beta(t, x)$  is continuous almost everywhere for  $t \in J$ , and
- (iii) for each real number  $r > 0$  there exists a function  $h_r \in L^1(J, \mathbb{R})$  such that

$$|\beta(t, x)| \leq h_r(t) \text{ a.e. } t \in J$$

for all  $x \in \mathbb{R}$  with  $|x| \leq r$ .

We consider the following set of assumptions:

- ( $H_0$ ) The functions  $\mu, \theta, \sigma, \eta : J \rightarrow J$  are continuous.
- ( $H_1$ ) The function  $q : J \rightarrow \mathbb{R}$  is continuous with  $Q = \sup_{t \in J} |q(t)|$ .
- ( $H_2$ ) The function  $v : J \times J \rightarrow \mathbb{R}$  is continuous and  $V = \sup_{t, s \in J} |v(t, s)|$ .
- ( $H_3$ ) The function  $k : J \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there is a function  $\beta_1 \in B(J, \mathbb{R})$  with bound  $\|\beta_1\|$  such that

$$|k(t, x) - k(t, y)| \leq \beta_1(t)|x - y| \text{ a.e. } t \in J$$

for all  $x, y \in \mathbb{R}$ .

(H<sub>4</sub>) The function  $f : J \times \mathbb{R} \rightarrow \mathbb{R} - \{0\}$  is continuous and there is a function  $\alpha_1 \in B(J, \mathbb{R})$  with bound  $\|\alpha_1\|$  such that

$$|f(t, x) - f(t, y)| \leq \alpha_1(t)|x - y| \text{ a.e. } t \in J$$

for all  $x, y \in \mathbb{R}$ .

(H<sub>5</sub>) The function  $g$  is  $L^1$ -Carathéodory.

(H<sub>6</sub>) There exists a function  $\phi \in L^1(J, \mathbb{R})$  and a continuous and nondecreasing function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ - \{0\}$  such that

$$|g(t, x)| \leq \phi(t)\psi(|x|) \text{ a.e. } t \in J$$

for all  $x \in \mathbb{R}$ .

**Theorem 3.1.** *Assume that the hypotheses (H<sub>0</sub>) – (H<sub>6</sub>) hold. If there exists a real number  $r > 0$  such that*

$$(7) \quad \|\alpha_1\| (Q + V\|\phi\|_{L^1}\phi(r)) + \|\beta_1\| < 1$$

and

$$(8) \quad r > \frac{K + FV\|\phi\|_{L^1}\psi(r)}{1 - [\|\alpha_1\|(Q + V\|\phi\|_{L^1}\psi(r)) + \|\beta_1\|]}$$

where  $F = \sup\{|f(t, 0)| : t \in J\}$  and  $K = \sup\{|k(t, 0)| : t \in J\}$ , then the FIE (4) has a solution on  $J$ .

*Proof.* Consider the closed ball  $\overline{B}_r(0)$  centered at origin 0 and of radius  $r$ , where the real number  $r$  satisfies the inequalities (7) and (8). Define three operators  $A$ ,  $B$  and  $C$  on  $BM(J, \mathbb{R})$  by

$$\begin{aligned} Ax(t) &= f(t, x(\theta(t))), t \in J \\ Bx(t) &= q(t) + \int_0^{\sigma(t)} v(t, s)g(s, x(\eta(s))) ds, t \in J, \text{ and} \\ Cx(t) &= k(t, x(\mu(t))), t \in J. \end{aligned}$$

Consider the operator equation

$$(9) \quad \lambda A\left(\frac{x}{\lambda}\right)(t)Bx(t) + \lambda C\left(\frac{x}{\lambda}\right)(t) = x(t), t \in J.$$

Then the FIE (4) is equivalent to the operator equation (9) with  $\lambda = 1$ .

We shall show that the operators  $A, B$  and  $C$  satisfy all the conditions of Theorem 2.3 on  $BM(J, \mathbb{R})$ . Let  $x, y \in BM(J, \mathbb{R})$ . Then by (H<sub>4</sub>),

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t, x(\theta(t))) - f(t, y(\theta(t)))| \\ &\leq \alpha_1(t)|x(\theta(t)) - y(\theta(t))| \\ &\leq \|\alpha_1\| \|x - y\|_{BM}. \end{aligned}$$

Taking the maximum over  $t$ ,

$$\|Ax - Ay\|_{BM} \leq \|\alpha_1\| \|x - y\|_{BM}.$$

This shows that  $A$  is Lipschitz with a Lipschitz constant  $\|\alpha_1\|$ . Similarly it is shown that  $C$  is a Lipschitz with a Lipschitz constant  $\|\beta_1\|$ . Next we shall show that the operator  $B$  is continuous and compact on  $\overline{B}_r(0)$ . Since  $g(t, x)$  is  $L^1_X$ -Carathéodory, by using the dominated convergence theorem (see Granas *et al* [12]), it can be shown that  $B$  is continuous on  $BM(J, \mathbb{R})$ . Let  $\{x_n\}$  be a sequence in  $\overline{B}_r(0)$ . Then we have  $\|x_n\| \leq r$  for each  $n \in \mathbb{N}$ . Then by  $(H_5)$ ,

$$\begin{aligned} |Bx_n(t)| &\leq |q(t)| + \left| \int_0^{\sigma(t)} |v(t, s)| |g(s, x(\eta(s)))| ds \right| \\ &\leq Q + \int_0^{\sigma(t)} h_r(s) ds \\ &\leq Q + V \|h_r\|_{L^1} \end{aligned}$$

This further, by taking supremum over  $t$ , yields that  $\|Bx_n\| \leq Q + V \|h_r\|_{L^1}$  for each  $n \in \mathbb{N}$ . As a result  $\{Bx_n : n \in \mathbb{N}\}$  is a uniformly bounded set in  $B(\overline{B}_r(0))$ . Let  $t, \tau \in J$ . Then by the definition of  $B$ , we obtain

$$\begin{aligned} |Bx_n(t) - Bx_n(\tau)| &\leq |q(t) - q(\tau)| \\ &\quad + \left| \int_0^{\sigma(t)} v(t, s)g(s, x(\eta(s)))ds - \int_0^{\sigma(\tau)} v(t, s)g(s, x(\eta(s)))ds \right| \\ &\leq |q(t) - q(\tau)| \\ &\quad + \left| \int_0^{\sigma(t)} v(t, s)g(s, x(\eta(s))) ds - \int_0^{\sigma(t)} v(\tau, s)g(s, x(\eta(s))) ds \right| \\ &\quad + \left| \int_0^{\sigma(t)} v(\tau, s)g(s, x(\eta(s))) ds - \int_0^{\sigma(\tau)} v(\tau, s)g(s, x(\eta(s))) ds \right| \\ &\leq |q(t) - q(\tau)| + \int_0^{\sigma(t)} |v(t, s) - v(\tau, s)| |g(s, x(\eta(s)))| ds \\ &\quad + \left| \int_{\sigma(\tau)}^{\sigma(t)} |v(\tau, s)| h_r(s) ds \right| \\ &\leq |q(t) - q(\tau)| + \int_0^{\sigma(t)} |v(t, s) - v(\tau, s)| h_r(s) ds + |p(t) - p(\tau)| \end{aligned}$$

where  $p(t) = V \int_0^{\sigma(t)} h_r(s) ds$ . Since  $q, p$  and  $k_s(t) = k(t, s)$  are continuous on  $J$ , they are uniformly continuous and consequently

$$|Bx_n(t) - Bx_n(\tau)| \rightarrow 0 \text{ as } t \rightarrow \tau.$$

Thus  $\{Bx_n : n \in \mathbb{N}\}$  is an equi-continuous set in  $B(\overline{B}_r(0))$ . Hence  $B(\overline{B}_r(0))$  is compact by Arzelà-Ascoli theorem for compactness. Thus  $B$  is a continuous and compact operator on  $B(\overline{B}_r(0))$ . Finally, we have

$$\begin{aligned} M\phi_A(r) + \phi_C(r) &= \|B(\overline{B}_r(0))\|\phi_A(r) + \phi_C(r) \\ &\leq [\|\alpha_1\|(Q + V\|\phi\|_{L^1}\psi(r)) + \|\beta_1\|]r \\ &< r \end{aligned}$$

for all  $r > 0$ , because  $\|\alpha_1\|(Q + V\|\phi\|_{L^1}\psi(r)) + \|\beta_1\| < 1$ .

Thus all the conditions of Theorem 2.3 are satisfied and hence an application of it yields that either the conclusion (i) or the conclusion (ii) holds. We shall show that the conclusion (ii) is not possible. Assume the contrary. Then there is an  $u \in BM(J, \mathbb{R})$  with  $\|u\| = r$  satisfying (9). Therefore, we have for any  $0 < \lambda < 1$ ,

$$\begin{aligned} |u(t)| &\leq \lambda \left| k\left(t, \frac{u(\mu(t))}{\lambda}\right) \right| + \left| \lambda f\left(t, \frac{u(\theta(t))}{\lambda}\right) \right| \\ &\quad \times \left[ |q(t)| + \int_0^{\sigma(t)} |v(t, s)| |g(s, u(\eta(t)))| ds \right] \\ &\leq \lambda \left[ \left| k\left(t, \frac{u(\mu(t))}{\lambda}\right) - k(t, 0) \right| + |k(t, 0)| \right] \\ &\quad + \lambda \left[ \left| f\left(t, \frac{u(\theta(t))}{\lambda}\right) - f(t, 0) \right| + |f(t, 0)| \right] \left[ Q + V \int_0^{\sigma(t)} |g(s, u(\eta(t)))| ds \right] \\ &\leq \|\beta_1\| \|u(\mu(t))\| + K + [\|\alpha_1\| \|u(\theta(t))\| + F] \left[ Q + V \int_0^{\sigma(t)} |g(s, u(\eta(t)))| ds \right] \\ &\leq \|\beta_1\| \|u(\mu(t))\| + K + \|\alpha_1\| (Q + V\|\phi\|_{L^1}\psi(|u(\theta(t))|)) \\ &\quad + FQ + FV \int_0^{\sigma(t)} \phi(s)\psi(|u(\eta(s))|) ds \\ &\leq \|\beta_1\| \|u\|_{BM} + K + \|\alpha_1\| (Q + V\|\phi\|_{L^1}\psi(\|u\|_{BM})) \\ &\quad + FQ + FV\|\phi\|_{L^1}\psi(\|u\|_{BM}). \end{aligned}$$

Taking the supremum over  $t$ ,

$$(10) \quad \|u\|_{BM} \leq \|\beta_1\| \|u\|_{BM} + K + \|\alpha_1\| (Q + V\|\phi\|_{L^1}\psi(\|u\|_{BM})) + FQ + FV\|\phi\|_{L^1}\psi(\|u\|_{BM}).$$

Substituting  $\|u\|_{BM} = r$  in the above inequality (10) yields

$$r \leq \frac{K + FQ + FV\|\phi\|_{L^1}\psi(r)}{1 - [\|\beta_1\| + \|\alpha_1\|(Q + V\|\phi\|_{L^1}\psi(r))]}$$

which is a contradiction to (8). Hence the conclusion (i) holds. As a result the operator equation (9) has a solution in  $B_r(0)$  for  $\lambda = 1$  and therefore FIE (3) has a solution on  $J$ . This completes the proof.  $\square$

As an application, we consider the following initial value problem of functional differential equation (in short FDE)

$$(11) \quad \left( \frac{x(t) - k(t, x(\mu(t)))}{f(t, x(\theta(t)))} \right)' = g(s, x(\eta(t))) \text{ a.e. } t \in J,$$

$$(12) \quad x(0) = x_0 \in \mathbb{R}$$

where  $f : J \times \mathbb{R} \rightarrow \mathbb{R} \setminus \{0\}$  is continuous,  $g, k : J \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\mu, \theta, \eta : J \rightarrow J$  are continuous with  $\theta(0) = 0 = \mu(0)$ .

By a solution of FDE (11)-(12) we mean a function  $x \in AC(J, \mathbb{R})$  that satisfies the equations (11)-(12) on  $J$ , where  $AC(J, \mathbb{R})$  is the space of all absolutely continuous real-valued functions on  $J$ .

**Theorem 3.2.** *Assume that the hypotheses  $(H_3)$ - $(H_6)$  hold. Further suppose that there exists a real number  $r > 0$  such that*

$$(13) \quad r > \frac{K + F \left( \left| \frac{x_0 - k(0, x_0)}{f(0, x_0)} \right| + \|\phi\|_{L^1} \psi(r) \right)}{1 - \left[ \|\alpha_1\| \left( \left| \frac{x_0 - k(0, x_0)}{f(0, x_0)} \right| + \|\phi\|_{L^1} \psi(r) \right) + \|\beta_1\| \right]}$$

where

$$(14) \quad \|\alpha_1\| \left( \left| \frac{x_0 - k(0, x_0)}{f(0, x_0)} \right| + \|\phi\|_{L^1} \psi(r) \right) + \|\beta_1\| < 1.$$

Then FDE (11)-(12) has a solution on  $J$ .

*Proof.* Set  $X = C(J, \mathbb{R})$ , where  $C(J, \mathbb{R})$  is the space of continuous real-valued functions on  $J$ . Clearly  $C(J, \mathbb{R})$  is a Banach algebra with respect to the norm and multiplication given in  $BM(J, \mathbb{R})$ . The FDE (11)-(12) is equivalent to the integral equation

$$(15) \quad \begin{aligned} x(t) &= k(t, x(\mu(t))) + [f(t, x(\theta(t)))] \\ &\times \left( \frac{x_0 - k(0, x_0)}{f(0, x_0)} + \int_0^t g(s, x(\eta(s))) ds \right), \quad t \in J. \end{aligned}$$

Now the desired conclusion follows by an application of Theorem 3.2 with  $BM(J, \mathbb{R}) = C(J, \mathbb{R})$ ,  $Q = \left| \frac{x_0 - k(0, x_0)}{f(0, x_0)} \right|$  and  $v(t, s) = 1$  for all  $t, s \in J$ , since  $C(J, \mathbb{R}) \subset AC(J, \mathbb{R})$ . □

## References

- [1] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**(1969), 456-444.

- [2] S. Chandrasekhar, Radiative Heat Transfer, Dover New York, 1960.
- [3] K. Deimling, Nonlinear Functional Analysis, Springer Verlag, 1985.
- [4] B. C. Dhage, A fixed point theorem and applications to nonlinear integral equations, Proc. Int. Symp. Nonlinear Anal. Appl. Bio-Math., Waltair, India (1987), 53-59.
- [5] B. C. Dhage, *On existence theorems for nonlinear integral equations in Banach algebras via fixed point technique*, East Asian Math. J., **17**(2001), 33-45.
- [6] B. C. Dhage, *On a fixed point theorem of Krasnoselskii-Shaefer type*, EJQTDE, **6**(2002), 1-9.
- [7] B. C. Dhage and D. O'Regan, *A fixed point theorem in Banach algebras with applications to functional integral equations*, Functional Diff. Equations, **7**(3-4)(2000), 259-267.
- [8] B. C. Dhage, *A functional integro-differential equation in Banach algebras*, Functional Diff. Equations, **11**(3-4)(2004), 321-332.
- [9] B. C. Dhage, *Some nonlinear alternatives in Banach algebras with applications I*, Nonlinear Studies, (accepted).
- [10] B. C. Dhage U. P. Dolhare and S. K. Ntouyas, *Existence theorems for nonlinear first order functional differential equations in Banach algebras*, Comm. Appl. Nonlinear Anal., **10**(4)(2003), 59-69.
- [11] J. Dugundji and A. Granas, Fixed Point Theory, Monographie Math., Warsaw, 1982.
- [12] A. Granas, R. B. Guenther and J. W. Lee, *Some general existence principles for Carathéodory theory of nonlinear differential equations*, J. Math. Pures et Appl., **70**(1991), 153-196.
- [13] M. A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, Pergamon Press, 1964.
- [14] L. A. Lusternik and V. J. Sobolev, Elements of Functional Analysis, Hindusthan Publ. Corpn Delhi, 1985.
- [15] D. R. Smart, Fixed Point Theorems, Cambridge Univ. Press, 1980.
- [16] E. Zeidler, Nonlinear Functional Analysis : Part I, Springer Verlag, 1985.