Honam Mathematical J. **27** (2005), No. 4, pp. 543–550

CANONICAL FORM OF AN TRANSITIVE INTUITIONISTIC FUZZY MATRICES

Hong-Youl Lee and Nae-Gyeong Jeong

Abstract. Some properties of a transitive fuzzy matrix are examined and the canonical form of the transitive fuzzy matrix is given using the properties. As a special case an open problem concerning idempotent matrices is solved. Thus we have the same result in a intuitionistic fuzzy matrix theory. In our results a nilpotent intuitionistic matrix and a symmetric intuitionistic matrix play an important role. We decompose a transitive intuitionistic fuzzy matrix into sum of a nilpotent intuitionistic matrix and a symmetric intuitionistic matrix. Then we obtain a canonical form of the transitive intuitionistic fuzzy matrix.

1. Introduction

In 1965, Zadeh[5] introduced the concept of fuzzy sets which formed the fundamental of fuzzy mathematics. Since then various workers have contributed to the development of the fuzzy theory. Atanassov[1, 2, 3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets.

The theory of intuitionistic fuzzy matrix is very useful in the discussion of intuitionistic fuzzy relations. we can represent basic propositions of the theory of intuitionistic fuzzy relations in the matrix operations.

Received October 19, 2005. Accepted November 7, 2005.

2000 Mathematics Subject Classification : Primary 05A15; Secondary 65F25.

Key words and phrases : Intuitionistic Matrix, Transitive Fuzzy Matrix.

Futhermore we can deal with the intuitionistic fuzzy matrix. The transitive intuitionistic fuzzy matrices correspond to the transitive intuitionistic fuzzy relations. We define the notion of intuitionistic fuzzy matrices as a generalization of fuzzy matrices. Also, we present the canonical form of an transitive intuitionistic fuzzy matrix. As a special case we give some result for the idempotent matrix and examine some properties transitive intuitionistic fuzzy matrices and show decompositions of transitive intuitionistic fuzzy matrix. In this paper, every notation is the standard in general of fuzzy theory.

2. Intuitionistic fuzzy matrices and definitions

We define some operations for fuzzy matrices whose elements are in the unit interval $[0,1]$. An *intuitionistic fuzzy matrix* $\mathcal{A}$ is

$$\mathcal{A} = [(A, B)] = [(a_{ij}, b_{ij})]$$

where A and B are fuzzy matrices, and $a_{ij} + b_{ij} \leq 1$ for all i, j.

Obviously, every fuzzy matrix $A = [(a_{ij})]$ is an intuitionistic fuzzy matrix of the form $[(a_{ij}, 1 - a_{ij})]$.

Let J be an $n \times n$ fuzzy matrix that have all entries 1, let I be an $n \times n$ identity fuzzy matrix and let $\mathcal{I} = [(I, J - I)]$. First, for $x, y \in [0,1]$ and w, z such that $x + w \leq 1$ and $y + z \leq 1$ we define $[x, w] \vee [y, z]$, $[x, w] \wedge [y, z]$ and $[x, w] \ominus [y, z]$ as follows:

$$[x, w] \vee [y, z] = [(max(x, y), min(w, z))],$$

$$[x, w] \wedge [y, z] = [(min(x, y), max(w, z))],$$

$$[x, w] \ominus [y, z] = \begin{cases} [(x, w)] & \text{if} \quad x > y, \quad w \leq z \\ [(0, 1)] & \text{if} \quad x \leq y, \quad w > z. \end{cases}$$

Next, we define the following matrix operations for $n \times n$ intuitionistic fuzzy matrices. Let $\mathcal{A} = [(a_{ij}, b_{ij})]$ and $\mathcal{B} = [(c_{ij}, d_{ij})]$ be $m \times n$ intuitionistic fuzzy matrices and let $\mathcal{C} = [(e_{ij}, f_{ij})]$ be an $n \times l$ intuitionistic fuzzy matrix:

(1) $\mathcal{A} \vee \mathcal{B} = [(a_{ij}, b_{ij}) \vee (c_{ij}, d_{ij})]$.

(2) $\mathcal{A} \wedge \mathcal{B} = [(a_{ij}, b_{ij}) \wedge (c_{ij}, d_{ij})]$.

(3) $\mathcal{A} \ominus \mathcal{B} = [(a_{ij}, b_{ij}) \ominus (c_{ij}, d_{ij})]$.

(4) $\mathcal{A} \times \mathcal{C} = [(\bigvee_{1 \leq k \leq n} (a_{ik} \wedge e_{kj}), \bigwedge_{1 \leq k \leq n} (b_{ik} \vee f_{kj}))]$.

(5) $\mathcal{A}^0 = \mathcal{I}$.

(6) $\mathcal{A}^{k+1} = \mathcal{A}^k \times \mathcal{A} \quad (k = 0, 1, 2, \dots)$.

(7) $\mathcal{A}^T = [(a_{ji}, b_{ji})]$ (the transpose of $\mathcal{A}$).

(8) $\Delta\mathcal{A} = \mathcal{A} \ominus \mathcal{A}^T$.

(9) $\nabla\mathcal{A} = \mathcal{A} \wedge \mathcal{A}^T$.

(10) $\mathcal{A} \leq \mathcal{C}$ if and only if $a_{ij} \leq e_{ij}$, $b_{ij} \geq f_{ij}$ for all i, j.

Then, by the simple calculation,

$$\mathcal{A}\mathcal{I} = \mathcal{I}\mathcal{A} = \mathcal{A}.$$

Therefore, $\mathcal{I}$ is the *identity intuitionistic fuzzy matrix.* Let P be an $n \times n$ permutation fuzzy matrix and $\mathcal{P} = [(P, J - P)]$. Then, by the simple calculation,

$$\mathcal{P}\mathcal{P}^T = \mathcal{P}^T\mathcal{P} = \mathcal{I}.$$

Therefore, $\mathcal{P}$ is a *permutation intuitionistic fuzzy matrix.* A matrix $\mathcal{A}$ is *transitive* if $\mathcal{A}^2 \leq \mathcal{A}$. A matrix $\mathcal{A}$ is *idempotent* if $\mathcal{A}^2 = \mathcal{A}$. Accordingly, any idempotnet matrix is transitive. A matrix $\mathcal{A}$ is *nilpotent* if $\mathcal{A}^n = \mathcal{O}$, where $\mathcal{O}$ is the intuitionistic zero matrix, $[(O, J)]$. A matrix $\mathcal{A}$ is *irreflexive* if $\mathcal{A} \wedge \mathcal{I} = \mathcal{O}$. A matrix $\mathcal{A}$ is *symmetric* if $\mathcal{A}^T = \mathcal{A}$.

Proposition 2.1. If an intuitionistic fuzzy matrix $\mathcal{A}$ is nilpotent, it is irreflexive.

Proof. Suppose that $\mathcal{A} = [(a_{ij}, b_{ij})]$ is an nilpotent intuitionistic fuzzy matrix and $\mathcal{I} = [0, 1]$. Generally,

$$\mathcal{A} \wedge \mathcal{I} = \begin{bmatrix} (a_{11}, b_{11}) & (0,1) & (0,1) & \cdots & (0,1) \\ (0,1) & (a_{22}, b_{22}) & (0,1) & \cdots & (0,1) \\ (0,1) & (0,1) & (a_{33}, b_{33}) & \cdots & (0,1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (0,1) & (0,1) & (0,1) & \cdots & (a_{nn}, b_{nn}) \end{bmatrix}.$$

Since $\mathcal{A}^n = \mathcal{O}$, $(a_{ii}, b_{ii}) \wedge (a_{ii}, b_{ii}) = (0, 1)$ for any i. Therefore, $\mathcal{A} \wedge \mathcal{I} = \mathcal{O}$ □

Proposition 2.2. If an intuitionistic fuzzy matrix $\mathcal{A}$ is irreflexive transitive, then it is nilpotent.

Proof. Suppose that $\mathcal{A}$ is an irreflexive transitive intuitionistic fuzzy matrix. Since $\mathcal{A}$ is irreflexive, $(a_{ii}, b_{ii}) = (0, 1)$ for any i. Therefore, $\mathcal{A}$ is nilpotent by the transitivity. □

3. Some results

First, we examine some basic properties of nilpotent intuitionistic matrices. They are useful in the following discussion. Next, we give the canonical form of a special type of matrix which is given by the sum of a nilpotent intuitionistic matrix and a symmetric intuitionistic matrix. Then we examine the properties of transitive intuitionistic matrices and decompose a transitive intuitionistic matrix into the sum of a nilpotent intuitionistic matrix and a symmetric intuitionistic matrix. Thus we obtain a theorem on canonical form of the transitive intuitionistic matrix. In the following, we deal only with square intuitionistic fuzzy matrices.

Lemma 3.1. If an intuitionistic fuzzy matrix $\mathcal{S} = [(s_{ij}, t_{ij})]$ is symmetric, then

$$\Delta(\mathcal{A} \vee \mathcal{S}) \leq \Delta\mathcal{A}$$

for any intuitionistic fuzzy matrix $\mathcal{A}$.

Proof. Let $\mathcal{U} = [(u_{ij}, v_{ij})] = \Delta(\mathcal{A} \vee \mathcal{S})$, that is,

$$(u_{ij}, v_{ij}) = ((a_{ij}, b_{ij}) \vee (s_{ij}, t_{ij})) \ominus ((a_{ji}, b_{ji}) \vee (s_{ij}, t_{ij})).$$

If $a_{ij} \vee s_{ij} > a_{ji} \vee s_{ij}$ and $b_{ij} \wedge t_{ij} < b_{ji} \wedge t_{ij}$, then

$$\begin{aligned}(u_{ij}, v_{ij}) &= (a_{ij} \vee s_{ij}, b_{ij} \wedge t_{ij})\\ &= (a_{ij}, b_{ij})\\ &= ((a_{ij}, b_{ij}) \ominus (a_{ji}, b_{ji}))\end{aligned}$$

On the other hand, if $a_{ij} \vee s_{ij} \leq a_{ji} \vee s_{ij}$ and $b_{ij} \wedge t_{ij} \geq b_{ji} \wedge t_{ij}$, then

$$(u_{ij}, v_{ij}) = (0, 1) \leq ((a_{ij}, b_{ij}) \ominus (a_{ji}, b_{ji}))$$

Thus we have $\mathcal{U} \leq \Delta\mathcal{A}$. □

Lemma 3.2. If an intuitionistic fuzzy matrix $\mathcal{A} = [(a_{ij}, b_{ij})]$ is nilpotent, then $\Delta\mathcal{A} = \mathcal{A}$ and $\nabla\mathcal{A} = \mathcal{O}$.

Proof. Let $\mathcal{A} = [(a_{ij}, b_{ij})]$ and $\mathcal{S} = [(s_{ij}, t_{ij})] = \Delta\mathcal{A}$. Then $(s_{ij}, t_{ij}) = ((a_{ij}, b_{ij}) \ominus (a_{ji}, b_{ji}))$. Since $\mathcal{A}$ is nilpotent, clearly $(a_{ij}, b_{ij}) \wedge (a_{ji}, b_{ji}) = (0, 1)$ (otherwise, $\mathcal{A}^2$ would not be irreflexive), so that $(s_{ij}, t_{ij}) = (a_{ij}, b_{ij})$. Thus $\nabla\mathcal{A} = \mathcal{O}$ and $\Delta\mathcal{A} = \mathcal{A}$. □

Lemma 3.3. For any intuitionistic fuzzy matrix $\mathcal{A} = [(a_{ij}, b_{ij})]$,

$$\mathcal{A} = \Delta\mathcal{A} \vee \nabla\mathcal{A}.$$

Proof. Let $\mathcal{S} = [(s_{ij}, t_{ij})] = \Delta\mathcal{A} \vee \nabla\mathcal{A}$. That is,

$$(s_{ij}, t_{ij}) = ((a_{ij}, b_{ij}) \ominus (a_{ji}, b_{ji})) \vee ((a_{ij}, b_{ij}) \wedge (a_{ji}, b_{ji})).$$

If $(a_{ij}, b_{ij}) > (a_{ji}, b_{ji})$, then

$$(s_{ij}, t_{ij}) = ((a_{ij}, b_{ij}) \vee (a_{ji}, b_{ji})) = (a_{ij}, b_{ij}).$$

If $(a_{ij}, b_{ij}) \leq (a_{ji}, b_{ji})$, then

$$(s_{ij}, t_{ij}) = ((0, 1) \vee (a_{ij}, b_{ij})) = (a_{ij}, b_{ij}).$$

Thus we have $\mathcal{S} = \mathcal{A}$. □

Lemma 3.4. Let $\mathcal{N}$ be a nilpotent and $\mathcal{S}$ be a symmetric intuitionistic fuzzy matrix. If $\mathcal{A} = \mathcal{N} \vee \mathcal{S}$, then $\Delta\mathcal{A} \leq \mathcal{N}$ and $\nabla\mathcal{A} = \mathcal{S}$.

Proof. By Lemma 3.1 and 3.2,

$$\Delta\mathcal{A} = \Delta(\mathcal{N} \vee \mathcal{S}) \leq \Delta\mathcal{N} = \mathcal{N}.$$

By the definition,

$$\begin{aligned} \nabla\mathcal{A} &= \mathcal{A} \wedge \mathcal{A}^T \\ &= (\mathcal{N} \vee \mathcal{S}) \wedge (\mathcal{N}^T \vee \mathcal{S}) \\ &= (\mathcal{N} \wedge \mathcal{N}^T) \vee (\mathcal{N} \wedge \mathcal{S}) \vee (\mathcal{S} \wedge \mathcal{N}^T) \vee \mathcal{S} = \mathcal{S}. \end{aligned}$$

□

It should be noted that when $\mathcal{A}$ is decomposable into the sum of a nilpotent matrix $\mathcal{N}$ and a symmetric matrix $\mathcal{S}$, the decomposition is not unique. However, $\Delta\mathcal{A}$ is minimal in the set of such nilpotent matrices, since $\Delta\mathcal{A} \leq \mathcal{N}$.

Theorem 3.5. Let $\mathcal{N}$ be a nilpotent and $\mathcal{S}$ be a symmetric intuitionistic fuzzy matrix. For an intuitionistic fuzzy matrix $\mathcal{A}$ given by

$\mathcal{A} = \mathcal{N} \vee \mathcal{S}$, there exists a permutation intuitionistic fuzzy matrix $\mathcal{P}$ such that $\mathcal{T} = [(u_{ij}, v_{ij})] = \mathcal{P} \times \mathcal{A} \times \mathcal{P}^T$ satisfies $u_{ij} \geq u_{ji}$ and $v_{ij} \leq v_{ji}$ for $i > j$.

Proof. Suppose that $\mathcal{A} = \mathcal{N} \vee \mathcal{S}$ where $\mathcal{N}$ is nilpotent and $\mathcal{S}$ is symmetric. Then

$$\begin{aligned} \mathcal{T} &= [(u_{ij}, v_{ij})] \\ &= \mathcal{P} \times \mathcal{A} \times \mathcal{P}^T \\ &= \mathcal{P} \times (\mathcal{N} \vee \mathcal{S}) \times \mathcal{P}^T \\ &= (\mathcal{P} \times \mathcal{N} \times \mathcal{P}^T) \vee (\mathcal{P} \times \mathcal{S} \times \mathcal{P}^T). \end{aligned}$$

Since $\mathcal{N}$ is nilpotent, $\mathcal{P} \times \mathcal{N} \times \mathcal{P}^T$ becomes strictly lower triangular for some permutation matrix $\mathcal{P}$. Thus, since $\mathcal{P} \times \mathcal{S} \times \mathcal{P}^T$ is symmetric, $\mathcal{T}$ satisfies $u_{ij} \geq u_{ji}$ and $v_{ij} \leq v_{ji}$ for $i > j$ by choosing such a permutation matrix $\mathcal{P}$. □

The following lemma, which shows basic properties transitive fuzzy matrix case, is well-known. Moreover, the proof of transitive intuitionistic fuzzy matrix case is similar. (See[4] for a proof)

Lemma 3.6. If $\mathcal{A}$ is a transitive intuitionistic fuzzy matrix, then $\Delta\mathcal{A}$ is irreflexive, so nilpotent. Moreover, $\nabla\mathcal{A}$ is symmetric and transitive, so idempotent.

Theorem 3.7. For a transitive intuitionistic fuzzy matrix $\mathcal{A}$, there exists a permutation matrix $\mathcal{P}$ such that $\mathcal{T} = [(u_{ij}, v_{ij})] = \mathcal{P} \times \mathcal{A} \times \mathcal{P}^T$ satisfies $u_{ij} \geq u_{ji}$ and $v_{ij} \leq v_{ji}$ for $i > j$.

Proof. By Theorem 3.5, Lemma 3.3 and 3.6, we have this result. □

References

[1] K. Atanassov, *Intuitionistic fuzzy sets, in:V. Ssurev, Ed., VII ITKR's Session,*

Sofia(June 1983 Central Sci. and Techn. Library, Bulg. Academy of Science), 1984.

[2] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), 87-96.

[3] K. Atanassov, *Review and new results on intuitionistic fuzzy sets, Preprint IM-MFAIS-1-88*, Sofia, 1988.

[4] L. A. Zadeh, *A fuzzy algorithmic approach to the definition of complex or imprecise concepts. Int.*, J. Math-Machine Studies **8** (1976), 249-291.

[5] L. A. Zadeh, *Fuzzy Sets*, Inform. and Control **8** (1965), 338-353.

Hong-Youl Lee
Department of Mathematics Education
Woosuk University
Wanju-gun, 565-701 Korea
Email : hylee@woosuk.ac.kr

Nae-Gyeong Jeong
Department of Mathematics
Seonam University
Namwon, 590-711 Korea
Email : jnk3611@seonam.ac.kr