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Triangular Normed Fuzzification of (Implicative) Filters in Lattice Implication Algebras

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Abstract. We obtain characterizations of T-fuzzy (implicative) filters and some properties of T-product of fuzzy (implicative) filters in lattice implication algebras. We also establish the extension property for T-fuzzy implicative filters.

1. Introduction

In the field of many-valued logic, lattice-valued logic plays an important role for two aspects: One is that it extends the chain-type truth-value field of some well-known presented logic [1] to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. Therefore Goguen [2], Pavelka [12] and Novak [11] researched

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on this lattice-valued logic formal systems. Moreover, in order to establish a logic system with truth value in a relatively general lattice, in 1990, during the study of the project "The Study of Abstract Fuzzy Logic" granted by National Natural Science Foundation in China, Xu established the lattice implication algebra by combining lattice and implication algebra, and investigated many useful structures [9], [10], [14], [15], [16]. Lattice implication algebra provided the foundation to establish the corresponding logic system from the algebraic viewpoint. For the general development of lattice implication algebras, the filter theory plays an important role. (see [16], [3], [4], [5], [7], [8].) Xu and Qin [17] introduced the notion of fuzzy (implicative) filters in lattice implication algebras.

In this paper, we discuss the triangular normed fuzzification of (implicative) filters in lattice implication algebras as a generalization of fuzzy (implicative) filters. We obtain characterizations of T-fuzzy (implicative) filters. We also establish an extension property for T-fuzzy implicative filters. Using a t-norm T, we define the T-product of fuzzy (implicative) filters of a lattice implication algebra, and investigate their properties. The notion of T-fuzzy (implicative) filters is a useful tool for studying further properties of lattice implication algebras. We can apply this notion to study the fuzzification of positive implicative filters, associative filters, and several kinds of ideals by using triangular norms.

2. Preliminaries

A *lattice implication algebra* [14] is defined to be a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution "$'$" and a binary operation "$\rightarrow$" satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(I2) $x \rightarrow x = 1$,

(I3) $x \rightarrow y = y' \rightarrow x'$,

(I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,

(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

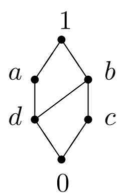
(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x,\ y,\ z \in L$

Example 2.1 ([6]). Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and

Cayley tables as follows:



x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Define $\vee$- and $\wedge$-operations on L as follows:

$$x \vee y := (x \rightarrow y) \rightarrow y, \;\; x \wedge y := ((x' \rightarrow y') \rightarrow y')',$$

for all $x, y \in L$. Then L is a lattice implication algebra.

In the sequel, the binary operation "$\rightarrow$" will be denoted by juxtaposition. We can define a partial ordering "$\leq$" on a lattice implication algebra L by $x \leq y$ if and only if $xy = 1$.

In a lattice implication algebra L, the following hold (see [14]):

(p1) $0x = 1$, $1x = x$ and $x1 = 1$.

(p2) $xy \leq (yz)(xz)$.

(p3) $x \leq y$ implies $yz \leq xz$ and $zx \leq zy$.

(p4) $x' = x0$.

(p5) $x \vee y = (xy)y$.

(p6) $((yx)y')' = x \wedge y = ((xy)x')'$.

(p7) $x \leq (xy)y$.

Generally, an aggregation operator is a mapping $F : I^n \rightarrow I$ $(n \geq 2)$, where $I = [0, 1]$. Essentially it takes a collection of arguments and provides an aggregated value. An important class of aggregation operators are the triangular norm operators, t-norm and t-conorm. These operators play a significant role in the theory of fuzzy subsets by generalizing the intersection (*and*) and union (*or*) operators, respectively.

By a *t-norm* T (see [13]) we mean a function $T : I \times I \rightarrow I$ satisfying the following conditions:

(T1) $T(x, 1) = x$,

(T2) $T(x, y) \leq T(x, z)$ whenever $y \leq z$,

(T3) $T(x, y) = T(y, x)$,

(T4) $T(x, T(y, z)) = T(T(x, y), z)$,

for all $x,\ y,\ z \in I$.

A few t-norms which are frequently encountered are "T_m", "$Prod$" and "min" defined by $T_m(x,y) = \max\{x+y-1, 0\}$, $Prod(x,y) = xy$ and

$$\min\{x,y\} = \begin{cases} x & \text{if } x \le y, \\ y & \text{if } y < x. \end{cases}$$

For a t-norm T, let Δ_T denote the set of elements $\alpha \in I$ such that $T(\alpha,\alpha) = \alpha$, that is,

$$\Delta_T := \{\alpha \in I \mid T(\alpha,\alpha) = \alpha\}.$$

Note that every t-norm T has a useful property:

(p8) $T(\alpha,\beta) \le \min\{\alpha,\beta\}$ for all $\alpha,\beta \in I$.

A t-norm T on I is said to be *continuous* if T is a continuous function from $I \times I$ to I with respect to the usual topology. A *fuzzy set* in a set L is a function $\mu : L \to [0,1]$. For $\alpha \in [0,1]$, the set $U(\mu;\alpha) := \{x \in L \mid \mu(x) \ge \alpha\}$ is called an *upper level subset* of μ. A fuzzy set μ in a set L is said to satisfy *imaginable property* if $\mathrm{Im}(\mu) \subseteq \Delta_T$.

A subset F of a lattice implication algebra L is called a *filter* [16] of L if it satisfies

(a1) $1 \in F$,

(a2) $(\forall x \in F)\,(\forall y \in L)\,(xy \in F \ \Rightarrow\ y \in F)$.

A subset F of a lattice implication algebra L is called an *implicative filter* [16] of L if it satisfies

(a1) $1 \in F$,

(a3) $(\forall x,y,z \in L)\,(x(yz) \in F,\ xy \in F \ \Rightarrow\ xz \in F)$.

Xu and Qin [17] considered the fuzzification of filters and implicative filters in lattice implication algebras.

A fuzzy set μ in a lattice implication algebra L is called a *fuzzy filter* [17] of L if it satisfies

(b1) $(\forall x \in L)\,(\mu(1) \ge \mu(x))$,

(b2) $(\forall x,y \in L)\,(\mu(y) \ge \min\{\mu(x),\ \mu(xy)\})$.

A fuzzy set μ in a lattice implication algebra L is called a *fuzzy implicative filter* [17] of L if it satisfies

(b1) $(\forall x \in L)\,(\mu(1) \ge \mu(x))$,

(b3) $(\forall x,y,z \in L)\,(\mu(xz) \ge \min\{\mu(x(yz)),\ \mu(xy)\})$.

Recall that every fuzzy implicative filter is a fuzzy filter, but the converse is not true in general.

3. Triangular normed fuzzy (implicative) filters

In what follows, let L and T denote a lattice implication algebra and a t-norm, respectively, unless otherwise specified.

We call a fuzzy set μ in L *upper* (resp. *lower*) if $\mu(x) \geq \frac{1}{2}$ (resp. $\mu(x) < \frac{1}{2}$) for all $x \in L$.

And a fuzzy set μ in a lattice implication algebra L is called an *upper fuzzy filter* of L with respect to a t-norm T, usually abbreviated to *upper T-fuzzy filter* of L, if it satisfies

(b1) $(\forall x \in L)\,(\mu(1) \geq \mu(x))$,

(c1) μ is upper,

(c2) $(\forall x, y \in L)\,(\mu(y) \geq T(\mu(x),\, \mu(xy)))$.

A fuzzy set μ in L is called a *fuzzy filter* of L with respect to a t-norm T, usually abbreviated to *T-fuzzy filter* of L, if it satisfies (b1) and (c2).

If we take $T = \min$, then a T-fuzzy filter of L is only a fuzzy filter of L.

Example 3.1. Let L be a lattice implication algebra in Example 2.1. Define a fuzzy set μ in L by $\mu(b) = \mu(c) = \mu(1) = 0.7$ and $\mu(0) = \mu(a) = \mu(d) = 0.07$. Then μ is a T_m-fuzzy filter of L which does not satisfy the imaginable property. On the other hand, a fuzzy set ν in L defined by

$$\nu(x) = \begin{cases} 1 & \text{if } x \in \{b, c, 1\}, \\ 0 & \text{otherwise}, \end{cases}$$

is an imaginable T_m-fuzzy filter of L.

Theorem 3.2. *Let F be a filter of L and let μ be a fuzzy set in L defined by*

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in F, \\ \beta & \text{otherwise}. \end{cases}$$

(i) *If $\alpha,\ \beta \in (0, 1)$ with $\alpha > \beta$, then μ is a T_m-fuzzy filter of L which is not imaginable.*

(ii) *If $\alpha = 1$ and $\beta = 0$, then μ is an imaginable T_m-fuzzy filter of L.*

Proof. (i) Let $x,\ y \in L$. If $y \in F$, then clearly

$$(\forall x \in L)\,(\mu(y) = \alpha \geq T_m(\mu(x),\, \mu(xy))).$$

Assume that $y \notin F$. Then $x \notin F$ or $xy \notin F$. Hence $\mu(x) = \beta$ or $\mu(xy) = \beta$, and so $\mu(y) \geq \beta \geq T_m(\mu(x),\, \mu(xy))$. Obviously, μ is not imaginable.

(ii) It is similar to the proof of (i) and clearly $\mathrm{Im}(\mu) \subseteq \Delta_{T_m}$. □

Theorem 3.3. *Every imaginable T-fuzzy filter is a fuzzy filter for every t-norm T.*

Proof. Let μ be an imaginable T-fuzzy filter of L. Then

$$(\forall x, y \in L)\,(\mu(y) \geq T(\mu(x),\, \mu(xy))).$$

Since μ is imaginable, it follows from (T2), (T3) and (p8) that

$$\begin{aligned} \min\{\mu(x),\, \mu(xy)\} &= T(\min\{\mu(x),\, \mu(xy)\},\, \min\{\mu(x),\, \mu(xy)\}) \\ &\leq T(\mu(x),\, \mu(xy)) \;\leq\; \min\{\mu(x),\, \mu(xy)\}. \end{aligned}$$

so that $\mu(y) \geq T(\mu(x),\, \mu(xy)) = \min\{\mu(x),\, \mu(xy)\}$. Hence μ is a fuzzy filter of L. □

Lemma 3.4. *If μ is a T-fuzzy filter of L satisfying the imaginable property, then μ is order preserving.*

Proof. Let $x,\ y \in L$ be such that $x \leq y$. Then $xy = 1$, and so

$$\mu(y) \geq T(\mu(xy),\, \mu(x)) = T(\mu(1),\, \mu(x)) \geq T(\mu(x),\, \mu(x)) = \mu(x).$$

This completes the proof. □

Theorem 3.5. *If μ is an imaginable T-fuzzy filter of L for every t-norm T, then*

(1) $$(\forall x, y, z \in L)\,(x(yz) = 1 \;\Rightarrow\; \mu(z) \geq T(\mu(x),\, \mu(y))).$$

Conversely, if μ satisfies the imaginable property and the condition (1)*, then μ is a T-fuzzy filter of L.*

Proof. Let $x,\ y,\ z \in L$ be such that $x(yz) = 1$. Then $x \leq yz$, and so $\mu(x) \leq \mu(yz)$ by Lemma 3.4. Using (c2), (T2) and (T3), we have

$$\mu(z) \geq T(\mu(y),\, \mu(yz)) \geq T(\mu(x),\, \mu(y)).$$

Now assume that μ satisfies the imaginable property and the condition (1). Since $x \leq x1$ for all $x \in L$, we have $\mu(1) \geq T(\mu(x),\, \mu(x)) = \mu(x)$ by (1). Since $x \leq (xy)y$ for all $x, y \in L$, it follows from (1) that $\mu(y) \geq T(\mu(x), \mu(xy))$. Hence μ is a T-fuzzy filter of L. □

A fuzzy set μ in L is called a *fuzzy implicative filter* of L with respect to a t-norm T, usually abbreviated to *T-fuzzy implicative filter* of L, if it satisfies

(b1) $(\forall x \in L)\,(\mu(1) \geq \mu(x))$,

(c3) $(\forall x, y, z \in L)\,(\mu(xz) \geq T(\mu(x(yz)),\, \mu(xy)))$.

Theorem 3.6. *Every T-fuzzy implicative filter is a T-fuzzy filter.*

Proof. Let μ be a T-fuzzy implicative filter of L. If we replace x by 1 in (c3) and use (p1), then

$$\mu(z) = \mu(1z) \geq T(\mu(1(yz)),\, \mu(1y)) = T(\mu(yz),\, \mu(y))$$

for all $y,\ z \in L$. Hence μ is a T-fuzzy filter of L. □

The following example shows that the converse of Theorem 3.6 is not true in general.

Example 3.7. Let $L := \{0, a, b, c, 1\}$. Define the partially ordered relation on L as $0 \leq a \leq b \leq c \leq 1$, and define

$$x \wedge y := \min\{x, y\},\ x \vee y := \max\{x, y\}$$

for all $x, y \in L$ and "$'$" and "$\rightarrow$" as follows:

x	x'
0	1
a	c
b	b
c	a
1	0

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Then $(L, \vee, \wedge, ', \rightarrow)$ is a lattice implication algebra (see Xu and Qin [16]). Let μ be a fuzzy set in L defined by $\mu(1) > \mu(x)$ for all $x \in L \setminus \{1\}$. Then μ is a T_m-fuzzy filter of L, but it is not a T_m-fuzzy implicative filter of L because

$$\begin{aligned}\mu(a0) = \mu(c) < 2\mu(1) &= \max\{2\mu(1), 0\}\\ &= T_m(\mu(1), \mu(1)) = T_m(\mu(a(b0)),\, \mu(ab)).\end{aligned}$$

Theorem 3.8. *If μ is an imaginable T-fuzzy filter of L satisfying the following inequality*:

$$(\forall x, y, z \in L)\, (\mu(yz) \geq T(\mu(x(y(yz))),\, \mu(x))), \tag{2}$$

then μ is an imaginable T-fuzzy implicative filter of L.

Proof. Note that $x(yz) = y(xz) \leq (xy)(x(xz))$ for all $x, y, z \in L$. Since μ is order preserving by Lemma 3.4, it follows from (2), (T2) and (T3) that

$$\mu(xz) \geq T(\mu((xy)(x(xz))),\, \mu(xy)) \geq T(\mu(x(yz)),\, \mu(xy)).$$

Hence μ is an imaginable T-fuzzy implicative filter of L. □

Theorem 3.9. *Let μ be a T-fuzzy filter of L that satisfies the imaginable property. Then the following are equivalent*:

(i) *μ is a T-fuzzy implicative filter of L.*

(ii) $(\forall x, y \in L)\,(\mu(xy) \geq \mu(x(xy)))$.

(iii) $(\forall x, y, z \in L)\,(\mu((xy)(xz)) \geq \mu(x(yz)))$.

Proof. (i) $\Rightarrow$ (ii) Assume that μ is a T-fuzzy implicative filter of L. Using (c3), (I2), (b1), (T2) and (T3), we have

$$\begin{aligned}\mu(xy) &\geq T(\mu(x(xy)),\, \mu(xx)) = T(\mu(x(xy)),\, \mu(1)) \\ &\geq T(\mu(x(xy)),\, \mu(x(xy))) = \mu(x(xy))\end{aligned}$$

for all $x,\ y \in L$. This proves (ii).

(ii) $\Rightarrow$ (iii) Suppose that (ii) is valid and let $x, y, z \in L$. Note that $x(yz) \leq x((xy)(xz))$ by (I1), (p2) and (p3). Since μ is order preserving by Lemma 3.4, it follows from (I1) and (ii) that

$$\begin{aligned}\mu((xy)(xz)) &= \mu(x((xy)z)) \geq \mu(x(x((xy)z))) \\ &= \mu(x((xy)(xz))) \geq \mu(x(yz)),\end{aligned}$$

which proves (iii).

(iii) $\Rightarrow$ (i) Assume that (iii) holds. Using (c2), (T2) and (T3), we get

$$\mu(xz) \geq T(\mu((xy)(xz)),\, \mu(xy)) \geq T(\mu(x(yz)),\, \mu(xy)).$$

This completes the proof. □

Theorem 3.10 *If μ is a T-fuzzy (implicative) filter of L, then $U(\mu; 1)$ is either empty or a (implicative) filter of L.*

Proof. Assume that $U(\mu; 1) \neq \emptyset$ and μ is a T-fuzzy (implicative) filter of L. Then there exists $x \in U(\mu; 1)$, and so $\mu(1) \geq \mu(x) = 1$, i.e., $1 \in U(\mu; 1)$. Let $x, y \in L$ be such that $x \in U(\mu; 1)$ and $xy \in U(\mu; 1)$. Then

$$\mu(y) \geq T(\mu(x),\, \mu(xy)) = T(1, 1) = 1,$$

and so $y \in U(\mu; 1)$. Now, let $x, y, z \in L$ be such that $x(yz) \in U(\mu; 1)$ and $xy \in U(\mu; 1)$. It follows from (c3) and (T1) that

$$\mu(xz) \geq T(\mu(x(yz)),\, \mu(xy)) = T(1, 1) = 1$$

so that $xz \in U(\mu; 1)$. Hence $U(\mu; 1)$ is a (implicative) filter of L. □

Lemma 3.11 ([17]). *A fuzzy set μ in L is a fuzzy (implicative) filter of L if and only if the nonempty level set $U(\mu; \alpha)$ of μ is a (implicative) filter of L.*

Theorem 3.12. *If μ is a fuzzy set in L whose nonempty level set $U(\mu; \alpha)$, $\alpha \in [0, 1]$, is a (implicative) filter of L, then μ is a T-fuzzy (implicative) filter of L.*

Proof. Assume that the nonempty level set $U(\mu;\alpha)$ of μ is an implicative filter of L. Then μ is a fuzzy implicative filter of L by Lemma 3.11. Thus

$$\mu(xz) \geq \min\{\mu(x(yz)),\, \mu(xy)\} \geq T(\mu(x(yz)),\, \mu(xy))$$

for all $x,\ y,\ z \in L$. Hence μ is a T-fuzzy implicative filter of L. □

Theorem 3.13 (Extension property for a T-fuzzy implicative filter)**.** *Let μ and ν be T-fuzzy filters of L that satisfy the imaginable property. Assume that $\mu(1) \geq \nu(1)$ and $\mu \leq \nu$, that is, $\mu(x) \leq \nu(x)$ for all $x \in L \setminus \{1\}$. If μ is a T-fuzzy implicative filter of L, then so is ν.*

Proof. Let $x,\ y,\ z \in L$. Using (I1) and Theorem 3.9, we have

$$\begin{aligned}&\nu((x(yz))((xy)(xz))) = \nu((xy)((x(yz))(xz)))\\ &\quad = \nu((xy)(x((x(yz))z))) \geq \mu((xy)(x((x(yz))z)))\\ &\quad \geq \mu(x(y((x(yz))z))) = \mu((x(yz))(x(yz))) = \mu(1) \geq \nu(1).\end{aligned}$$

It follows from (c2), (b1), (T2) and (T3) that

$$\begin{aligned}&\nu((xy)(xz)) \geq T(\nu(x(yz)),\, \nu((x(yz))((xy)(xz))))\\ &\quad \geq T(\nu(x(yz)),\, \nu(1)) \geq T(\nu(x(yz)),\, \nu(x(yz))) = \nu(x(yz))\end{aligned}$$

so from Theorem 3.9 that ν is a T-fuzzy implicative filter of L. □

Theorem 3.14. *Let $\{\mu_i \mid i \in \Lambda\}$ be a class of T-fuzzy implicative filters of L. Then $\bigcap_{i\in\Lambda} \mu_i$ is a T-fuzzy implicative filter of L where $\bigcap_{i\in\Lambda} \mu_i$ is defined by $\Big(\bigcap_{i\in\Lambda} \mu_i\Big)(x) = \inf_{i\in\Lambda} \mu_i(x)$ for all $x \in L$.*

Proof. For any $x \in L$, we have

$$\Big(\bigcap_{i\in\Lambda} \mu_i\Big)(1) = \inf_{i\in\Lambda} \mu_i(1) \geq \inf_{i\in\Lambda} \mu_i(x) = \Big(\bigcap_{i\in\Lambda} \mu_i\Big)(x).$$

Let $x,\ y,\ z \in L$. Then

$$\begin{aligned}\Big(\bigcap_{i\in\Lambda} \mu_i\Big)(xz) &= \inf_{i\in\Lambda} \mu_i(xz) \geq \inf_{i\in\Lambda} T(\mu_i(x(yz)),\, \mu_i(xy))\\ &\geq T(\inf_{i\in\Lambda} \mu_i(x(yz)),\, \inf_{i\in\Lambda} \mu_i(xy))\\ &= T(\Big(\bigcap_{i\in\Lambda} \mu_i\Big)(x(yz)),\, \Big(\bigcap_{i\in\Lambda} \mu_i\Big)(xy)).\end{aligned}$$

Hence $\bigcap_{i\in\Lambda} \mu_i$ is a T-fuzzy implicative filter of L. □

Let f be a mapping defined on L. If ν is a fuzzy set in $f(L)$, then the fuzzy set $\mu = \nu \circ f$ in G, i.e., the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in L$, is called the *preimage* of ν under f.

Theorem 3.15. *Let $f : L \to M$ be an onto homomorphism of lattice implication algebras, ν a T-fuzzy (implicative) filter of M and μ the preimage of ν under f. Then μ is a T-fuzzy (implicative) filter of L. Moreover, if ν satisfies the imaginable property then so does μ.*

Proof. For any $x \in L$, we get

$$\mu(x) = \nu(f(x)) \leq \nu(1) = \nu(f(1)) = \mu(1).$$

Let $x,\ y \in L$. Then $\mu(y) = \nu(f(y)) \geq T(\nu(a),\, \nu(af(y)))$ for any $a \in M$. Let x_a be an arbitrary preimage of a under f. Then

$$\begin{aligned} \mu(y) \geq T(\nu(a),\, \nu(af(y))) &= T(\nu(f(x_a)),\, \nu(f(x_a)f(y))) \\ &= T(\nu(f(x_a)),\, \nu(f(x_a y))) = T(\mu(x_a),\, \mu(x_a y)). \end{aligned}$$

Since a is arbitrary, the above inequality is true for all $x \in L$, i.e.,

$$(\forall x, y \in L)\, (\mu(y) \geq T(\mu(x),\, \mu(xy))),$$

which proves that μ is a T-fuzzy filter of L. Next, let $x,\ y,\ z \in L$. If ν is T-fuzzy implicative, then

$$\mu(xz) = \nu(f(xz)) = \nu(f(x)f(z)) \ \geq T(\nu(f(x)(bf(z))),\, \nu(f(x)b))$$

for any $b \in M$. Consider an arbitrary preimage y_b of b under f. Then

$$\begin{aligned} \mu(xz) &\geq T(\nu(f(x)(bf(z))),\, \nu(f(x)b)) \\ &= T(\nu(f(x)(f(y_b)f(z))),\, \nu(f(x)f(y_b))) \\ &= T(\nu(f(x(y_b z))),\, \nu(f(xy_b))) \\ &= T(\mu(x(y_b z)),\, \mu(xy_b)). \end{aligned}$$

Since b is arbitrary, this inequality is true for all $y \in L$, that is,

$$(\forall x,\ y,\ z \in L)\, (\mu(xz) \geq T(\mu(x(yz)),\, \mu(xy))).$$

Hence μ is a T-fuzzy implicative filter of L. Now if ν satisfies the imaginable property and $\alpha \in \mathrm{Im}(\mu)$, then $\alpha = \mu(x) = \nu(f(x))$ for some $x \in L$. Therefore $\mathrm{Im}(\mu) \subseteq \mathrm{Im}(\nu) \subseteq \Delta_T$, and so μ satisfies the imaginable property. □

We say that a fuzzy set μ in L has the *sup property* if, for any subset A of L, there exists $a_0 \in A$ such that $\mu(a_0) = \sup\limits_{a \in A} \mu(a)$.

Theorem 3.16. *Let $f : L \to M$ be a homomorphism of a lattice implication algebra L onto a lattice implication algebra M. Let μ be a T-fuzzy implicative filter of L which has the sup property. Then the image, say ν, of μ under f is a T-fuzzy* *implicative filter of M, where ν is defined by $\nu(a) = \sup_{x\in f^{-1}(a)} \mu(x)$ for all $a \in M$.*

Proof. Since $1 \in f^{-1}(1)$, we have

$$\nu(1) = \sup_{t\in f^{-1}(1)} \mu(t) = \mu(1) \geq \mu(x)$$

for all $x \in L$, and so $\nu(1) \geq \sup_{t\in f^{-1}(a)} \mu(t) = \nu(a)$ for all $a \in M$. For any $a, b, c \in M$, let $x_a \in f^{-1}(a)$, $x_b \in f^{-1}(b)$ and $x_c \in f^{-1}(c)$ be such that $\mu(x_a x_c) = \sup_{t\in f^{-1}(ac)} \mu(t)$, $\mu(x_a(x_b x_c)) = \sup_{t\in f^{-1}(a(bc))} \mu(t)$, and $\mu(x_a x_b) = \sup_{t\in f^{-1}(ab)} \mu(t)$. Then

$$\begin{aligned}\nu(ac) &= \sup_{t\in f^{-1}(ac)} \mu(t) = \mu(x_a x_c) \geq T(\mu(x_a(x_b x_c)), \mu(x_a x_b)) \\ &= T\Big(\sup_{t\in f^{-1}(a(bc))} \mu(t), \sup_{t\in f^{-1}(ab)} \mu(t)\Big) = T(\nu(a(bc)), \nu(ab)).\end{aligned}$$

Hence ν is a T-fuzzy implicative filter of L. □

We now present some methods of constructions of T-fuzzy (implicative) filters.

Let μ and ν be two fuzzy sets in L. Then we define the *T-product* of μ and ν, denoted by $[\mu \cdot \nu]_T$, by $[\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x))$ for all $x \in L$.

Theorem 3.17. *Let μ and ν be T-fuzzy implicative filters of L. If a t-norm T^* dominates T, i.e., if*

$$T^*(T(\alpha, \gamma), T(\beta, \delta)) \geq T(T^*(\alpha, \beta), T^*(\gamma, \delta))$$

for all $\alpha, \beta, \gamma, \delta \in [0, 1]$, then T^*-product $[\mu \cdot \nu]_{T^*}$ is a T-fuzzy implicative filter of L.

Proof. For every $x \in L$, we obtain

$$[\mu \cdot \nu]_{T^*}(1) = T^*(\mu(1), \nu(1)) \geq T^*(\mu(x), \nu(x)) = [\mu \cdot \nu]_{T^*}(x).$$

Let $x, y, z \in L$. Then

$$\begin{aligned}[\mu \cdot \nu]_{T^*}(xz) &= T^*(\mu(xz), \nu(xz)) \\ &\geq T^*(T(\mu(x(yz)), \mu(xy)), T(\nu(x(yz)), \nu(xy))) \\ &\geq T(T^*(\mu(x(yz)), \nu(x(yz))), T^*(\mu(xy), \nu(xy))) \\ &= T([\mu \cdot \nu]_{T^*}(x(yz)), [\mu \cdot \nu]_{T^*}(xy)).\end{aligned}$$

Therefore $[\mu \cdot \nu]_{T^*}$ is a T-fuzzy implicative filter of L. □

Theorem 3.18. *Let $L = L_1 \times L_2$ be the direct product of lattice implication algebras L_1 and L_2. If μ_1 and μ_2 are T-fuzzy implicative filters of L_1 and L_2 respectively,* *then* $\mu = \mu_1 \times \mu_2$ *is a* T*-fuzzy implicative filter of* L *defined by* $\mu(x) = (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2))$ *for all* $(x_1, x_2) = x \in L$.

Proof. For any $x = (x_1, x_2) \in L$, we have

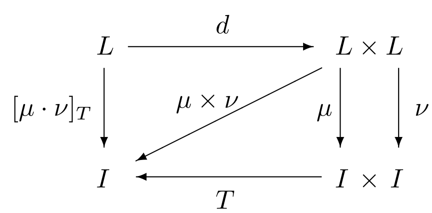
$$\begin{aligned} \mu(x) &= (\mu_1 \times \mu_2)(x_1, x_2) = T(\mu_1(x_1), \mu_2(x_2)) \\ &\leq T(\mu_1(1), \mu_2(1)) = (\mu_1 \times \mu_2)(1, 1) = \mu(1). \end{aligned}$$

Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in L$. Then

$$\begin{aligned} \mu(xz) &= (\mu_1 \times \mu_2)((x_1, x_2)(z_1, z_2)) \\ &= (\mu_1 \times \mu_2)(x_1 z_1, x_2 z_2) = T(\mu_1(x_1 z_1), \mu_2(x_2 z_2)) \\ &\geq T(T(\mu_1(x_1(y_1 z_1)), \mu_1(x_1 y_1)), T(\mu_2(x_2(y_2 z_2)), \mu_2(x_2 y_2))) \\ &= T(T(\mu_1(x_1(y_1 z_1)), \mu_2(x_2(y_2 z_2))), T(\mu_1(x_1 y_1), \mu_2(x_2 y_2))) \\ &= T((\mu_1 \times \mu_2)(x_1(y_1 z_1), x_2(y_2 z_2)), (\mu_1 \times \mu_2)(x_1 y_1, x_2 y_2)) \\ &= T((\mu_1 \times \mu_2)((x_1, x_2)((y_1, y_2)(z_1, z_2))), (\mu_1 \times \mu_2)((x_1, x_2)(y_1, y_2))) \\ &= T(\mu(x(yz)), \mu(xy)). \end{aligned}$$

Hence $\mu = \mu_1 \times \mu_2$ is a T-fuzzy implicative filter of L. □

For two T-fuzzy implicative filters μ and ν of L, the relationship between T-fuzzy implicative filter $\mu \times \nu$ and $[\mu \cdot \nu]_T$ can be viewed via the following diagram.



where $I = [0, 1]$ and $d : L \to L \times L$ is defined by $d(x) = (x, x)$. It is easy to verify that $[\mu \cdot \nu]_T$ is the preimage of $\mu \times \nu$ under d.

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