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On op-idempotents

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Abstract. In this paper, we introduce the concept of op-idempotents. It is shown that every exchange ring can be characterized by op-idempotents

A ring R is an exchange ring if for every right R-module A and two decompositions $A = M \oplus N = \oplus_{i \in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\oplus_{i \in I} A_i')$. Clearly, regular rings, π-regular rings, semi-perfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero (cf. [2, Theorem 7.2]) are all exchange rings. We say that an element $e \in R$ is a op-idempotent provided that $e^2 = -e$. Let $R = \mathbb{Z}/3\mathbb{Z}$. Then R is an exchange ring and $\overline{2} \in R$ is a op-idempotent, while it is not an idempotent. Also we know that every non-zero Boolean ring is an exchange ring without any non-trivial op-idempotent. Thus op-idempotents are different from idempotents in exchange rings. In this paper, we observe that every exchange ring can be characterized by its op-idempotents.

Throughout the paper, all rings are associative with identities. We always use $J(R)$ to denote the Jacobson radical of R.

Lemma 1. *Let R be a ring. Then the following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any $a \in R$, there exists a op-idempotent $e \in R$ such that $e \in Ra$ and $1 + e \in R(1 + a)$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, we have $-a \in R$. By [6, Theorem 2.1], there exists an idempotent $f \in R$ such that $f \in R(-a)$ and $1 - f \in R(1 + a)$. Set $e = -f$. Then we see that $e \in R$ is a op-idempotent. Furthermore, we get $e \in Ra$ and $1 + e \in R(1 + a)$, as required.

(2) $\Rightarrow$ (1) For any $a \in R$, we get $-a \in R$. So there exists an op-idempotent $e \in R(-a)$ such that $1 + e \in R(1 - a)$. Let $f = -e$. Then $f = f^2 \in R$. In addition, we have $f \in Ra$ and $1 - f \in R(1 - a)$. By [6, Theorem 2.1], R is an exchange ring. $\square$

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Lemma 2. *The following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any $a \in R$, there exists a op-idempotent $e \in R$ such that $e - a \in R(a + a^2)$.*

Proof. (1) $\Rightarrow$ (2) For any $a \in R$, it follows by Lemma 1 that there exists a op-idempotent $e \in Ra$ such that $1 + e \in R(1 + a)$. Hence $e - a = e(1 + a) - (1 + e)a \in R(a + a^2)$.

(2) $\Rightarrow$ (1) For any $a \in R$, we have $-a \in R$; hence, there exists a op-idempotent $e \in R$ such that $e - (-a) \in R\big(-a + (-a)^2\big)$. This infers that $e + a \in R(a - a^2)$. Assume that $e + a = ra(1 - a)$ for some $r \in R$. Then $e = r(1 - a)a - a \in Ra$ and $1 + e = (1 + ra)(1 - a) \in R(1 - a)$. Let $f = -e$. Then $f = f^2 \in Ra$ and $1 - f \in R(1 - a)$. In view of [6, Theorem 2.1], we complete the proof. $\square$

We say that every op-idempotent lifts modulo a left ideal I of R in case $x + x^2 \in I$ implies that $e - x \in I$ for some op-idempotent $e \in R$.

Theorem 3. *Let R be a ring. Then the following are equivalent*:

(1) *R is an exchange ring.*

(2) *Every op-idempotent lifts modulo any left ideal of R.*

Proof. (1) $\Rightarrow$ (2) Let I be a left ideal of R. Suppose that $x + x^2 \in I$. By virtue of Lemma 2, there exists a op-idempotent $e \in R$ such that $e - x \in R(x + x^2) \subseteq I$. That is, every op-idempotent lifts modulo I, as required.

(2) $\Rightarrow$ (1) Let $a \in R$, and let $I = R(a + a^2)$. Clearly, $a + a^2 \in I$. So we have a op-idempotent $e \in R$ such that $e - a \in I$. That is, $e - a \in R(a + a^2)$. Using Lemma 2, we conclude that R is an exchange ring. $\square$

Since a ring R is an exchange ring if and only if so is the opposite ring R^{op}, by Theorem 3, we deduce that a ring R is an exchange ring if and only if every op-idempotent lifts modulo any right ideal of R. Recall that an element $u \in R$ is full in case $RuR = R$. In [1], Ara and Goodearl studied stable rank of full corners of a ring. Now we investigate exchange rings by using full elements.

Lemma 4. *Let R be an exchange ring. Given $ax + b = 1$ in R, there exists $y \in R$ such that $a + by \in R$ is a full element.*

Proof. Assume that $ax + b = 1$ with $a, x, b \in R$. Since R is an exchange ring, by [6, Theorem 2.1], there exists $e = e^2 \in R$ such that $e = bs$ and $1 - e = (1 - b)t$ for some $s, t \in R$. So $1 - e = axt$, hence $(1 - e)aR = (1 - e)R$. Thus we can find $u, v \in R$ such that $1 = (1 - e)au + ev$. It is easy to verify that $(1 - e)\big((1 - e)a + e\big)u + e\big((1 - e)a + e\big)v = (1 - e)au + ev = 1$, whence $R\big((1 - e)a + e\big)R = R$. Set $y = s(1 - a)$. Therefore $R(a + by)R = R\big(a + bs(1 - a)\big)R = R\big(a + e(1 - a)\big)R = R$, as asserted. $\square$

Lemma 5. *Let R be an exchange ring, and let $e, f \in R$ be op-idempotents. If $eR \cong fR$, then there exists a full element $u \in R$ such that $eu = uf$.*

Proof. Since $\psi : eR \cong fR$, we have $f = \psi(er) = \big(f\psi(e)e\big)\big(erf\big)$. Let $a = erf$ and $b = f\psi(e)e$. Then $f = ba$. Clearly, $ab = (erf)\big(f\psi(e)e\big) = \psi^{1}(f)f\big(f\psi(e)e\big) = \psi^{-1}\big(f\psi(-e)\big) = \psi^{-1}\big(-\psi(-e)\big) = e$. So we can find $a \in eRf$ and $b \in fRe$ such that $e = ab$ and $f = ba$. In addition, $aba = ea = -a$ and $bab = fb = -b$. As $a(-b) + (1 + ab) = 1$ with $a, b \in R$, it follows by Lemma 4 that there exists $y \in R$ such that $v := a + (1 + ab)y \in R$ is a full element. Obviously, $b = -bab = -bvb$. Let $u = (1 + ab + vb)v(1 + ba + bv)$. As $(1 + ab + vb)^2 = 1 = (1 + ba + bv)^2$, we have $RuR = RvR = R$. Furthermore, we get $eu = ab(1 + ab + vb)v(1 + ba + bv) = -abv(1 + ba + bv) = -a = -(1 + ab + vb)vba = (1 + ab + vb)v(1 + ba + bv)ba = uf$, as required. □

Theorem 6. *A ring R is an exchange ring if and only if for any $a \in R$, there exist op-idempotents $e, f \in R$ such that*

(1) $e \in aR, 1 + e \in (1 + a)R$.

(2) $f \in Ra, 1 + f \in R(1 + a)$.

(3) $euv = uvf$ *with full elements* $u, v \in R$.

Proof. One direction is obvious by Lemma 1. Conversely, assume now that R is an exchange ring. For any $a \in R$, we have an idempotent $e' \in R$ such that $e' \in aR, 1 - e' \in (1 - a)R$. Set $g' = 1 - e'$ and $b = 1 - a$. Then $e' \in aR$ and $g' \in bR$. Suppose that $e' = ar'$ and $g' = bs'$. Set $r = r'e'$ and $s = s'g'$. Then we have $rar = r, sbs = s, rbs = 0$ and $sar = 0$. Let $r'' = 1 - sb + rb$ and $s'' = 1 - ra + sa$. Similarly to [7, Proposition], we get $r''ar'' = r'', s''bs'' = s''$ and $r''a + s''b = 1$. Let $f' = r''a$. Then we have an idempotent $f' \in Ra$ such that $1 - f' = s''b \in R(1 - a)$.

Clearly, $s'bR \cong bs'R = g'R$. By Lemma 5, we have $g'u = us'b$ and $RuR = R$. Hence $e'u = (1 - g')u = u(1 - s'b)$. On the other hand, $1 - s'b = ar''$; hence, we get $(1 - s'b)R \cong r''aR = f'R$. By Lemma 5 again, we have $(1 - s'b)v = vf'$ and $RvR = R$. Therefore $e'uv = u(1 - s'b)v = uvf'$ and $RuR = RvR = R$. Let $e = -e'$ and $f = -f'$. Then $e, f \in R$ are both op-idempotents. Applying the argument above to $-a \in R$, we complete the proof. □

It is well known that every finitely generated projective right module over an exchange ring is generated by some idempotents. Now we extend this fact to op-idempotents as follows.

Proposition 7. *Let P be a finitely generated projective right module over an exchange ring R. Then there exist op-idempotents $e_1, \cdots, e_n \in R$ such that $P \cong e_1R \oplus \cdots \oplus e_nR$.*

Proof. Clearly, P has the finitely exchange property. Set $M = P \oplus Q$. Then $M = P \oplus Q = \bigoplus_{i=1}^{n} R_i$ with all $R_i \cong R$. By the finite exchange property of P, we have $Q_i (1 \leq i \leq n)$ such that $M = P \oplus \big(\bigoplus_{i=1}^{n} Q_i\big)$, where all Q_i are direct summands of R_i respectively. Assume that $Q_i \oplus P_i = R_i$ for all i. Then we have

$P \oplus \big(\bigoplus_{i=1}^{n} Q_i\big) = \big(\bigoplus_{i=1}^{n} P_i\big) \oplus \big(\bigoplus_{i=1}^{n} Q_i\big)$. Hence $P \cong P_1 \oplus \cdots \oplus P_n$, where P_i is isomorphic to a direct summand of R as a right R-module for all i.

Suppose that M is a finitely generated projective right module over an exchange ring R. Assume that there is a right R-module Q such that $P \oplus Q \cong R$. Let $e : R \cong P \oplus Q \to P$ given by $e(p,q) = -p$ for any $p \in P, q \in Q$. Then $P \cong eR$. So we have op-idempotents e_i such that $P_i \cong e_iR$. Therefore $P \cong e_1R \oplus \cdots \oplus e_nR$ with all op-idempotents $e_i \in R$. □

Recall that an ideal I of a ring R is an exchange ideal provided that for any $x \in I$, there exists an idempotent $e \in I$ such that $e - x \in R(x - x^2)$. Clearly, every strongly π-regular ideal of a ring is an exchange ideal. Now we study exchange rings by virtue of exchange ideals.

Theorem 8. *Let I be an exchange ideal of a ring R. Then the following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any $a \in R$, there exists a op-idempotent $e \in R$ such that $e \in Ra + I$ such that $1 + e \in R(1 + a) + I$.*

Proof. (1) ⇒ (2) is trivial by Lemma 1.

(2) ⇒ (1) For any $x + I \in R/I$, there exists a op-idempotent $e \in R$ such that $e \in Rx + I$ such that $1 + e \in R(1 + x) + I$. So there exists a op-idempotent $e + I \in (R/I)(x + I)$ such that $(1 + I) + (e + I) \in (R/I)\big((1 + I) + (x + I)\big)$. In view of Lemma 1, R/I is an exchange ring.

Given $x - x^2 \in I$, there exists a op-idempotent $e \in R$ such that $e \in R(-x) + I$ such that $1 + e \in R(1 - x) + I$. Let $f = -e$. Then $f = f^2$, $f \in Rx + I$ and $1 - f \in R(1 - x) + I$. This infers that $f - x = f(1 - x) - (1 - f)x \in R(x - x^2) + I \subseteq I$. That is, every idempotent lifts modulo I. According to [1, Theorem 2.2], R is an exchange ring. □

Corollary 9. *Let R be a ring. Then the following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any $a \in R$, there exists a op-idempotent $e \in R$ such that $e \in Ra + J(R)$ such that $1 + e \in R(1 + a) + J(R)$.*

Proof. Since $J(R)$ is an exchange ideal of R, we obtain the result by Theorem 8. □

Analogously [8, Theorem 3], we show that if for any $a \in R$ there exists a op-idempotent $e \in R$ such that $Ra + J(R) = Re + J(R)$, then R is an exchange ring.

Theorem 10. *Let I be an exchange ideal of a ring R. Then the following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any* $a \in R$*, there exists a op-idempotent* $e \in R$ *such that* $e - a \in R(a + a^2) + I$.

Proof. (1) $\Rightarrow$ (2) is clear by Lemma 2.

(2) $\Rightarrow$ (1) For any $x \in R$, there exists an idempotent $e \in R$ such that $e - x \in R(x+x^2)+I$. Assume now that $e-x = r(x+x^2)+s$ for $r \in R, s \in I$. Then we have $e = \big(1+r(1+x)\big)x+s \in Rx+I$ such that $1+e = (1+rx)(1+x)+s \in R(1+x)+I$. According to Theorem 8, R is an exchange ring. □

Corollary 11. *Let* I *be an exchange ideal of a ring* R*. If for any* $a \in R$ *there exist a op-idempotent* $e \in R$ *and a unit* $u \in R$ *such that* $a \equiv e + u \pmod I$*. Then* R *is an exchange ring.*

Proof. Given any $x \in R$, we have a op-idempotent $e \in R$ and a unit $u \in R$ such that $x \equiv e+u \pmod I$. It is easy to verify that $u\big(x+u^{-1}(1+e)u\big) \equiv eu+ue+u^2+u \equiv x^2+x \pmod I$. Set $f = -u^{-1}(1+e)u$. Then $f^2 = -f \in R$. In addition, we have $x - f \in R(x+x^2)+I$. According to Theorem 8, R is an exchange ring. □

It follows by Corollary 11 that a ring R is an exchange ring if and only if for any $a \in R$ there exist a op-idempotent $e \in R$ and a unit $u \in R$ such that $a \equiv e + u \pmod{J(R)}$.

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