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ISOMETRIES ON THE SPACE OF CONTINUOUS FUNCTIONS WITH FINITE CODIMENSIONAL RANGE

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Abstract. For compact Hausdorff spaces X and Y let $C(X)$ and $C(Y)$ denote the complex Banach spaces of complex valued continuous functions on X and Y, respectively. Also let $T : C(X) \to C(Y)$ be a linear isomerty. Necessary and sufficient conditions are given such that range of T has finite codimension.

Let X and Y be compact Hausdorff spaces and $T : C(X) \to C(Y)$ be a linear isometry such that the codimension of $ranT$ in Y be $n \geq 1$. If $X = Y$, $n = 1$ and $\bigcap_{m=1}^{\infty} ranT^m = \{0\}$ then T is called a shift isometry see [1] and [2].

Now we give the well known theorem of Holsztynski [3] on which the results of this paper is based.

Theorem 1. Let X and Y be two compact Hausdorff spaces. Let $T : C(X) \to C(Y)$ be a linear isometry. Then there is a closed set $Y_0 \subseteq Y$ and a continuous function $\varphi : Y_0 \to X$ from Y_0 onto X, and there exists a function $\alpha \in C(Y)$ such that $||\alpha|| = 1$, $|\alpha(y)| = 1$ for

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$y \in Y_0$, and

$$(Tf)(y) = \alpha(y) f(\varphi(y)) \quad \text{for all} \quad y \in Y_0 \quad \text{and} \quad f \in C(X).$$

Theorem 2. Suppose that T is a linear isometry of $C(X)$ into $C(Y)$ such that $ranT$ has codimension $n \geq 1$. Then there is a closed set $Y_0 = Y \backslash A$ where A is a subset of Y with at most n points each of which an isolated point, and there is a continuous surjective map $\varphi : Y_0 \rightarrow X$ and a unimodular function $\alpha \in C(Y_0)$ such that

$$(Tf)(y) = f(\varphi(y))\alpha(y)$$

for all $y \in Y_0$.

Proof. By the Holsztynski theorem there exists a non-empty closed subset Y_0 of Y, a continuous map φ from Y_0 onto X, and a unimodular function $\alpha \in C(Y_0)$ such that $(Tf)(y) = f(\varphi(y))\alpha(y)$ for all $y \in Y_0$. Suppose $A = Y \backslash Y_0$ contained $n+1$ distinct points $\{a_1, a_2, \cdots, a_{n+1}\}$. Let $K_i = \{a_i\}$ and $V_i = Y \backslash Y_0 \cup (A - \{a_i\})$ for $i = 1, \cdots, n+1$. Applying the Urysohn's lemma we obtain functions $f_1, \cdots, f_{n+1}$ in $C(Y)$ such that $f_i(a_i) = ||f_i|| = 1$ and $f_i(a_j) = 0, \quad i \neq j$ and $f_i(y) = 0$ for all $y \in Y_0$. Assume that f_i is in $ranT$ for some i. Then there exists a function $h_i \in C(X)$ such that

$$(Th_i)(y) = h_i(\varphi(y))\alpha(y) = f_i(y).$$

Consequently, $h_i(\varphi(y)) = 0$ for all $y \in Y_0$. Since φ is surjective, $h_i \equiv 0$, and so $f_i \equiv 0$ which is a contradiction. Since $\{f_1, \cdots, f_{n+1}\}$ is a linearly independent subset of $C(Y)$, we see that $ranT$ can not have codimension n. But Y_0 is a closed subset of the Hausdorff Space Y, and so the points a_i's are isolated points (if exist). □

Theorem 3. Let T be a linear isometry of $C(X)$ into $C(Y)$ such that $ranT$ has codimension $n \geq 1$, and let Y_0, φ, A, and α be as in

Theorem 2. Then $\varphi^{-1}(x)$ has at most $n+1$ elements for each $x \in X$. Furthermore, if $x_0 \in X$ and

$$\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_{n+1}) = x_0$$

for some distinct points $a_1, a_2, \cdots, a_{n+1}$ in Y_0, then $\varphi^{-1}(x)$ is a singleton for each $x \in X \backslash \{x_0\}$. In addition, if $A \neq \emptyset$ then φ is injective and hence a homeomorphism.

Proof. Suppose that $y_1, y_2, \cdots, y_{n+2}$ are distinct points of Y_0 such that

$$\varphi(y_1) = \varphi(y_2) = \cdots = \varphi(y_{n+2}).$$

Using Urysohn's lemma we obtain functions $f_i, i = 1, \cdots, n+1$ in $C(Y)$ such that

$$f_i(y_i) = 1 \text{ and } f_i(y_j) = 0, \quad j = 1, 2, \cdots, n+2, \ j \neq i.$$

Since $\{f_1, f_2, \cdots, f_{n+1}\}$ is a linearly independent set and $codim(ranT) = n$, there exist constants $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ not all zero, such that

$$g = \lambda_1 f_1 + \cdots + \lambda_{n+1} f_{n+1} \in ranT.$$

Let $g = Th$ for some $h \in C(X)$. So

$$|g(y_i)| = |(Th)(y_i)| = |h(\varphi(y_i))|$$

has a constant value C for every $i = 1, \cdots, n+2$. But $g(y_{n+2}) = 0$ and so $g(y_i) = 0, \ i = 1, \cdots, n+1$. This implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1} = 0$, which is a contradiction. Therefore, $\varphi^{-1}(x)$ has at most $n+1$ elements.

Now suppose that x_1 and x_2 are distinct points in X such that

$$\varphi^{-1}(x_1) = \{a_1, a_2, \cdots, a_{n+1}\}$$

where a_i's are distinct and $\varphi^{-1}(x_2) = \{b_1, b_2\}$ where $b_1 \neq b_2$. Consider continuous functions $g_1, g_2, \cdots, g_{n+1}$ on Y such that for $i = 1, \cdots, n$

$$g_i(b_1) = g_i(b_2) = 0, \quad g_i(a_i) = 1, \quad g_i(a_j) = 0 \quad (j \neq i),$$

$$g_{n+1}(a_i) = g_{n+1}(b_2) = 0, \quad g_{n+1}(b_1) = 1.$$

Since the codimension of range of T is n there exist non-zero constants $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ such that $\lambda_1 g_1 + \cdots + \lambda_{n+1} g_{n+1} \in ranT$. As above we can show that

$$|(\lambda_1 g_1 + \cdots + \lambda_{n+1} g_{n+1})(a_i)| = c, \quad i = 1, \cdots, n+1$$

and

$$|(\lambda_1 g_1 + \cdots + \lambda_{n+1} g_{n+1})(b_i)| = d, \quad i = 1, 2$$

for some constants c and d. It follows that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1} = 0$, a contradiction.

If $A \neq \emptyset$ then Theorem 2 implies that A has at most n isolated points. So the function defined by

$$f(y) = \begin{cases} 0, & (y \in Y_0) \\ 1, & (y \in A) \end{cases}$$

is continuous. Now, by the same argument as in the proof of Theorem 2 one can show that f does not belong to the $ranT$.

Suppose that there exist distinct points y_1 and y_2 in Y_0 such that $\varphi(y_1) = \varphi(y_2)$. Let g be a continuous function on Y such that $g(y) = 0$ on $A \cup \{y_1\}$ and $g(y_2) = 1$. If $g \in ranT$ then $g = Th$ for some $h \in C(X)$ and so $0 = g(y_1) = (Th)(y_1) = h(\varphi(y_1))\alpha(y_1)$; also, $1 = (Th)(y_2) = h(\varphi(y_2))\alpha(y_2)$. Therefore,

$$0 = |h(\varphi(y_1))| = |h(\varphi(y_2))| = 1$$

which is a contradiction. So there exist non-zero constants λ_1 and λ_2 such that $\lambda_1 f + \lambda_2 g \in ranT$. It implies that $\lambda_1 = \lambda_2 = 0$, again a contradiction. Hence, φ is a homeomorphism. $\square$

Example 1. Let $X = Y$ be the one point compactification of the natural nubmers, so $C(X)$ may be identified with the sequence space c of all convergent sequence. Define the operator T on $C(X) \cong c$ by

$$T(x_1, x_2, \cdots) = (\overbrace{x_1, x_1, \cdots, x_1}^{n\ times}, x_2, x_3, \cdots).$$

It is clear that T is an isometry whose range has codimension n. In fact,

$$(x_1, x_2, x_3, \cdots) + ranT = (x_1 - x_{n+1}, x_2 - x_{n+1}, \cdots, x_n - x_{n+1}, 0, 0, 0, \cdots) + ranT.$$

Let $A_k = \{1, 2, \cdots, k\}$ for $k = 1, \cdots, n$ and $A_0 = \emptyset$. For a fixed k, put $Y_0 = Y \backslash A_k$ and define $\varphi : Y_0 \to X$ by

$$\varphi(k+1) = \varphi(k+2) = \cdots = \varphi(n) = 1, \varphi(n+i) = i+1, \quad i \geq 1.$$

If $\alpha(i) = 1$, $(i \geq 1)$ then we see that

$$(Tf)(i) = \alpha(i) f(\varphi(i)), \quad (i \in Y_0).$$

We recall that the surjective map $P : Y \to X$ is said to be a quotient map, provided that a subset U of X is open if and only if $P^{-1}(U)$ is open in Y.

Corollary 1. Suppose that in Theorms 2 and 3 $A = \emptyset$ and $\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_{n+1}) = x_0$ for some distinct points $a_1, a_2, \cdots, a_{n+1}$ in Y and $x_0 \in X$. Then the map $\varphi : Y \to X$ is a quotient map.

Proof. Suppose V is a non-empty set in X for which $\varphi^{-1}(V)$ is open and let $x \in V$. If $x \neq x_0$ then, by Theorem 3, $\varphi^{-1}(x)$ is a singleton and so there are open neighborhoods U of $\varphi^{-1}(x)$ and W of $\{a_1, a_2, \cdots, a_{n+1}\}$ such that $U \cap W = \emptyset$. Hence $a_i \notin U \cap \varphi^{-1}(V)$ for $i = 1, \cdots, n+1$, which implies that the restriction of φ to the compact set $\overline{U \cap \varphi^{-1}(V)}$ is injective, and so is a homeomorphism. It follows that $\varphi(U) \cap V$ is an open neighborhood of x in V, because its inverse image $\varphi^{-1}(\varphi(U) \cap V) = U \cap \varphi^{-1}(V)$ is open. Thus, x is an interior point of V.

Now, suppose that $x = x_0$, and let W_1 and W_2 be disjoint open sets such that $a_1 \in W_1$ and $a_i \in W_2$, $i = 2, 3, \cdots, n+1$. Put $U = W_1 \cap \varphi^{-1}(V)$. So we have $a_1 \in \overline{U}$ and $a_i \notin \overline{U}$, $i = 2, \cdots, n+1$. Then the restriction of φ to $\overline{U}$ is a homeomorphism and consequently, $\varphi(U)$

is an open subset of V containing x. This shows that φ is a quotient map. $\square$

Corollary 2. Suppose that $A = \emptyset$ and $\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_{n+1}) = x_0$ for some distinct points $a_1, a_2, \cdots, a_{n+1}$ in Y. Then Y is homeomorphic to the quotient space $Y/\approx$ where $\approx$ is the equivalence relation defined as

$$y_1 \approx y_2 \text{ iff either } y_1 = y_2 \text{ or } \varphi(y_1) = \varphi(y_2) = x_0.$$

Proof. The result follows from preceding corollary and Theorem 11.2 of chapter 2 of [4]. $\square$

A topological space X is called extemely disconnected if the closure of each open set is open (as well as closed). It is known that if X is an extremely disconnected compact Hausdorff space and U and V are two disjoint open sets in X then $\overline{U}$ and $\overline{V}$ are disjoint.

Theorem 4. Let Y be an extremely disconnected compact Hausdorff space with no isolated points and X be any compact Hausdorff space. Then there is no an isometry T of $C(X)$ into $C(Y)$ so that $codim(ranT) = n \geq 1$, and $\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_{n+1})$ for distinct points $a_1, a_2, \cdots, a_{n+1}$ in Y_0.

Proof. Since Y has no isolated points, $A = \emptyset$. If there where such T, then it would follow from Corollary 2 that the qoutient space $Y/\approx$ is homeomorphic to Y, which implies that $Y/\approx$ is extremely disconnected. Suppose that W_1 and W_2 are disjoint open sets in Y with $a_1 \in W_1$ and $a_2 \in W_2$. Let $q : Y \to Y/\approx$ be the quotient map and let [a] denote the equivalence class of a in Y. Then $q : Y - \{a_1, a_2, \cdots, a_{n+1}\} \to (Y/\approx) - \{[a_1]\}$ is a homeomorphism; also $q(W_1) - \{[a_1]\}$ and $q(W_2) - \{[a_2]\}]$ are disjoint open sets in $Y/\approx$. On the other hand, $[a_1] \in q(W_1)$ and since

$Y/\approx$ has no isolated points, $[a_1] \in \overline{q(W_1)} = \overline{q(W_1) - [a_1]}$. Similarly, $[a_1] = [a_2] \in \overline{q(W_2) - [a_2]}$. But this is contradiction. □

Remark 1. If $(Z, \sum, \mu)$ is a non-atomic measure space then $L^\infty(Z, \sum, \mu)$ is isometrically isomorphic to $C(Y)$, where Y is an extremely disconnected compact Hausdorff space with no isolated points. So we can apply the preceding theorem.

Theorem 5. Let $X = [a, b]$ and $Y = [c, d]$, then there is no a linear isometry T of $C(X)$ into $C(Y)$ with $codim(ranT) = n \geq 1$ such that $\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_{n+1}) = x_0$ for some distinct points $a_1, a_2, \cdots, a_{n+1}$ in Y.

Proof. Since Y has no isolated points, $A = \emptyset$. Suppose that such T exists. Without loss of generality, assume that $a_1 < a_2 < \cdots < a_{n+1}$. Then using Theorem 3 we see that any point $t \in (a_1, a_2)$, $\varphi(t) \neq \varphi(a_1)$. The intermediate value theorem implies that φ must take on every value between $\varphi(a_1)$ and $\varphi(t)$ on the interval (a_1, t) and again on the interval (t, a_2) but, by Theorem 3, $\varphi^{-1}(x)$ is a singleton for each $x \in X\backslash\{x_0\}$, which is a contradiction. □

Remark 2. When $n = 1$, φ is not injective because otherwise it is a homeomorphism and so T is onto, which is a contradiction. Consequently, there exists a unique pair of points a_1, a_2 with $a_1 < a_2$ in $Y = [c, d]$ such that $\varphi(a_1) = \varphi(a_2)$. Therefore we have the well known result which states that there is no linear isomerty T of $C(X)$ into $C(Y)$ such that $ranT$ has codimension 1.

Theorem 6. Let $\varphi : Y \to X$ be continuous and surjective; moreover, $\alpha \in C(Y)$ be such that $|\alpha(y)| = 1$ for every $y \in Y$. Suppose, furthermore, that there exist distinct points $a_1, a_2, \cdots, a_{n+1}$ in $Y (n \geq 1)$ with

$\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_{n+1}) = x_0$ and $\varphi : Y\backslash\{a_1, \cdots, a_{n+1}\} \to X\backslash\{x_0\}$ is a homeomorphism. Then $T : C(X) \to C(Y)$ defined by

$$(Tf)(y) = \alpha(y)f(\varphi(y)), \quad f \in C(X)$$

is a linear isometry such that $ranT$ has codimension n.

Proof. Obviously, T is linear and isometry; so it suffices to show that $ranT$ has codimension n. Urysohn's lemma, now, gives us functions $g_1, g_2, \cdots, g_n$ in $C(Y)$ so that $g_i(a_i) = 1$ and $g_i(a_j) = 0$, $j \neq i$ where $i = 1, \cdots, n$ and $j = 1, \cdots, n+1$. If

$$\beta_1 g_1 + \beta_2 g_2 + \cdots + \beta_n g_n + ranT = ranT$$

for some constants $\beta_1, \cdots, \beta_n$ then there is a function $f \in C(X)$ such that

$$\beta_1 g_1(y) + \beta_2 g_2(y) + \cdots + \beta_n g_n(y) = (Tf)(y) = \alpha(y)f(\varphi(y)), \ y \in Y.$$

Taking $y = a_{n+1}$ and then $y = a_i, i = 1, \cdots, n$ we conclude that

$$\beta_1 = \beta_2 = \cdots = \beta_n = 0.$$

On the other hand, let $g \in C(Y)$ and put

$$\alpha_i = g(a_i) - \frac{\alpha(a_i)g(a_{n+1})}{\alpha(a_{n+1})} \quad i = 1, \cdots, n.$$

So

$$\frac{g(a_i) - \alpha_i}{\alpha(a_i)} = \frac{g(a_{n+1})}{\alpha(a_{n+1})}, \quad i = 1, \cdots, n.$$

If

$$h(t) = \frac{g(\varphi^{-1}(t)) - \sum_{i=1}^{n} \alpha_i g_i(\varphi^{-1}(t))}{\alpha(\varphi^{-1}(t))} \quad \text{for} \quad t \neq x_0$$

and $h(x_0) = g(a_{n+1})/\alpha(a_{n+1})$ then $h \in C(X)$ and $g - \sum_{i=1}^{n} \alpha_i g_i = Th$. It follows that $g + ranT = \sum_{i=1}^{n} \alpha_i g_i + ranT$. Hence, $ranT$ has codimension n. $\square$

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