A Note on an Analogous Continued Fraction of Ramanujan

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ABSTRACT. We give an integral representation for an analogous continued fraction of Ramanujan, for this we first prove an interesting identity.

1. Preliminaries

One of the most celebrated theorems given by Ramanujan is the Rogers-Ramanujan continued fraction

$\frac{C(q)}{1 + \frac{q^2}{1 + \frac{q^4}{1 + \cdots}}} = \prod_{n=0}^{\infty} \frac{1 - q^{5n+2}}{1 - q^{5n+4}}.$

In his first letter to Hardy, Ramanujan besides this result, also gave several astounding corollaries of this result. Hardy found the results very interesting and was not able to prove them. This may be termed as the starting point of the unique relation between Hardy and Ramanujan.

Ramanujan gave, in the words of Andrews [1, p.188], “the following outrageous formula”.

$\frac{q^{1/5}}{C(q)} = \frac{\sqrt{5} - 1}{2} \exp \left\{ -\frac{1}{5} \int_0^1 \frac{(1-t)^5(1-t^{10}) \cdots dt}{(1-t^5)(1-t^{10}) \cdots t} \right\}.$

Andrews [1, p.206] has given a proof of (2), by using the following result, also due to Ramanujan,

$\frac{(q; q)^{5/5}_{\infty}}{(q^5; q^5)^{\infty}_{\infty}} = 1 - 5 \left[ \frac{q}{1-q} - \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \frac{4q^4}{1-q^4} + \frac{6q^5}{1-q^5} \right].$

This can be written as

$\frac{1}{5q} - \sum_{m=0}^{\infty} \frac{(5m + 1)q^{5m}}{1 - q^{5m+1}} + \frac{(5m + 4)q^{5m+3}}{1 - q^{5m+4}} - \frac{(5m + 2)q^{5m+1}}{1 - q^{5m+2}} - \frac{(5m + 3)q^{5m+2}}{1 - q^{5m+3}} = \frac{1}{5q} \frac{(q; q)^{5/5}_{\infty}}{(q^5; q^5)^{\infty}_{\infty}}.$

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Taking the logarithmic derivative of (2) we have (4). This simple proof was
given by Andrews [1,p. 206]. Now it only remains to prove (4). The proof of this
identity was given by Bailey [3, pp 29-32].

We consider an analogous continued fraction

\[
C(-q, q) = 1 + \frac{(1 + 1/q)q}{1 + (1 + 1/q + 1/q^{3} + \cdots)}
\]

In this paper we give an identity analogous to (4) and give an integral representation
for \(C(-q, q)\) analogous to (2).

2. Notations

We shall use the following usual basic hypergeometric notations:

For \(|q| < 1,

\[
(a)_{0} = 1, \\
(aq^{k})_{n} = (1 - a)(1 - aq^{k}) \cdots (1 - aq^{k(n-1)}), \quad 0 < n < \infty \\
(a_{1}, a_{2}, \ldots a_{r}; q^{k})_{n} = (a_{1}; q^{k})_{n}(a_{2}; q^{k})_{n} \cdots (a_{k}; q^{k})_{n}, \\
\]

\[
\left[a_{1}, \ldots, a_{r}; b_{1}, \ldots, b_{r}; q; z\right]_{\infty} = \sum_{n=-\infty}^{\infty} \frac{(a_{1}, a_{2}, \ldots, a_{r}; q)_{n}}{(b_{1}, b_{2}, \ldots, b_{r}; q)_{n}} z^{n},
\]

where \(|b_{1}b_{2}\cdots b_{r}/a_{1}a_{2}\cdots a_{r}| < |z| < 1.

3. The identity

\[
1 - 8 \sum_{m=0}^{\infty} \left[ \frac{(4m + 1)q^{4m+1}}{1 - q^{4m+1}} - \frac{2(4m + 2)q^{4m+2}}{1 - q^{4m+2}} - \frac{(4m + 3)q^{4m+3}}{1 - q^{4m+3}} \right]
\]

\[
= \frac{(-q^{2}; q^{2})_{\infty}(q; q)_{\infty}^{2}(q^{4}; q^{4})_{\infty}}{(-q^{4}; q^{4})_{\infty}}
\]

\[
= \frac{(q, q)_{\infty}^{4}}{(-q, q)_{\infty}^{4}}.
\]

This identity is analogue to the identity of Ramanujan [4].


\[
6\psi_{6}\left[qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, d, e; a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq/d, aq/e; q: qa^{2}/bcde\right]
\]

\[
= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, qa; q)_{\infty}}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^{2}/bcde; q)_{\infty}}.
\]
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Taking \( c = -a^{\frac{1}{2}}, b = a^{\frac{1}{2}} w, c = a^{\frac{1}{2}} w^2, d = a^{\frac{1}{2}} w^3 \) where \( w = e^{\frac{i\pi}{4}} \) in (8), we have

\[
\psi_{44} \left[ qa^{\frac{1}{2}}, wa^{\frac{1}{2}}, w^2 a^{\frac{1}{2}}, w^3 a^{\frac{1}{2}} \right] ; q^4 = \frac{(aq, qw, q, -qw^3, qw^3, q, -qw, q, q/a; q)_\infty}{(qa^2 w^3, qa^2 w^2, qa^2 w, -qa^2 \pi w^3, qa^2 \pi w^2, qa^2 \pi w, -qa^2 \pi, q; q)_\infty}. \tag{9}
\]

The left side of (9)

\[
= 1 + \sum_{n=-\infty}^{\infty} \frac{(qa^2; q)_n (a^{\frac{1}{2}} w; q)_n (-a^{\frac{1}{2}}; q)_n (-a^{\frac{1}{2}} w; q)_n q^n}{(a^2; q)_n (-qa^2 w; q)_n (qa^2 \pi w; q)_n}
\]

\[
= 1 + \sum_{n=-\infty}^{\infty} \frac{(1 - a^{\frac{1}{2}} q^n)(1 + a)(1 + a^{\frac{1}{2}})q^n}{(1 - a^{\frac{1}{2}})(1 + a^2 q^n)(1 + a^{\frac{1}{2}})q^n}
\]

\[
= 1 + \frac{(1 + a)(1 + a^{\frac{1}{2}})^2}{1 - a} \sum_{n=-\infty}^{\infty} \frac{(1 - a^{\frac{1}{2}} q^n)^2 q^n}{1 - a^2 q^{4n}}
\]

\[
= 1 + \frac{(1 + a)(1 + a^{\frac{1}{2}})^2}{1 - a} \sum_{n=1}^{\infty} \left[ \frac{q^n a(a - 1) - 2q^{2n} (a^{\frac{1}{2}} - a^{\frac{5}{2}})}{1 - a^2 q^{4n}(a^{\frac{1}{2}} - q^{4n})}
\]

\[
+ \frac{q^{3n}(a^3 - 1) + q^{5n}(a^3 - 1) + 2q^{6n}(a^{\frac{1}{2}} - a^{\frac{5}{2}}) + q^{7n}(-a + a^2)}{(1 - a^2 q^{4n})(a^{\frac{1}{2}} - q^{4n})} \right]
\]

\[
= 1 - (1 + a)(1 + a^{\frac{1}{2}})^2 \times \sum_{n=1}^{\infty} \left[ \frac{aq^n - 2a^{\frac{1}{2}} (a + 1)q^{2n}}{(1 - a^2 q^{4n})(a^{\frac{1}{2}} - q^{4n})}
\]

\[
+ \frac{q^{3n}(a^2 + a + 1) + q^{5n}(a^2 + a + 1) - 2a^{\frac{1}{2}}(a + 1)q^{6n} + aq^{7n}}{(1 - a^2 q^{4n})(a^{\frac{1}{2}} - q^{4n})} \right].
\]

Taking limit as \( a \to 1 \), the left side of (9)

\[
= 1 - 8 \sum_{n=1}^{\infty} \left[ \frac{q^n - 4q^{2n} + 3q^{3n} + 3q^{5n} - 4q^{6n} + q^{7n}}{(1 - q^{4n})^2} \right].
\]

Now

\[
\sum_{n=1}^{\infty} \frac{q^{an}}{(1 - q^{4n})^2} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (m + 1)q^{(4m+\alpha)n}
\]

\[
= \sum_{m=0}^{\infty} \frac{(m + 1)q^{4m+\alpha}}{1 - q^{4m+\alpha}}.
\]
Using (10), the left side of (9)

\[1 - 8 \sum_{m=0}^{\infty} \left[ \frac{(4m + 1)q^{4m+1}}{1 - q^{4m+1}} - \frac{2(4m + 2)q^{4m+2}}{1 - q^{4m+2}} + \frac{(4m + 3)q^{4m+3}}{1 - q^{4m+3}} \right].\]

The right side of (9)

\[= \prod_{n=1}^{\infty} \frac{(1 - q^n)^2(1 - aq^n)(1 - q^n/a)(1 - q^{2n})(1 + q^n)(1 - \sqrt{a}q^n)(1 - q^n/\sqrt{a})}{(1 - a^2q^{4n})(1 - q^{4n}/a)(1 + qa^n)(1 + q^n/\sqrt{a})(1 - q^{2n})}.\]

Taking the limit as \(a \to 1\), the right side of (9)

\[= \prod_{n=1}^{\infty} \frac{(1 - q^n)^2(1 + q^{2n})(1 - q^n)^2}{(1 - q^{4n})(1 - q^{2n})(1 + q^n)^2} = \frac{(q; q)^4_{\infty}}{(-q; q)^4_{\infty}},\]

which proves (7).

4. Integral representation

**Theorem.** We shall prove

\[\frac{q^2}{[C(-q, q) - 1]} = A \exp \left\{ -\frac{1}{8} \int_{q}^{1} \frac{(t; t)_{\infty}^4}{(-t; t)_{\infty}^4} \frac{dt}{t} \right\}.\]

**Proof.** Now

\[C(-q, q) = 1 + \frac{(q^2; q^4)_{\infty}^2}{(q; q)_{\infty}^4 (q^3; q^4)_{\infty}}.\]

Taking logarithmic derivative

\[\frac{C'(-q, q)}{[C(-q, q) - 1]} = -\sum_{n=0}^{\infty} \left[ \frac{2(4n + 2)q^{4n+1}}{1 - q^{4n+2}} - \frac{(4n + 1)q^{4n}}{1 - q^{4n+1}} - \frac{(4n + 3)q^{4n+2}}{1 - q^{4n+3}} \right] = \frac{1}{8q} - \frac{1}{8q} \frac{(q; q)^4_{\infty}}{(-q; q)^4_{\infty}}.\]

So

\[\frac{1}{8q} \frac{C'(-q, q)}{[C(-q, q) - 1]} = \frac{1}{8q} \frac{(q; q)^4_{\infty}}{(-q; q)^4_{\infty}}.\]

Hence by (8)

\[\frac{q^2}{[C(-q, q) - 1]} = A \exp \left\{ -\frac{1}{8} \int_{q}^{1} \frac{(t; t)_{\infty}^4}{(-t; t)_{\infty}^4} \frac{dt}{t} \right\}.\]
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which proves the theorem and is the integral representation of the function.

In the end we give another identity:

\[(q^4; q^4)\infty [C(-q, q) - 1] = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2}}{(q; q^4)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2 + 4n + 2}}{(q^3; q^4)_{n+1}}.\]

Using the quintuple identity in the form

\[\frac{(x^2; q)\infty (q/x^2; q)\infty (q; q)\infty}{(x; q)\infty (q/x; q)\infty} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} x^{2n}}{(x; q)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n^2 + 3n + 2)/2} x^{-2n-2}}{(q/x; q)_{n+1}}\]

and making \(q \to q^4\) and taking \(x = q\) in (15) we have immediately (14).

The analogous identity of Ramanujan on the base 5

\[(q^5; q^5)\infty C(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2 - n)/2}}{(q; q^5)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2 + 11n + 6)/2}}{(q^4; q^5)_{n+1}}\]

was proved by Andrews [1, p.200]. It can also be proved directly by making \(q \to q^5\) and taking \(x = q\) in (15).

5. Conclusion

In this paper we have given the analogue of the ‘astounding’ identity of Ramanujan and ‘outrageous’ integral representation. In a subsequent paper we shall find the analogue of other remarkable identities of Ramanujan [5].

References


