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NOTES ON $\overline{WN_{n,0,0_{[2]}}}$ I

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Abstract. The Weyl-type non-associative algebra $\overline{WN_{g_n,m,s_r}}$ and its subalgebra $\overline{WN_{n,m,s_r}}$ are defined and studied in the papers [8], [9], [10], [12].
We will prove that the Weyl-type non-associative algebra $\overline{WN_{n,0,0_{[2]}}}$ and its corresponding semi-Lie algebra are simple. We find the non-associative algebra automorphism group $Aut_{non}(\overline{WN_{1,0,0_{[2]}}})$.

1. Preliminaries

Let $\mathbf{N}$ be the set of all non-negative integers and $\mathbf{Z}$ be the set of all integers. Let $\mathbf{F}$ be a field of characteristic zero. Let $\mathbf{F}^\bullet$ be the multiplicative group of non-zero elements of $\mathbf{F}$. Let $\mathbf{F}[x_1, \cdots, x_{m+s}]$ be the polynomial ring with the variables $x_1, \cdots, x_{m+s}$. Let $g_1, \cdots, g_n$ be given polynomials in $\mathbf{F}[x_1, \cdots, x_{m+s}]$. For $n, m, s \in \mathbf{N}$, let us define the commutative, associative $\mathbf{F}$-algebra $F_{g_n,m,s} = \mathbf{F}[e^{\pm g_1}, \cdots, e^{\pm g_n}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}, x_{m+1}, \cdots, x_{m+s}]$ which is called a stable algebra in the paper [5] with the standard basis

$$\mathbf{B} = \{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} | a_1, \cdots, a_n, i_1, \cdots, i_m \in \mathbf{Z}, \\ i_{m+1}, \cdots, i_{m+s} \in \mathbf{N}\}$$

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and with the obvious addition and the multiplication [5], [8] where we take appropriate $g_1, \cdots, g_n$ so that $\mathbf{B}$ can be the standard basis of $F_{g_n,m,s}$. ∂_w, $1 \le w \le m+s$, denotes the usual partial derivative with respect to x_w on $F_{g_n,m,s}$. For partial derivatives $\partial_u, \cdots, \partial_v$ of $F_{g_n,m,s}$, the composition $\partial_u^{j_u} \circ \cdots \circ \partial_v^{j_v}$ of them is denoted $\partial_u^{j_u} \cdots \partial_v^{j_v}$ where $j_u, \cdots, j_v \in \mathbf{N}$. Let us define the vector space $WN(g_n, m, s)$ over $\mathbf{F}$ which is spanned by the standard basis

$$\begin{aligned}(1)\qquad &\{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} | a_1, \cdots, a_n, i_1, \cdots, i_m \in \mathbf{Z},\\ &i_{m+1}, \cdots, i_{m+s} \in \mathbf{N}, j_u, \cdots, j_v \in \mathbf{N}, 1 \le u, \cdots, v \le m+s\}\end{aligned}$$

Thus we may define the multiplication $*$ on $WN(g_n, m, s)$ as follows:

$$\begin{aligned}(2)\qquad & e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} * \\ &\qquad e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \\ = \ & e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} \\ &\qquad (e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}}) \partial_h^{j_h} \cdots \partial_w^{j_w}\end{aligned}$$

for any basis elements $e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1,m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v}$ and $e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2,m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \in WN(g_n, m, s)$. Thus we can define the Weyl-type non-associative algebra $\overline{WN_{g_n,m,s}}$ with the multiplication $*$ in (2) and with the set $WN(g_n, m, s)$ [1], [14]. For $r \in \mathbf{N}$, let us define the the non-associative subalgebra $\overline{WN_{g_n,m,s_r}}$ of the non-associative algebra $\overline{WN_{g_n,m,s}}$ spanned by

$$\begin{aligned}&\{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v} | a_1, \cdots, a_n, i_1, \cdots, i_m \in \mathbf{Z},\\ &\qquad i_{m+1}, \cdots, i_s \in \mathbf{N}, j_u, \cdots, j_v \in \mathbf{N},\\ (3)\qquad &\qquad j_u + \cdots + j_v \le r, 1 \le u, \cdots, v \le m+s\}\end{aligned}$$

The non-associative subalgebra $\overline{WN_{g_n,m,s_1}}$ of the non-associative algebra $\overline{WN_{g_n,m,s}}$ is the the non-associative algebra $\overline{N_{g_n,m,s}}$ in the paper [1]. There is no left or right identity of $\overline{WN_{g_n,m,s}}$. The the non-associative algebra $\overline{WN_{g_n,m,s}}$ is $\mathbf{Z}^n$-graded as follows:

$$\overline{WN_{g_n,m,s}} = \bigoplus_{(a_1,\cdots,a_n)} WN_{(a_1,\cdots,a_n)} \tag{4}$$

where $WN_{(a_1,\cdots,a_n)}$ is the vector subspace of $\overline{WN_{g_n,m,s}}$ with the basis

$$\{e^{a_1 g_1}\cdots e^{a_n g_n} x_1^{i_1}\cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u}\cdots\partial_v^{j_v} | i_1,\cdots,i_m \in \mathbf{Z},$$
$$i_{m+1},\cdots,i_{m+s}, j_u,\cdots,j_v \in \mathbf{N}, 1 \le u,\cdots,v \le m+s\}.$$

An element in $WN_{(a_1,\cdots,a_n)}$ is called an $(a_1,\cdots,a_n)$-homogenous element and $WN_{(a_1,\cdots,a_n)}$ is called the $(a_1,\cdots,\ a_n)$-homogeneous component. For any basis element $e^{a_1 g_1}\cdots e^{a_n g_n} x_1^{i_1}\cdots x_{m+s}^{i_{m+s}}\ \partial_t$ of $\overline{WN_{g_n,m,s}}$, let us define the homogeneous degree $deg_N(e^{a_1 g_1}\cdots e^{a_n g_n} x_1^{i_1}\cdots\ x_{m+s}^{i_{m+s}} \partial_u^{j_u}\cdots\partial_v^{j_v})$ of it as follows:

$$deg_N(e^{a_1 g_1}\cdots e^{a_n g_n} x_1^{i_1}\cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u}\cdots\partial_v^{j_v}) = \sum_{u=1}^{m+s} |i_u|$$

where $|i_u|$ is the absolute value of i_u, $1 \le u \le m+s$. Throughout this paper, for any basis element $e^{a_\mu g_\mu}\cdots e^{a_\nu g_\nu} x_\lambda^{i_\lambda}\cdots x_\sigma^{i_\sigma}\partial_u^{j_u}\cdots\partial_v^{j_v}$, we write it such that $1 \le \mu \le \cdots \le \nu \le n$, $1 \le \lambda \le \cdots \le \sigma \le m$, and $1 \le u \le \cdots \le v \le m+s$. For any element $l \in \overline{WN_{g_n,m,s}}$, we may define $deg_N(l)$ as the highest homogeneous degree of the basis terms of l. Thus for any basis elements l_1 and l_2 of $\overline{WN_{0,0,s}}$, we may write $l_1 + l_1$ or $l_2 + l_1$ well orderly with unambiguity. For any element $l \in \overline{WN_{0,0,s}}$, we may define $deg_N(l)$ as the highest homogeneous degree of each monomial of l. For any $l \in \overline{WN_{g_n,m,s}}$, let us define $\#(l)$ as the number of different homogeneous components of l. $\overline{WN_{n,m,s}}$ (resp. $\overline{WN_{g_n,m,s_r}}$) has the subalgebra WT (resp. WT_r) spanned by $\{\partial_u^{j_u}\cdots\partial_v^{j_v} | (resp.\ \ j_u+\cdots+j_v \le r,)\ \ j_u,\cdots,j_v \in \mathbf{N}, 1 \le u,v \le s_1\}$ which is the right annihilator of $\overline{WN_{g_n,m,s}}$ (resp. $\overline{WN_{g_n,m,s_r}}$). Let us define the non-associative subalgebra $\overline{WN_{n,0,0_{[2]}}}$ of the non-associative algebra $\overline{WN_{g_n,m,s}}$ is spanned by $\{e^{a_1 x_1}\cdots e^{a_n x_n}\partial_u^2 | 1 \le u \le n\}$. The non-associative algebra $\overline{WN_{n,0,0}}_{[2]}$ is $\mathbf{Z}^n$-graded as follows:

$$(5) \qquad \overline{WN_{n,0,0}}_{[2]} = \bigoplus_{(a_1,\cdots,a_n)} N_{(a_1,\cdots,a_n)}$$

where $N_{(a_1,\cdots,a_n)}$ is the vector subspace of $\overline{WN_{n,0,0}}_{[2]}$ with the basis $\{e^{a_1x_1}\cdots e^{a_nx_n}\partial_v^2 | 1 \le v \le n\}$. The non-associative algebra $\overline{WN_{g_n,m,s}}$ contains the matrix ring $M_n(\mathbf{F})$ [1]. A non-associative algebra A is simple, if it has no proper two sided ideal which is not zero ideal [14]. For any element l in a non-associative algebra A, l is full, if the ideal $< l >$ generated by l is A. Generally, the algebra $\overline{WN_{0,0,s_r}}$ or $\overline{WN_{0,0,s}}$ is not Lie admissible [1], [8], since the Jacobi identity does not hold using the commutator of the non-associative algebra $\overline{WN_{0,0,s_r}}$ or the non-associative algebra $\overline{WN_{0,0,s}}$ for $r > 1$. For any $\mathbf{F}$-algebra A and an element $l \in A$, an element $l_1 \in A$ is a left (resp. right) stabilizing element of l, if $l_1 * l = cl$ (resp. $l * l_1 = cl$) where $c \in \mathbf{F}$. For any element $l_1 \in A$, $l \in A$ is a locally left (resp. right) unity of $l_1 \in A$, if $l * l_1 = l_1$ (resp. $l_1 * l = l_1$) holds and throughout the paper, we read it as that l is a left unity of l_1, etc.. The Weyl-type non-associative algebra $\overline{WN_{g_n,m,s_r}}$ and its subalgebra $\overline{WN_{n,m,s_r}}$ contains the matrix ring $M_s(\mathbf{F})$, i.e., $x_u\partial_v$ in The Weyl-type non-associative algebra $\overline{WN_{g_n,m,s_r}}$ or its subalgebra $\overline{WN_{n,m,s_r}}$ corresponds e_{uv} where e_{uv} is the unit matrix of $M_s(\mathbf{F})$ such that its uv-entry is one and its other terms are zero.

2. Simplicity of $\overline{WN_{n,0,0}}_{[2]}$

Even if the non-associative algebra $\overline{WN_{n,0,0}}_{[2]}$ has right annihilators, we have the following results. The non-associative algebra $\overline{WN_{n,0,0}}_{[2]}$ has no idempotent.

Remark 1. An (non-associative, Lie, or associative) algebra A is simple if and only if every element of the (non-associative, Lie, or associative) algebra A is full.

Lemma 1. *For any* ∂_u^2, $1 \leq u \leq n$, *in the non-associative algebra* $\overline{WN_{n,0,0_{[2]}}}$, ∂_u^2 *is full.*

Proof. Let I be a non-zero ideal of the non-associative algebra $\overline{WN_{n,0,0_{[2]}}}$ which contains ∂_u in the lemma. For any basis element $e^{a_1x_1}\cdots e^{a_nx_n}\partial_v^2$ of $\overline{WN_{n,0,0_{[2]}}}$ with $a_u \neq 0$,

$$\partial_u^2 * e^{a_1x_1}\cdots e^{a_nx_n}\partial_v^2 = a_u^2 e^{a_1x_1}\cdots e^{a_nx_n}\partial_v^2 \in I$$

This implies that $\overline{WN_{n,0,0_{[2]}}} \subset I$, i.e., $\overline{WN_{n,0,0_{[2]}}} = I$. This implies that ∂_u is full. Therefore we have proven the lemma. □

Theorem 1. *The non-associative algebra* $\overline{WN_{n,0,0_{[2]}}}$ *is simple.*

Proof. Let I be a non-zero ideal of the non-associative algebra $\overline{WN_{n,0,0_{[2]}}}$. Let l be any non-zero element in I. Let us prove the theorem by induction on $\#(l)$ of l. If $\#(l) = 1$, then there is nothing to prove by Lemma 1. Let us assume that we have proven the theorem $\#(l) = k$ where k is a positive integer. If l has at least two different partial derivatives ∂_u^2 and ∂_v^2, then we have that

$$\#(l * e^{a_ux_u}\partial_u^2) \leq k - 1$$

This implies that we have proven the theorem by induction. Let us assume that l has only one partial derivative, say ∂_u^2. Without loss of generality, we may assume that l has two different basis terms i.e. $e^{a_1x_1}\cdots e^{a_nx_n}\partial_u^2$ and $e^{b_1x_1}\cdots e^{b_nx_n}\partial_v^2$ such that $a_1 \neq b_1$. This implies that

$$\#(\partial_1^2 * (e^{-a_1x_1}\partial_1 * l)) \leq k - 1$$

This implies that $(\partial_1^2*(e^{-a_1x_1}\partial_1^2*l)) \in I$ and $\#(\partial_1^2*(e^{-a_1x_1}\partial_1^2*l)) \leq k-1$. Thus we have that $I = \overline{WN_{n,0,0_{[2]}}}$ by induction. Therefore we have proven the theorem. □

Corollary 1. *The non-associative algebra* $\overline{WN_{1,0,0}}_{[2]}$ *is simple.*

Proof. The proof of the corollary is straightforward by Theorem 1. Thus let us omit the proof of the corollary. □

Theorem 2. *The semi-Lie algebra* $\overline{WN_{n,0,0}}_{[2]_{[,]}}$ *is simple.*

Proof. Since every element of the semi-Lie algebra $\overline{WN_{n,0,0}}_{[2]_{[,]}}$ is full, the proof of the theorem is straightforward by Theorem 1. So let omit it. □

Corollary 2. *The Lie algebra* $\overline{WN_{0,0,n}}_{1_{[,]}}$ *is simple.*

Proof. Since the Lie algebra $\overline{WN_{n,0,0}}_{1_{[,]}}$ is isomorphic to the Lie algebra $\overline{WN_{0,0,n}}_{1_{[,]}}$, the Lie algebra $\overline{WN_{0,0,n}}_{1_{[,]}}$ is simple. □

The semi-Lie algebra $\overline{WN_{0,0,n}}_{[2]_{[,]}}$ is called the Witt type semi-Lie algebra [13]. $WN_{1,0,0_{r[,]}}$ is self-centralizing [7].

3. Automorphism group $Aut_{non}(\overline{WN_{1,0,0}}_{[2]})$

For any non-associative algebras A and B, an additive $\mathbf{F}$-map θ from A to B is an algebra homomorphism, if $\theta(l_1 * l_2) = \theta(l_1) * \theta(l_2)$ holds for any $l_1, l_1 \in A$. For any algebra homomorphism from A to B is a monomorphism, if θ is injective. For any algebra homomorphism from A to B is a endomorphism, if θ is surjective. Note that by Corollary 1, the non-associative algebra $\overline{WN_{1,0,0}}_{[2]}$ spanned by $\{e^{ax}\partial^2 | a \in \mathbf{Z}\}$ is simple. For any Lie algebras L_1 over $\mathbf{F}$, an $\mathbf{F}$-map θ from L_1 to L_2 is Lie homomorphism, if $\theta([l_1, l_2]) = [\theta(l_1), \theta(l_2)]$ holds for any $l_1, l_2 \in L_1$. For the Lie homomorphism θ, from L_1 to L_2 (resp. L_1) if it is bijective, then we call it isomorphism (resp. automorphism) etc.. The matrix ring $M_m(\mathbf{F})$ is not imbedded in the the non-associative algebra $\overline{WN_{n,0,0}}_{[2]}$

as **F**-algebras, since any subalgebra A of the the non-associative algebra $\overline{WN_{n,0,0}}_{[2]}$ does not have the identity element of A such that the dimension of A is greater than 3.

Proposition 1. *For any non-associative algebra endomorphism θ of $\overline{WN_{1,0,0}}_{[2]}$, if θ is non-zero, then θ is injective.*

Proof. Let θ be a non-associative algebra endomorphism θ of $\overline{WN_{1,0,0}}_{[2]}$. $Ker(\theta)$ is an ideal of $\overline{WN_{1,0,0}}_{[2]}$. By Corollary 1, either $Ker(\theta) = 0$ holds or $Ker(\theta) = \overline{WN_{1,0,0}}_{[2]}$ holds. Since θ is not the zero map, $Ker(\theta) = 0$. This implies that θ is injective. So we have proven the proposition. □

Note 1. For any basis element $e^{ax}\partial$ of $\overline{WN_{1,0,0}}_{[2]}$, if we define **F**-linear maps θ_{+,d_1} and θ_{-,d_2} of $\overline{WN_{1,0,0}}_{[2]}$, as follows:

$$\theta_{+,d_1}(e^{kx}\partial^2) = d_1^k e^{kx}\partial^2$$

and

$$\theta_{-,d_2}(e^{kx}\partial^2) = d_2^k e^{-kx}\partial^2$$

then θ_{+,d_1} and θ_{-,d_2} can be linearly extended to non-associative algebra automorphisms of $\overline{WN_{1,0,0}}_{[2]}$ where $d_1, d_2 \in \mathbf{F}^{\bullet}$.

Lemma 2. *For any non-associative algebra automorphism θ of $\overline{WN_{1,0,0}}_{[2]}$, $\theta(\partial^2) = c\partial^2$ holds where c is a non-zero scalar.*

Proof. Let θ be the non-associative algebra automorphism θ of $\overline{WN_{1,0,0}}_{[2]}$ in the lemma. Since ∂ is a basis of the right annihilator of $\overline{WN_{1,0,0}}_{[2]}$, ∂ is invariant under any automorphism of $\overline{WN_{1,0,0_2}}[2]$. This implies that $\theta(\partial^2) = c\partial^2$ holds where c is a non-zero scalar. □

Lemma 3. *For any θ in the non-associative algebra automorphism group $Aut_{non}(\overline{WN_{1,0,0}}_{[2]})$ of $\overline{WN_{1,0,0}}_{[2]}$ is θ is either θ_{+,d_1} or θ_{-,d_2} where $d_1, d_2 \in \mathbf{F}$.*

Proof. Let θ be the non-associative algebra automorphism of $\overline{WN_{1,0,0}}_{[2]}$ in the lemma. By Lemma 3, $\theta(\partial^2) = c\partial^2$ holds where c is a non-zero scalar. By Lemma 3 and $\theta(\partial^2 * e^x\partial^2) = \theta(e^x\partial^2)$, we have that

$$c\partial^2 * \theta(e^x\partial^2) = \theta(e^x\partial^2) \tag{6}$$

$\theta(e^x\partial^2)$ can be written as follows:

$$\theta(e^x\partial^2) = C(b_1)e^{b_1x}\partial^2 + \cdots + C(b_t)e^{b_tx}\partial^2 \tag{7}$$

where $C(b_1), \cdots, C(b_t) \in \mathbf{F}$ and $b_1 > \cdots > b_t$. By (6) and (7), we have that $cb_1^2 = 1$. If $c \neq 1$, then $b_1 \notin \mathbf{N}$. This implies that $c = 1$ and $b_1 = \pm 1$.
Case I. Let us assume that $c = 1$ and $b_1 = -1$ hold. Since $\theta(e^{-x}\partial^2 * e^x\partial^2) = \theta(\partial^2)$, we may put $\theta(\partial^2) = \partial^2$ and $\theta(e^x\partial^2) = d_1e^{-x}\partial^2$ where $d_1 \in \mathbf{F}^\bullet$. We also have that

$$\theta(e^{-x}\partial^2) = d_1^{-1}e^{-x}\partial^2 \tag{8}$$

By $\theta(e^x\partial^2 * e^x\partial^2) = e^{2x}\partial^2$, we have that

$$\theta(e^{2x}\partial^2) = d_1^2e^{2x}\partial^2 \tag{9}$$

By (8) and (9), we may assume that $\theta(e^{kx}\partial^2) = d_1^ke^{kx}\partial^2$ holds by induction on $k \in \mathbf{N}$ of $e^{kx}\partial^2$. By $\theta(e^x\partial^2 * e^{kx}\partial^2) = ke^{(k+1)x}\partial^2$, we have that $\theta(e^{(k+1)x}\partial^2) = d_1^{k+1}e^{(k+1)x}\partial^2$. This proves that $\theta(e^{kx}\partial^2) = d_1^ke^{kx}\partial^2$ holds for any $k \in \mathbf{N}$. Symmetrically, we can prove that

$$\theta(e^{kx}\partial^2) = d_1^ke^{kx}\partial^2 \tag{10}$$

holds for any negative integer k by (8). This implies that θ is the non-associative algebra automorphism θ_{+,d_1} which is defined in Note 1.
Case II. Let us assume that $c = 1$ and $b_1 = -1$ hold. Without loss of generality, we may put $\theta(\partial^2) = \partial^2$ and $\theta(e^x\partial^2) = d_2e^{-x}\partial^2$ where $d_2 \in \mathbf{F}$. By $\theta(e^x\partial^2 * e^x\partial^2) = e^{2x}\partial^2$, we have that

$$\theta(e^{2x}\partial^2) = d_2^2e^{-2x}\partial^2 \tag{11}$$

By induction on $k \in \mathbf{N}$ of $e^{kx}\partial^2$, we can prove that

$$\theta(e^{kx}\partial^2) = d_2^ke^{-kx}\partial^2 \tag{12}$$

By $\theta(e^{-x}\partial^2 * e^x\partial^2) = \partial^2$, we have that $\theta(e^{-x}\partial^2) * d_2e^{-x}\partial^2) = \partial^2$. This implies that $\theta(e^{-x}\partial^2) = d_2^{-1}e^x\partial^2$. By induction on $k \in \mathbf{N}$ of $e^{kx}\partial^2$, we can prove that

$$\theta(e^{-kx}\partial^2) = d_2^{-k}e^{kx}\partial^2 \tag{13}$$

This implies that θ is the non-associative algebra automorphism θ_{-,d_2} which is defined in Note 1. By Case I and Case II, we have proven the lemma. □

Theorem 3. *The non-associative algebra automorphism group* $Aut_{non}(\overline{WN_{1,0,0_{[2]}}})$ *of* $\overline{WN_{1,0,0_{[2]}}}$ *is generated by* θ_{+,d_1} *and* θ_{-,d_2} *where* $d_1, d_2 \in \mathbf{F}^\bullet$.

Proof. Let θ be the non-associative algebra automorphism of $\overline{WN_{1,0,0_{[2]}}}$. By Lemma 3, θ is either θ_{+,d_1} or θ_{-,d_2} where $d_1, d_2 \in \mathbf{F}$.
So $Aut_{non}(\overline{WN_{1,0,0_{[2]}}})$ of $\overline{WN_{1,0,0_{[2]}}}$ is generated by θ_{+,d_1} and θ_{-,d_2}. Therefore we have proven the theorem. □

Corollary 3. *The non-associative algebra automorphism group* $Aut_{non}(\overline{WN_{1,0,0_{[2]}}})$ *of the non-associative algebra* $\overline{WN_{1,0,0_{[2]}}}$ *is a non-abelian group.*

Proof. By Theorem 3, The non-associative algebra automorphism group $Aut_{non}(\overline{WN_{1,0,0_{[2]}}})$ of the non-associative algebra $\overline{WN_{1,0,0_{[2]}}}$ is generated by θ_{+,d_1} and θ_{-,d_2} where $d_1, d_2 \in \mathbf{F}^\bullet$. Thus it is enough to check that $\theta_{+,d_1} \circ \theta_{-,d_2} \neq \theta_{-,d_2} \circ \theta_{+,d_1}$ where $\circ$ is the composition of the non-associative algebra automorphisms θ_{+,d_1} and θ_{-,d_2}. But it is trivial to check the inequality by taking some basis element of the non-associative algebra $\overline{WN_{1,0,0_{[2]}}}$. So let omit its proof. This completes the proof of the corollary. □

Proposition 2. *The non-associative algebra* $\overline{WN_{1,0,0_{[2]}}}$ *is not isomorphic to the non-associative algebra* $\overline{WN_{0,1,0_{[2]}}}$ *as non-associative algebras.*

Proof. Since the non-associative algebra $\overline{WN_{0,1,0}}_{[2]}$ has a right unity and the non-associative algebra $\overline{WN_{1,0,0}}_{[2]}$ does not have a right unity, the proof of the proposition is straightforward. □

Another proof of Proposition 2 can be proved by reviewing Theorem 2 and Theorem 1 in the paper [8], since they have different non-associative algebra automorphism groups.

Since $WN_{1,0,0_{1}}{}_{[,]}$ is self-centralizing, the semi-Lie algebra $WN_{1,0,0_{1}}{}_{[,]}$ has similar semi-Lie automorphisms θ_{+,d_1} and θ_{-,d_2} as the non-associative algebra $WN_{1,0,0_1}$ in Note 1. Thus we have the following theorem and leu us omit its proof because of the proof of Theorem 3 and the above comments.

Theorem 4. *The semi-Lie algebra automorphism group* $Aut_{semi-Lie}(\overline{WN_{1,0,0}}_{[2]_{[,]}})$ *of* $\overline{WN_{1,0,0}}_{[2]_{[,]}}$ *is generated by* θ_{+,d_1} *and* θ_{-,d_2} *where* $d_1, d_2 \in \mathbf{F}^{\bullet}$.

Proposition 3. *The semi-Lie algebra* $\overline{WN_{1,0,0}}_{[2]_{[,]}}$ *(resp. the non-associative algebra* $\overline{WN_{0,1,0}}_{[2]}$*) does not hold its Jacobian conjecture.*

Proof. It is easy to define a non-zero endomorphism θ of $\overline{WN_{1,0,0}}_{[2]_{[,]}}$ (resp. $\overline{WN_{0,1,0}}_{[2]}$) which is not surjective. This completes its proof. □

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