Honam Mathematical J. **27** (2005), No. 4, pp. 609–619

THE EXISTENCE OF M SOLUTIONS OF THE NONLINEAR ELLIPTIC EQUATION; USING THE VARIATIONAL METHOD

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Abstract. We are concerned with the multiplicity of solutions of the nonlinear elliptic equation with Dirichlet boundary condition. We reveal the existence of m solutions of the nonlinear elliptic equation by a critical point theory, under some condition.

1. INTRODUCTION

In this paper we are concerned with the multiple solutions of the nonlinear elliptic equation with Dirichlet boundary condition

$$-\Delta u = g(u) \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0, \qquad \text{on } \partial\Omega.$$

Here Ω be a smooth bounded region in R^n with smooth boundary $\partial\Omega$ and $\lambda_1 < \lambda_2 \le \ldots \le \lambda_k \le \ldots$ be the eigenvalues of $-\Delta$ with Dirichlet boundary condition in Ω. Also we assumed that $g : R \to R$ be a differentiable function such that $g(0) = 0$, and

$$g'(\infty) = \lim_{|u|\to\infty} \frac{g(u)}{u} \in R.$$

Received November 10, 2005. Revised December 22, 2005.

2000 Mathematics Subject Classification : 34C15, 34C25, 35Q72.

Key words and phrases : Elliptic equation, critical point theory, multiple solutions.

* This work was financially supported by the Kunsan National University's Academic Program in the year 2005.

This type problem was studied by several authors. Castro and Lazer in [3] showed that if the interval $(g'(0), g'(\infty)) \cup (g'(\infty), g'(0))$ contains the eigenvalues $\lambda_k, \ldots, \lambda_j$ and $g'(t) < \lambda_{j+1}$ for all $t \in R$, then (1.1) has at least three solutions. The proofs in [3] are based on global Lyapunov-Schmidt arguments applied to variational problems. Castro and Cossio in [4] proved that problem (1.1) has at least five solutions if g is a differentiable function such that $g(0) = 0$, $g'(0) < \lambda_1$, $g'(\infty) \in (\lambda_k, \lambda_{k+1})$ with $k \geq 2$, and $g'(t) \leq \gamma < \lambda_{k+1}$. They proved this by using Lyapunov-Schmidt reduction arguments, the mountain pass lemma, and characterizations of the local degree of critical points. Chang in [5] also approached the same problems using Morse theory, and Hofer in [10] obtained the existence of five solutions when $g'(\infty) < \lambda_1$. For other results in the study of this problem we refer the reader to [12], [13], among others.

In section 2 we recall a critical point theory which will play a crucial role in our argument. In section 3 we define an invariant subspace X which can be applied in the critical point theory. In section 4 we prove the main results of this paper.

2. CRITICAL POINT THEORY

Let H be a real Hilbert space, and let Z_2 act on H orthogonally. Let Fix_{Z_2} be the set of fixed points of the action, i.e.,

$$Fix_{Z_2} = \{u \in H |\ \sigma \cdot u = u, \qquad \forall \sigma \in Z_2\}.$$

A set $A \subset H$ is called Z_2-invariant, if $\sigma \cdot u \in A$, $\forall u \in A$, $\forall \sigma \in Z_2$. A function $I : H \to R^1$ is called Z_2-invariant, if $I(\sigma \cdot u) = I(u)$, $\forall u \in H, \forall \sigma \in Z_2$. Let $C(B, D)$ be the set of continuous functions from B into D. If B is an invariant set we say $h \in C(B, D)$ is an equivariant map if $h(\sigma \cdot u) = \sigma \cdot h(u)$, $\forall u \in B$, $\forall \sigma \in Z_2$. Let S_r be the sphere centered at the origin of radius r. Let X be a Z_2-invariant subspace of H and

$I : X \to R$ be a functional of the form

$$I(u) = \frac{1}{2}L(u)u - \psi(u),$$

where $L : X \to X$ is linear, continuous, symmetric and equivariant, $\psi : X \to R$ is of class C^1 and invariant and $D\psi : X \to X$ is compact. The following result follows from [5].

Theorem 2.1. Assume that $I \in C^1(X, R)$ is Z_2-invariant and there exist two closed invariant linear subspaces V, W of X and $r > 0$ with the following properties:
(a) $V + W$ is closed and of finite codimension in X;
(b) $Fix_{Z_2} \subseteq V + W$;
(c) $L(W) \subseteq W$;
(d) $\sup_{S_r \cap V} I < +\infty$ and $\inf_W I > -\infty$;
(e) $u \notin Fix_{Z_2}$ whenever $DI(u) = 0$ and

$$\inf_W I \le I(u) \le \sup_{S_r \cap V} I.$$

(f) I satisfies $(PS)_c$ whenever $\inf_W I \le c \le \sup_{S_r \cap V} I$.
Then I possesses at least

$$\frac{1}{2}(dim(V \cap W) - codim_X(V + W))$$

distinct critical orbits in $I^{-1}([\inf_W I, \sup_{S_r \cap V} I])$.

3. INVARIANT SPACE

For each positive integer k let ϕ_k denote an eigenfunction corresponding to the eigenvalue λ_k. Let H be the Sobolev space $H_0^1(\Omega)$ which is the completion of the inner product space consisting of real C^1 functions having support contained in Ω with inner product

$$(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) dx,$$

and a norm

$$\|u\| = [\int_\Omega |\nabla u(x)|^2 dx]^{\frac{1}{2}}.$$

As it is well known, the set $\{\phi_k\}$ can be assumed to be complete and orthonormal in H. Let Z_2 act on H orthogonally. Then H has two invariant orthogonal subspaces Fix_{Z_2} and $Fix_{Z_2}^{\perp}$. Let us set

$$X = Fix_{X_2}^{\perp}.$$

The Z_2 action has the representation $u \mapsto -u$, $\forall u \in X$. Thus Z_2 acts freely on the invariant subspace X. We note that X is a closed invariant linear subspace of H compactly embedded in $L^2(\Omega)$. Moreover $(-\Delta)(X) \subseteq X$, $-\Delta : X \to X$ is an isomorphism. We need the following some properties. Since $\lambda_k \to +\infty$, we have:

Proposition 3.1. (i) $(-\Delta)u \in X$ implies $u \in X$.
(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$.

Proposition 3.2. Assume that $g : X \to X$ satisfies the assumptions of Theorem 1.1. Then all solutions in $L^2(\Omega)$ of

$$-\Delta u = g(u) \qquad \text{in } L^2(\Omega)$$

belong to X.

Proof. Let $g(u) = \sum h_k \phi_k \in L^2(\Omega)$. Then

$$(-\Delta)^{-1}(g(u)) = \sum \frac{1}{\lambda_k} h_k \phi_k.$$

Hence we have

$$\|(-\Delta)^{-1} g(u)\|^2 = \sum \lambda_k \frac{1}{\lambda_k^2} h_k^2 \leq C \sum h_k^2$$

for some $C > 0$, which means that

$$\|(-\Delta)^{-1} g(u)\| \leq C_1 \|u\|_{L^2(\Omega)}.$$

□

With the aid of Proposition 3.2 it is enough that we investigate the existence of solutions of (1.1) in the subspace X of $L^2(\Omega)$. Let $I : X \to R$ be the functional defined by,

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - G(u), \tag{3.1}$$

where $G(s) = \int_0^s g(\sigma)d\sigma$. Under the assumptions of Theorem 1.1, $I(u)$ is well defined. By the following Proposition, I is of class C^1 and the weak solutions of (1.1) coincide with the critical points of $I(u)$.

Proposition 3.3. Assume that $g(u)$ satisfies the assumptions of Theorem 1.1. Then $I(u)$ is continuous and Frèchet differentiable in X and

$$DI(u)(h) = \int_\Omega \nabla u \cdot \nabla h - g(u)h \tag{3.2}$$

for $h \in X$. Moeover $\int_\Omega G(u)dx$ is C^1 with respect to u. Thus $I \in C^1$.

It is easily checked that $-\Delta$ and g are equivariant, so I is invariant. Moreover $(-\Delta)(X) \subseteq X$, $-\Delta : X \to X$ is an isomorphism and $DI(X) \subseteq X$. Therefore critical points on X are critical points on H.

4. MAIN RESULTS

The main results of this paper are the followings.

Theorem A. Assume that $\lambda_k < g'(\infty) < \lambda_{k+1}$, $\lambda_{k+m} < g'(0) < \lambda_{k+m+1}$ for $k \geq 0$,$m \geq 1$, and $g'(t) \leq \gamma < \lambda_{k+m+1}$. Then problem (1.1) has at least m nontrivial solutions.

Theorem B. Assume that $\lambda_k < g'(0) < \lambda_{k+1}$, $\lambda_{k+m} < g'(\infty) < \lambda_{k+m+1}$ for $k \geq 0$, $m \geq 1$, and $g'(t) \leq \gamma < \lambda_{k+m+1}$, Then problem (1.1) has at least m nontrivial solutions.

First, we consider the case $\lambda_k < g'(\infty) < \lambda_{k+1}$, $\lambda_{k+m} < g'(0) < \lambda_{k+m+1}$ for $k \geq 0$, $m \geq 1$, and $g'(t) \leq \gamma < \lambda_{k+m+1}$. Let X_k be the subspace of X spanned by $\phi_1, \dots, \phi_k$ whose eigenvalues are $\lambda_1, \dots, \lambda_k$. Let $X_k^\perp$ be the orthogonal complement of X_k in X. Let $r = \frac{\lambda_k+\lambda_{k+1}}{2}$ and let $L : X \to X$ be the linear continuous operator such that

$$(Lu, v) = \int_\Omega (-\Delta u) \cdot v dx - r \int_\Omega uv dx.$$

Then L is symmetric, bijective and equivariant. The spaces X_k, $X_k^\perp$ are the negative space of L and the positive space of L. Moreover, there exists $\nu > 0$ such that

$$\forall u \in X_k \quad : \qquad (Lu, u) \leq (\lambda_k - r) \int_\Omega u^2 dx \leq -\nu \|u\|^2,$$

$$\forall u \in X_k^\perp \quad : \qquad (Lu, u) \geq (\lambda_{k+1} - r)) \int_\Omega u^2 dx \geq \nu \|u\|^2.$$

We can write

$$I(u) = \frac{1}{2}(Lu, u) - \psi(u),$$

where

$$\psi(u) = \int_\Omega [G(u) - \frac{1}{2} r u^2] dx.$$

Since X is compactly embedded in L^2, the map $D\psi : X \to X$ is compact.

Lemma 4.1. Under the same assumptions of Theorem A, $I(u)$ satisfies the $(P.S.)_M$ condition for any $M \in R$.

Lemma 4.2. Under the same assumptions of Theorem A, the function $I(u)$ is bounded from above on X_k;

$$\sup_{u \in X_k} I(u) < \infty \tag{4.1}$$

and from below on $X_k^\perp$;

$$\inf_{u \in X_k^\perp} I(u) > -\infty. \tag{4.2}$$

Proof. For some constant $d \geq 0$, we have $G_r(s) \geq \frac{1}{2}\alpha s^2 - d$, where $G_r(s) = \int_0^s g_r(\sigma)d\sigma$. For $u \in X_k$,

$$(Lu, u) \leq (\lambda_k - r)\int_\Omega u^2 dx = \frac{\lambda_k - \lambda_{k+1}}{2}\int_\Omega u^2,$$

$$\int_\Omega G_r(u) \geq \frac{\alpha}{2}\int_\Omega u^2 - d|\Omega|,$$

so that

$$I(u) \leq \frac{1}{2} \cdot \frac{\lambda_k - \lambda_{k+1}}{2}\int_\Omega u^2 - \frac{\alpha}{2}\int_\Omega u^2 + d|\Omega| < d|\Omega|,$$

since $\frac{\lambda_k - \lambda_{k+1}}{2} < \alpha$. Thus $\sup_{u \in X_k} I(u) < \infty$. Thus the functional I is bounded from above on X_k. Next we will prove that (4.2) holds. For some constant $\bar{d} \geq 0$, we have $G_r(s) \geq \frac{1}{2}\beta s^2 + \bar{d}$. For $u \in X_k^\perp$,

$$(Lu, u) \geq (\lambda_{k+1} - r)\int_\Omega u^2 = \frac{\lambda_{k+1} - \lambda_k}{2}$$

and

$$\int_{Omega} G_r(u) \leq \frac{\beta}{2}\int_\Omega u^2 + \bar{d}|\Omega|,$$

so that

$$I(u) \geq \frac{1}{2} \cdot \frac{\lambda_{k+1} - \lambda_k}{2}\int_\Omega u^2 - \frac{\beta}{2}\int_\Omega u^2 - \bar{d}|\Omega| > -\bar{d}|\Omega|,$$

since $\beta < \frac{\lambda_{k+1} - \lambda_k}{2}$. Thus $\inf_{u \in X_k^\perp} I(u) > -\infty$. □

Lemma 4.3. Let $G_0 : R \to R$ be a continuous function such that

$$\inf_{s \in R} \frac{G_0(s)}{1 + s^2} > -\infty, \qquad \lim_{s \to 0} \frac{G_0(s)}{s^2} \geq 0.$$

Then

$$\lim_{\substack{u \to 0 \\ u \in X}} \frac{1}{\|u\|^2}\int_\Omega G_0(u)dx \geq 0.$$

Proof. Let

$$h(s) = \begin{cases} (\frac{G_o(s)}{s^2})^- & \text{if } s \neq 0, \\ 0 & \text{if } s = 0. \end{cases}$$

Then $h : R \to R$ is bounded, continuous, with $h(0) = 0$ and $G_0(s) \geq -h(s)s^2$. If (u_n) is a sequence in H with $u_n \to 0$, then up to a subsequence, $u_n \to 0$ a.e., and $v_n = \frac{u_n}{\|u_n\|}$ is strongly convergent in $L^2(\Omega)$. Since

$$\frac{1}{\|u_n\|^2}\int_\Omega G_0(u_n)dx \geq -\int_\Omega h(u_n)v_n^2dx,$$

the claim follows. □

Lemma 4.4. Under the same assumptions of Theorem A, there exists a ball S_ρ with radius $\rho > 0$ centered at 0 such that

$$\sup_{u\in S_\rho\cap X_{k+m}} I(u) < 0.$$

It is easy to prove the lemma.

Lemma 4.5. Assume that the assumptions of Theorem A are satisfied. Let $u \notin Fix_{Z_2}$ be a critical point of I. Then $\inf_{u\in X_k^\perp} I(u) \leq I(u) \leq \sup_{u\in S_\rho\cap X_{k+m}} I(u)$.

Proof. Since the domain of the functional I is the space X and $Fix_{Z_2} \cap X = \emptyset$, $Fix_{Z_2} \cap I^{-1}[\inf_{u\in X_k^\perp} I(u), \sup_{u\in S_\rho\cap X_{k+m}}] = \emptyset$. Let $u \in X = Fix_{Z_2}^\perp$ be a critical point of I. Then by the assumptions of Theorem A, if $u \in X_k \cup X_{k+m}^\perp$, then $I(u) = 0$ and u can not be any critical point of I. Thus $u \in X_k^\perp \cap X_{k+m}$, and

$$DI(u)u = \int_\Omega [(-\Delta u)\cdot u - g(u)u]dx = 0.$$

Then we have

$$\begin{aligned} I(u) &= \int_\Omega [(-\Delta u)\cdot u - \int_\Omega G(u)]dx \\ &= \int_\Omega \frac{1}{2}[g(u)u - G(u)]dx. \end{aligned}$$

For $u \in X_k^{\perp}$,

$$\begin{aligned} I(u) &= \int_\Omega [\frac{1}{2}g(u)u - \frac{1}{2}ru^2]dx - \int_\Omega [G(u) - \frac{1}{2}ru^2]dx \\ &\geq \int_\Omega [\frac{1}{2}\lambda_{k+1}u^2 - \frac{1}{2}ru^2 - \frac{1}{2}\beta u^2]dx - \bar{d}|\Omega| \\ &\quad -\bar{d}|\Omega|. \end{aligned}$$

Thus

$$I(u) \geq \inf_{u \in X_k^{\perp}} I(u) > -\bar{d}|\Omega|.$$

For $u \in X_{k+m}$, by Lemma 4.3, there exists $\rho > 0$ such that

$$\begin{aligned} I(u) &= \int_\Omega [\frac{1}{2}g(u)u - \frac{1}{2}g'(0)u^2]dx - \int_\Omega [G(u) - \frac{1}{2}ru^2]dx \\ &\leq \int_\Omega [\frac{1}{2}\lambda_{k+m}u^2 - \frac{1}{2}g'(0)u^2]dx - \int_\Omega [G(u) - \frac{1}{2}g'(0)u^2]dx \\ &< -\int_\Omega [G(u) - \frac{1}{2}g'(0)u^2]dx < 0 \text{for } u \in S_\rho. \end{aligned}$$

Thus we have

$$I(u) \leq \sup_{u \in S_\rho \cap X_{k+m}} I(u) < 0.$$

Thus for $u \in X_k^{\perp} \cap X_{k+m}$

$$\inf_{u \in X_k^{\perp}} I(u) \leq I(u) \leq \sup_{u \in S_\rho \cap X_{k+m}} I(u) < 0.$$

Thus we prove the lemma. □

Proof of Theorem A, B. Now we want to apply Theorem 2.1. If we set $V = X_{k+m}$ and $W = X_k^{\perp}$, then V and W are closed invariant subspaces of X. By Proposition 3.3, I is $C^1(X, R)$. By Lemma 4.2 and Lemma 4.4, assumption (d) of Theorem 2.1 is satisfied. By Lemma 4.5, assumption (e) of Theorem 2.1 is satisfied. By Lemma 4.1, assumption of (f) of Theorem 2.1 is satisfied. By Theorem 2.1 problem (1.1) has at least $\frac{1}{2}dim(V \cap W) = m$ nontrivial solutions. In case $\lambda_k < g'(0) < \lambda_{k+1}$, $\lambda_{k+m} < g'(\infty) < \lambda_{k+m+1}$ for $k > 0$, $m \geq 1$, and $g'(t) \leq \gamma < \lambda_{k+m+1}$, denote by $r = \frac{1}{2}[\lambda_{k+m} + \lambda_{k+m+1}]$. We introduce L, V and W as in the previous case. Then we can apply the same argument of the functional I and the conclusion follows also in this case.

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