KYUNGPOOK Math. J. 45(2005), 199-209

Properties of Topological Ideals and Banach Category Theorem

V. Renuka Devi
Department of Mathematics, Govindammal Aditanar College, Tiruchendur-628 215, Tamil Nadu, India

D. Sivaraj
Department of Mathematics, Aditanar College, Tiruchendur-628 216, Tamil Nadu, India
e-mail : ttn_sivaraj@sancharnet.in

T. Tamizh Chelvam
Department of Mathematics, M. S. University, Tirunelveli, Tamil Nadu, India

Abstract. An ideal space is $\mathcal{I}$-resolvable if it has two disjoint $\mathcal{I}$-dense subsets. We answer the question: If X is $\mathcal{I}$-resolvable, then is X $(\mathcal{I} \cup \mathcal{N})$-resolvable?, posed by Dontchev, Ganster and Rose. We give three generalizations of the well known Banach Category Theorem and deduce the Banach category Theorem as a corollary. Characterizations of completely codense ideals and $\mathcal{I}$-locally closed sets are given and their properties are discussed.

1. Introduction and preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski ([10]) and Vaidyanathaswamy ([16]). An *ideal* $\mathcal{I}$ on a topological space (X, τ) is a collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. A *σ-ideal* is an ideal which satisfies (iii) If $A_i \in \mathcal{I}$ for $i \in \mathbb{N}$, then $\cup\{A_i \mid i \in \mathbb{N}\} \in \mathcal{I}$, where $\mathbb{N}$ is the set of all natural numbers. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $()^\star$:$\wp$(X)$\rightarrow \wp(X)$, called the *local function* ([10]) of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $cl^\star()$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the *⋆-topology*, finer than τ is defined by $cl^\star(A) = A \cup A^\star(\mathcal{I}, \tau)$ ([15]). When there is no chance for confusion, we will simply write $A^\star$ for $A^\star(\mathcal{I}, \tau)$ and $\tau^\star$ or $\tau^\star(\mathcal{I})$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an *ideal space*. A subset A of an ideal space

Received February 13, 2004, and, in revised form, October 30, 2004.
2000 Mathematics Subject Classification: 54A05, 54A10.
Key words and phrases: codense, completely codense, regular and normal ideals, $\mathcal{I}$-resolvability, $\mathcal{I}$-open, $\mathcal{I}$-dense, $\mathcal{I}$-locally closed.

$(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-*open* ([9]) if $A \subset int(A^\star)$. A is said to be $\mathcal{I}$-*dense* ([4]) if $A^\star = X$. Recall that A is said to be $\star$-*dense in itself* if $A \subset A^\star$, A is $\tau^\star$-*closed* if $A^\star \subset A$ ([8]) and A is $\star$-*perfect* if $A = A^\star$. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X,τ) and $\mathcal{M}$ is the ideal of all first category(meager) subsets in (X,τ). If $\mathcal{I}$ is any ideal, then the set of all countable union of members of $\mathcal{I}$ is a σ-ideal and is denoted by $\mathcal{I}_\sigma$. $\mathcal{I}_\sigma$ is called the *countable extension* of $\mathcal{I}$. Clearly, $\mathcal{M} = \mathcal{N}_\sigma$.

By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A \subset X$, $cl(A)$ and $int(A)$ will, respectively, denote the closure and interior of A in (X,τ) and $cl^\star(A)$ and $int^\star(A)$ will, respectively, denote the closure and interior of A in $(X,\tau^\star)$. A subset A of a space (X,τ) is an α-*set* ([12]) if $A \subset int(cl(int(A)))$. The family of all α-sets in (X,τ) is denoted by τ^α. τ^α is a topology on X which is finer than τ. The closure of A in (X,τ^α) is denoted by $cl_\alpha(A)$. An open subset A of a space (X,τ) is said to be *regularopen* if $A = int(cl(A))$. The complement of a regularopen set is *regularclosed*. The family of all regularopen subsets of (X,τ) is denoted by RO(X,τ) or simply RO(X). A subset A of a space (X,τ) is said to be *preopen* ([11]) if $A \subset int(cl(A))$. The family of all preopen sets is denoted by PO(X,τ) or simply, PO(X). The largest preopen set contained in A is called the *preinterior* of A and is denoted by pint(A) and pint$(A) = A \cap int(cl(A))$. A is preopen if and only if there is a regularopen set G such that $A \subset G$ and $cl(A) = cl(G)$ [5, Proposition 2.1]. An ideal $\mathcal{I}$ in a space (X,τ) is said to be *compatible* with respect to τ ([8])(*supercompact* [15]), denoted by $\mathcal{I} \sim \tau$, if for every subset A of X and for each $x \in A$, there exists a neighborhood U of x such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. Given a space (X,τ) and ideals $\mathcal{I}$ and $\Im$ on X, the *extension* of $\mathcal{I}$ via $\Im$ ([9]), denoted by $\mathcal{I} \star \Im$, is the ideal given by $\mathcal{I} \star \Im = \big\{A \subset X \mid A^\star(\mathcal{I}) \in \Im\big\}$. In particular, $\mathcal{I} \star \mathcal{N} = \big\{A \subset X \mid int(A^\star(\mathcal{I})) = \phi\big\}$ is an ideal containing both $\mathcal{I}$ and $\mathcal{N}$ and $\widetilde{\mathcal{I}} = \mathcal{I} \star \mathcal{N} \sim \tau$. Also, if $\mathcal{F}$ is the family of all closed sets, $\langle \mathcal{I} \cap \mathcal{F} \rangle = \big\{A \subset X \mid$ there exists $B \in \mathcal{I} \cap \mathcal{F}$ such that $A \subset B\big\} = \big\{A \subset X \mid cl(A) \in \mathcal{I}\big\}$ is an ideal generated by $\mathcal{I} \cap \mathcal{F}$ ([9]). The following lemmas will be useful in the sequel.

Lemma 1.1 ([14]). *If $(X,\tau,\mathcal{I})$ is an ideal space, then the following are equivalent.*

(a) *$\mathcal{I}$ is codense.*

(b) *For every $A \in \tau$, $A \subset A^\star$.*

(c) *For every $A \in SO(X)$, $A \subset A^\star$.*

(d) *For every regularclosed set F, $F = F^\star$.*

Proof. (a) and (b) are equivalent by Theorem 6.1 of [8].

(b) $\Rightarrow$ (c). Suppose $A \in$ SO(X). Then there exists an open set H such that $H \subset A \subset cl(H)$. For any subset H of X, we have $H^\star = cl(H^\star) \subset cl(H)$, by Theorem 2.3(c) of [8]. Since H is open, $H \subset H^\star$ and so $H^\star = cl(H^\star) = cl(H)$. Therefore, $A \subset cl(H) = cl(H^\star) = H^\star \subset A^\star$ which implies that $A \subset A^\star$.

(c) $\Rightarrow$ (d). If F is regularclosed, then F is semiopen and closed. Since F is semiopen, by hypothesis, $F \subset F^{\star}$. Since F is closed, F is $\tau^{\star}$-closed and so $F = cl^{\star}(F) = F \cup F^{\star}$, which implies that $F^{\star} \subset F$. Hence $F = F^{\star}$.

(d) $\Rightarrow$ (a) is clear. □

Lemma 1.2 ([14]). *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. If $A \subset A^{\star}$, then $A^{\star} = cl(A^{\star}) = cl(A) = cl^{\star}(A)$.*

Proof. For any subset A of X, we have $A^{\star} = cl(A^{\star}) \subset cl(A)$, by Theorem 2.3(c) of [8]. $A \subset A^{\star}$ implies that $cl(A) \subset cl(A^{\star})$ and so $A^{\star} = cl(A^{\star}) = cl(A)$. Clearly, for every subset A of X, $cl^{\star}(A) \subset cl(A)$. Let $x \notin cl^{\star}(A)$. Then there exists a $\tau^{\star}$- open set G containing x such that $G \cap A = \phi$. By Theorem 3.1 of [8], there exists $V \in \tau$ and $I \in \mathcal{I}$ such that $x \in V - I \subset G$.

$$\begin{aligned} G \cap A = \phi &\Rightarrow (V - I) \cap A = \phi \Rightarrow (V \cap A) - I = \phi \Rightarrow ((V \cap A) - I)^{\star} = \phi \\ &\Rightarrow (V \cap A)^{\star} - I^{\star} = \phi \ [8,\ \text{Theorem2.3(f)}] \\ &\Rightarrow (V \cap A)^{\star} = \phi \ [8,\ \text{Theorem2.3(h)}] \\ &\Rightarrow V \cap A^{\star} = \phi \ [8,\ \text{Theorem2.3(g)}] \ \Rightarrow V \cap A = \phi. \end{aligned}$$

Since V is an open set containing x, $x \notin cl(A)$ and so $cl(A) \subset cl^{\star}(A)$. Hence $cl(A) = cl^{\star}(A)$. □

Lemma 1.3 ([14]). *If $(X, \tau, \mathcal{I})$ is an ideal space such that $\mathcal{I} \subset \mathcal{N}$, then $A^{\star}(\mathcal{N}) = (A^{\star}(\mathcal{I}))^{\star}(\mathcal{N})$ for every subset A of X.*

Proof. Since $\mathcal{I} \subset \mathcal{N}$, $A^{\star}(\mathcal{N}) \subset A^{\star}(\mathcal{I}) \subset cl(A)$ by Theorem 2.3(b) of [8]. Since $A^{\star}(\mathcal{N}) = cl(int(cl(A)))$[8], $cl(int(cl(A))) \subset A^{\star}(\mathcal{I}) \subset cl(A)$ which implies that $cl(int(cl(A))) \subset cl(int(A^{\star}(\mathcal{I}))) \subset cl(int(cl(A)))$. Therefore, $A^{\star}(\mathcal{N}) = cl(int(A^{\star}(\mathcal{I}))) = cl(int(cl(A^{\star}(\mathcal{I})))) = (A^{\star}(\mathcal{I}))^{\star}(\mathcal{N})$. □

Lemma 1.4 ([14]). *If $(X, \tau, \mathcal{I})$ is an ideal space, then $A^{\star}(\mathcal{I}\star\mathcal{N}) = (A^{\star}(\mathcal{I}\star\mathcal{N}))^{\star}(\mathcal{N})$ for every subset A of X.*

First, we prove the following Lemma 1.5.

Lemma 1.5. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{N} \subset \mathcal{I}$ and $\mathcal{I} \sim \tau$, then*

(i) *$A^{\star}(\mathcal{I})$ is regularclosed for every subset A of X and*

(ii) *$A^{\star}(\mathcal{I}) = (A^{\star}(\mathcal{I}))^{\star}(\mathcal{N})$ for every subset A of X.*

Proof. (i) Since $\mathcal{N} \subset \mathcal{I}$, $A^{\star}(\mathcal{I}) \subset A^{\star}(\mathcal{N})$ by Theorem 2.3(b) of [8]. Therefore, $A^{\star}(\mathcal{I}) \subset A^{\star}(\mathcal{N}) = cl(int(cl(A))) \subset cl(A)$. Replacing A with $A^{\star}(\mathcal{I})$, we have $(A^{\star}(\mathcal{I}))^{\star}(\mathcal{I}) \subset cl(int(cl(A^{\star}(\mathcal{I})))) \subset cl(A^{\star}(\mathcal{I}))$. Since $\mathcal{I} \sim \tau$, $(A^{\star}(\mathcal{I}))^{\star}(\mathcal{I}) = A^{\star}(\mathcal{I})$ by Theorem 4.6(b) and Theorem 2.3(d and f) of [8]. Since $A^{\star}(\mathcal{I})$ is closed, we have $A^{\star}(\mathcal{I}) \subset cl(int(A^{\star}(\mathcal{I}))) \subset A^{\star}(\mathcal{I})$ and so $A^{\star}(\mathcal{I}) = cl(int(A^{\star}(\mathcal{I})))$. Therefore, $A^{\star}(\mathcal{I})$ is regularclosed.

(ii) Since $\tau \cap \mathcal{N} = \{\phi\}$ and $A^\star(\mathcal{I})$ is regularclosed, by the above Lemma 1.1(d), $A^\star(\mathcal{I})$=$(A^\star(\mathcal{I}))^\star(\mathcal{N})$. □

Proof of Lemma 1.4. By Theorem 4.11 of [8], $\mathcal{N} \sim \tau$. By Theorem 3.1 of [9], $\mathcal{I} \star \mathcal{N} \sim \tau$. Since $\mathcal{N} \subset \mathcal{I} \star \mathcal{N}$, by the above Lemma 1.5, $A^\star(\mathcal{I} \star \mathcal{N})$ is regularclosed and $A^\star(\mathcal{I} \star \mathcal{N}) = (A^\star(\mathcal{I} \star \mathcal{N}))^\star(\mathcal{N})$. □

2. Completely codense ideals

An ideal $\mathcal{I}$ on a space (X, τ) is said to be *codense* ([4]) if $\tau \cap \mathcal{I} = \{\phi\}$. An ideal $\mathcal{I}$ is said to be *completely codense* ([4]) if PO(X)$\cap\mathcal{I} = \{\phi\}$. An ideal $\mathcal{I}$ is said to be *regular* ([3]) if RO(X)$\cap\mathcal{I} = \{\phi\}$. An ideal $\mathcal{I}$ is said to be *normal* ([3]) if $\mathcal{N} \subset \mathcal{I}$ and $\mathcal{I}$ is regular. Every completely codense ideal is codense and every codense ideal is regular. The converse implications are not true. The following theorem gives characterizations of completely codense ideals.

Theorem 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.*

(a) *$\mathcal{I}$ is completely codense.*

(b) *$A \subset A^\star$ for every $A \in$PO(X).*

(c) *$pint(A) = \phi$ for every $A \in \mathcal{I}$.*

(d) *Every dense set is $\mathcal{I}$-dense.*

Proof. (a) ⇒ (b). Suppose $A \in$ PO(X) and $x \notin A^\star$. Then there exists an open set G containing x such that $G \cap A \in \mathcal{I}$. Since $A \in$ PO(X), $G \cap A \in$ PO(X) and so by hypothesis, $G \cap A = \phi$ which implies that $x \notin A$.

(b) ⇒ (c). Let $A \in \mathcal{I}$ such that pint$(A) \neq \phi$. Then there exists a non-empty preopen set G such that $G \subset A$ and so $G^\star \subset A^\star = \phi$. Since $G \subset G^\star$, $G = \phi$ which is a contradiction. Therefore, pint$(A) = \phi$.

(c) ⇒ (a). Let $A \in PO(X) \cap \mathcal{I}$. Then $A \in PO(X) \Rightarrow A \subset int(cl(A))$. $A \in \mathcal{I}$ $\Rightarrow$ pint$(A) = \phi \;\Rightarrow\; A \cap int(cl(A)) = \phi \;\Rightarrow A = \phi$.

(a) and (d) are equivalent by Theorem 4.10 of [4]. □

Corollary 2.2. *Let $(X, \tau, \mathcal{I})$ be an ideal space with a completely codense ideal $\mathcal{I}$. If $A \in$PO(X), then*

(a) *$A^\star(\mathcal{I}) = A^\star(\mathcal{N})$ and $A^\star(\mathcal{I})$ is regularclosed, and*

(b) *$cl(A) = cl^\star(A) = cl_\alpha(A)$.*

Proof. (a) If $A \in$ PO(X), by Theorem 2.1(b), $A \subset A^\star \subset cl(A)$ and so $A^\star = cl(A)$ which implies that $A^\star$ is regularclosed. Since $A \subset int(cl(A))$, we have $cl(A) \subset cl(int(cl(A))) \subset cl(A)$ and so $A^\star = cl(A) = cl(int(cl(A))) = A^\star(\mathcal{N})$.

(b) $cl(A) = cl^\star(A)$ by Theorem 2.1(b) and Lemma 1.2. Therefore, $cl^\star(A) = A \cup A^\star(\mathcal{I}) = A \cup A^\star(\mathcal{N}) = cl_\alpha(A)$. □

Theorem 2.3. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{N} \subset \mathcal{I}$, then the following are equivalent.*

(a) *$\mathcal{I}$ is codense.*

(b) *$\mathcal{I}$ is regular.*

Proof. (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (a). Let $G \in \tau \cap \mathcal{I}$. $G \in \tau \Rightarrow int(cl(G)) = G \cup (int(cl(G)) - G) \in \mathcal{I} \cup \mathcal{N} = \mathcal{I}$. Since $int(cl(G)) \in$ RO(X), $int(cl(G)) = \phi$ and so $G = \phi$. Hence $\mathcal{I}$ is codense.□

Corollary 2.4. *Every normal ideal is codense.*

In general, codense ideals need not be normal. If (X, τ) is a space with a non-empty nowhere dense set, then the ideal $\{\phi\}$ is codense but not normal. But for any codense ideal $\mathcal{I}$, it is clear that $\tilde{\mathcal{I}}$ is always normal. Since in open hereditarily irresolvable(o.h.i) spaces [1, Theorem 3.7] or in submaximal spaces (i.e., dense sets are open)[4, Theorem 4.15], codense ideals are completely codense and an ideal $\mathcal{I}$ is completely codense if and only if $\mathcal{I} \subset \mathcal{N}$ [4, Theorem 4.13], the following Corollary 2.5 follows from Corollary 2.4.

Corollary 2.5. *If X is an o.h.i space or a submaximal space, then an ideal $\mathcal{I}$ is normal if and only if $\mathcal{I} = \mathcal{N}$.*

Theorem 2.6. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then*

(a) $\{\phi\} \star \mathcal{I} = \langle \mathcal{I} \cap \mathcal{F} \rangle$.

(b) *If $\mathcal{I}$ is regular, then $\Im = \langle \mathcal{I} \cap \mathcal{F} \rangle$ is completely codense and the converse holds, if $\mathcal{N} \subset \mathcal{I}$.*

Proof. (a). $A \in \{\phi\} \star \mathcal{I} \Leftrightarrow A^\star(\{\phi\}) \in \mathcal{I} \Leftrightarrow cl(A) \in \mathcal{I} \Leftrightarrow A \in \langle \mathcal{I} \cap \mathcal{F} \rangle$.

(b) Suppose $A \in PO(X) \cap \Im$. $A \in PO(X) \Rightarrow A \subset int(cl(A))$. $A \in \Im \Rightarrow cl(A) \in \mathcal{I} \Rightarrow int(cl(A)) \in \mathcal{I}$. Since $\mathcal{I}$ is regular, $int(cl(A)) = \phi$ which implies that $A = \phi$. Therefore, $\Im$ is completely codense.

Conversely, suppose $\Im$ is completely codense and $\mathcal{N} \subset \mathcal{I}$. If $A \in RO(X) \cap \mathcal{I}$, then $A \in$ RO(X) and $A \in \mathcal{I}$. $A \in$ RO(X) implies that $cl(A) - A \in \mathcal{N}$ and so $cl(A) = (cl(A) - A) \cup A \in \mathcal{I} \cup \mathcal{N} = \mathcal{I}$. Therefore, $A \in \Im$. Since $A \in$ RO(X), $A \in$ PO(X) and so $A \in PO(X) \cap \Im$ which implies that $A = \phi$. Hence $\mathcal{I}$ is regular. □

The following example shows that $\mathcal{I}$ need not be regular even if $\Im = \langle \mathcal{I} \cap \mathcal{F} \rangle$ is completely codense.

Example 2.7. Let $X = \{a, b, c, d\}$ with the topology $\tau = \{\phi, \{c\}, \{a, b\}, \{a, b, c\}, X\}$. If $\mathcal{I} = \{\phi, \{a\}, \{b\}, \{a, b\}\}$, then $\Im = \langle \mathcal{I} \cap \mathcal{F} \rangle = \{\phi\}$ is completely codense but $\mathcal{I}$ is not regular, since $\{a, b\}$ is a regularopen set which is an element of $\mathcal{I}$.

The largest ideal contained in the ring of closed subsets of a space (X, τ) is called the *characteristic* ideal [16, Page 175], usually denoted by $\Gamma(X, \tau)$ or simply

Γ. The following theorem deals with the extension of completely codense ideals.

Theorem 2.8. *Let (X, τ) be a space with ideals $\mathcal{I}$ and $\Im$. If $\mathcal{I}$ and $\Im$ are completely codense, then $\mathcal{I} \star \Im$ is completely codense. The converse is true, if $\Im \subset \Gamma(X, \tau)$.*

Proof. Suppose $A \in (\mathcal{I} \star \Im) \cap \mathrm{PO(X)}$. $A \in \mathrm{PO(X)} \Rightarrow A \subset A^\star(\mathcal{I})$, by Theorem 2.1(b). $A \in \mathcal{I} \star \Im \Rightarrow A^\star(\mathcal{I}) \in \Im \Rightarrow A \in \Im$. Since, $\Im$ is completely codense, $A = \phi$. Therefore $\mathcal{I} \star \Im$ is completely codense. Conversely, if $\mathcal{I} \star \Im$ is completely codense, since $\mathcal{I} \subset \mathcal{I} \star \Im$, $\mathcal{I}$ is completely codense. To prove that $\Im$ is completely codense, we prove that $\Im \subset \mathcal{N}$. $A \in \Im \Rightarrow A$ is $\tau^\star(\mathcal{I})$-closed $\Rightarrow A^\star(\mathcal{I}) \subset A \Rightarrow A^\star(\mathcal{I}) \in \Im \Rightarrow A \in \mathcal{I} \star \Im \Rightarrow A \in \mathcal{N}$. Therefore, $\Im$ is completely codense. □

The following Corollary 2.9 follows from the fact that $\mathcal{N}$ is completely codense and $\mathcal{I} \subset \mathcal{I} \cup \mathcal{N} \subset \mathcal{I} \star \mathcal{N}$ for any arbitrary ideal $\mathcal{I}$.

Corollary 2.9. *Let $(X, \tau, \mathcal{I})$ be an ideal space. Then*

(a) *$\mathcal{I}$ is completely codense if and only if $\mathcal{I} \star \mathcal{N}$ is completely codense, and*

(b) *$\mathcal{I}$ is completely codense if and only if $\mathcal{I} \cup \mathcal{N}$ is completely codense.*

The following Example 2.10 shows that $\Im$ need not be completely codense, if $\mathcal{I} \star \Im$ is completely codense but clearly $\Im$ is completely codense, if $\Im \subset \mathcal{I}$.

Example 2.10. Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, X\}$. If $\mathcal{I} = \{\phi, \{c\}\}$ and $\Im = \{\phi, \{a\}\}$, then $\mathcal{I}$ and $\mathcal{I} \star \Im = \{\phi, \{c\}\}$ are completely codense ideals but $\Im$ is regular which is not completely codense.

3. Banach category theorem

A space is said to *resolvable* ([7]) if X is the union of two disjoint dense subsets. An ideal space $(X, \tau, \mathcal{I})$ is said to be *$\mathcal{I}$-resolvable* ([4]) if X has two disjoint $\mathcal{I}$ dense subsets. Every resolvable space is $\mathcal{N}$-resolvable and if $\mathcal{I}$ and $\Im$ are ideals with $\mathcal{I} \subset \Im$ and X is $\Im$-resolvable, then X is $\mathcal{I}$-resolvable. $\mathcal{I}$-resolvable spaces are introduced and studied by Dontchev, Ganster and Rose in [4] and they raised the question: If X is $\mathcal{I}$-resolvable, then is X $(\mathcal{I} \cup \mathcal{N})$-resolvable?. To answer this question, we need the following theorem.

Theorem 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\lambda = \mathcal{I} \cap \mathcal{N}$ and $\beta = \mathcal{I} \cup \mathcal{N}$, then for every subset A of X,*

(a) $(A^\star(\mathcal{I}))^\star(\mathcal{N}) = (A^\star(\mathcal{I} \star \mathcal{N}))^\star(\mathcal{N}) = (A^\star(\tilde{\mathcal{I}}))^\star(\mathcal{N}) = A^\star(\tilde{\mathcal{I}})$.

(b) $(A^\star(\mathcal{I}))^\star(\mathcal{N}) = (A^\star(\beta))^\star(\mathcal{N})$.

(c) $(A^\star(\beta))^\star(\mathcal{N}) \subset A^\star(\mathcal{N})$.

(d) $A^\star(\mathcal{N}) = (A^\star(\lambda))^\star(\mathcal{N})$.

Proof. (a) Let $x \notin A^\star(\tilde{\mathcal{I}})$. Then there exists an open set G containing x such that $G \cap A \in \tilde{\mathcal{I}}$ and so $(G \cap A)^\star(\mathcal{I}) \in \mathcal{N}$ which implies that $G \cap A^\star(\mathcal{I}) \in \mathcal{N}$. Hence $G \cap (A^\star(\mathcal{I}))^\star(\mathcal{N}) = \phi$ which implies that $x \notin (A^\star(\mathcal{I}))^\star(\mathcal{N})$. Therefore,$(A^\star(\mathcal{I}))^\star(\mathcal{N}) \subset A^\star(\tilde{\mathcal{I}}) \subset A^\star(\mathcal{I})$ and so $((A^\star(\mathcal{I}))^\star(\mathcal{N}))^\star(\mathcal{N}) \subset (A^\star(\tilde{\mathcal{I}}))^\star(\mathcal{N}) \subset (A^\star(\mathcal{I}))^\star(\mathcal{N})$. Since $\mathcal{N} \sim \tau$, it follows that $(A^\star(\tilde{\mathcal{I}}))^\star(\mathcal{N}) = (A^\star(\mathcal{I}))^\star(\mathcal{N})$. $(A^\star(\tilde{\mathcal{I}}))^\star(\mathcal{N}) = A^\star(\tilde{\mathcal{I}})$ by Lemma 1.4.

(b) Since $\mathcal{I} \subset \beta \subset \tilde{\mathcal{I}}$, we have $A^\star(\tilde{\mathcal{I}}) \subset A^\star(\beta) \subset A^\star(\mathcal{I})$ and so $(A^\star(\tilde{\mathcal{I}}))^\star(\mathcal{N}) \subset (A^\star(\beta))^\star(\mathcal{N}) \subset (A^\star(\mathcal{I}))^\star(\mathcal{N}) = (A^\star(\tilde{\mathcal{I}}))^\star(\mathcal{N})$ by (a). Therefore, $(A^\star(\tilde{\mathcal{I}}))^\star(\mathcal{N}) = (A^\star(\beta))^\star(\mathcal{N}) = (A^\star(\mathcal{I}))^\star(\mathcal{N})$.

(c) Since $\mathcal{N} \subset \beta$, $A^\star(\beta) \subset A^\star(\mathcal{N})$ and so $(A^\star(\beta))^\star(\mathcal{N}) \subset (A^\star(\mathcal{N}))^\star(\mathcal{N}) = A^\star(\mathcal{N})$, since $\mathcal{N} \sim \tau$.

(d) Since $\lambda \subset \mathcal{N}$, it follows from Lemma 1.3. □

Theorem 3.2. *An ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable if and only if it is $\tilde{\mathcal{I}}$-resolvable.*

Proof. Suppose X is $\mathcal{I}$-resolvable. Let A be any $\mathcal{I}$-dense subset of X. Then $A^\star(\mathcal{I}) = X$ and so $(A^\star(\mathcal{I}))^\star(\mathcal{N}) = X^\star(\mathcal{N}) = X$. By Theorem 2.1(a), $A^\star(\tilde{\mathcal{I}}) = X$. It follows that X is $\tilde{\mathcal{I}}$-resolvable. Since $\mathcal{I} \subset \mathcal{I} \star \mathcal{N}$, the converse is clear. □

Corollary 3.3. *An ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-resolvable if and only if it is $(\mathcal{I} \cup \mathcal{N})$-resolvable.*

In the remaining part of this section, we generalize the well known Banach Category Theorem. For topological spaces, Kuratowski ([10]) proved that every topology is compatible with the σ-ideal $\mathcal{M}$ of meager sets. Jankovic and Hamlett ([9]) extended the result of Kuratowski by proving the following:

Theorem A (Generalized Banach Category Theorem). *For any ideal space $(X, \tau, \mathcal{I})$, $(\tilde{\mathcal{I}})_\sigma \sim \tau$.*

Theorem 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $\beta = \mathcal{I} \cup \mathcal{N}$. Then*

(a) $\beta \star \mathcal{N} = \tilde{\mathcal{I}}$

(b) $\beta \star \mathcal{N} \sim \tau$ *and*

(c) $A^\star(\tilde{\mathcal{I}}) = A^\star(\beta \star \mathcal{N}) = (A^\star(\beta \star \mathcal{N}))^\star(\mathcal{N})$ *for every subset A of X.*

Proof. (a) $A \in \mathcal{I} \star \mathcal{N} \Rightarrow A^\star(\mathcal{I}) \in \mathcal{N} \Rightarrow A^\star(\beta) \in \mathcal{N}$, since $\mathcal{I} \subset \beta \Rightarrow A \in \beta \star \mathcal{N}$.Therefore, $\mathcal{I} \star \mathcal{N} \subset \beta \star \mathcal{N}$. $A \in \beta \star \mathcal{N} \Rightarrow A^\star(\beta) \in \mathcal{N} \Rightarrow A^\star(\tilde{\mathcal{I}}) \in \mathcal{N} \Rightarrow (A^\star(\mathcal{I}))^\star(\mathcal{N}) = \phi$ from Theorem 3.1(a). Since $\mathcal{N} \sim \tau$, $A^\star(\mathcal{I}) \in \mathcal{N}$ which implies that $A \in \mathcal{I} \star \mathcal{N}$. Therefore, $\beta \star \mathcal{N} \subset \mathcal{I} \star \mathcal{N}$. Hence $\beta \star \mathcal{N} = \mathcal{I} \star \mathcal{N}$.

(b) follows from the fact that $\mathcal{I} \star \mathcal{N} \sim \tau$.

(c) follows from Lemma 1.4. □

Now, using Theorem 3.4(a), we can generalize the Banach Category Theorem as follows.

Theorem 3.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $\beta = \mathcal{I} \cup \mathcal{N}$, then $(\tilde{\beta})_\sigma \sim \tau$.*

Theorem 3.6. *If $(X,\tau,\mathcal{I})$ is an ideal space, $\mathcal{I}$ is compatible and $\beta = \mathcal{I} \cup \mathcal{N}$, then $\beta = \beta \star \mathcal{N}$.*

Proof. Clearly, $\beta \subset \beta \star \mathcal{N}$. If $A \in \beta \star \mathcal{N}$, then $A^\star(\beta) \in \mathcal{N}$ and so $(A^\star(\beta))^\star(\mathcal{N}) = \phi$. By Theorem 3.1(b), $(A^\star(\mathcal{I}))^\star(\mathcal{N}) = \phi$ which implies that $A^\star(\mathcal{I}) \in \mathcal{N}$ and so $A \cap A^\star(\mathcal{I}) \in \mathcal{N}$. Since $\mathcal{I} \sim \tau$, $A - A^\star(\mathcal{I}) \in \mathcal{I}$. Hence, A $=(A - A^\star(\mathcal{I})) \cup (A \cap A^\star(\mathcal{I})) \in \mathcal{I} \cup \mathcal{N} = \beta$. This completes the proof. □

Corollary 3.7. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then, $\mathcal{N} \subset \mathcal{I}$ and $\mathcal{I} \sim \tau \Leftrightarrow \mathcal{I} = \mathcal{I} \star \mathcal{N}$* [14, Theorem 9(a)].

Corollary 3.8. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $\beta = \mathcal{I} \cup \mathcal{N}$ and $\mathcal{I} \sim \tau$, then $\beta = \beta \star \mathcal{N} = \mathcal{I} \star \mathcal{N}$ and so $\beta \sim \tau$.*

Proof. Follows from Theorems 3.4 and 3.6. □

Using Corollary 3.8, we can also generalize the Banach Category Theorem as follows.

Theorem 3.9. *If $(X,\tau,\mathcal{I})$ is an ideal space, $\beta = \mathcal{I} \cup \mathcal{N}$ and $\mathcal{I} \sim \tau$, then $\beta_\sigma \sim \tau$.*

If $(X,\tau,\mathcal{I})$ is an ideal space and $\mathcal{I} \sim \tau$, then clearly $\mathcal{I} \star \mathcal{I} = \mathcal{I}$. Also for an arbitrary ideal $\mathcal{I}$, $\tilde{\mathcal{I}} \star \mathcal{N} = \tilde{\mathcal{I}}$ [13, Theorem 4.1]. Since $\tilde{\mathcal{I}} \sim \tau$, $\tilde{\mathcal{I}} \star \tilde{\mathcal{I}} = \tilde{\mathcal{I}}$. In the following Theorem 3.10 , we prove that the extension of a compatible ideal $\mathcal{I}$ via $\tilde{\mathcal{I}}$ is $\tilde{\mathcal{I}}$.

Theorem 3.10. *If $(X,\tau,\mathcal{I})$ is an ideal space and $\mathcal{I} \sim \tau$, then $\mathcal{I} \star \tilde{\mathcal{I}} = \tilde{\mathcal{I}}$.*

Proof. $\mathcal{I} \star \tilde{\mathcal{I}} = \{A \mid A^\star(\mathcal{I}) \in \tilde{\mathcal{I}}\} = \{A \mid (A^\star(\mathcal{I}))^\star(\mathcal{I}) \in \mathcal{N}\} = \{A \mid A^\star(\mathcal{I}) \in \mathcal{N}\}$, since $\mathcal{I} \sim \tau = \{A \mid A \in \mathcal{I} \star \mathcal{N}\} = \tilde{\mathcal{I}}$. □

We have the following generalization of Banach Category Theorem using Theorem 3.10.

Theorem 3.11. *If (X,τ) is a space with a compatible ideal $\mathcal{I}$, then the countable extension of the compatible extension of $\mathcal{I}$ via $\tilde{\mathcal{I}}$ is always compatible with τ i.e., $(\mathcal{I} \star \tilde{\mathcal{I}})_\sigma \sim \tau$.*

The well known Banach Category Theorem (Corollary 3.12 below) is obtained as an immediate corollary to Theorems 3.5, 3.9 and 3.11, if we take $\mathcal{I} = \{\phi\}$.

Corollary 3.12. *Let (X,τ) be a space and let $\mathcal{M}$ denote the ideal of meager sets. Then $\mathcal{M} \sim \tau$.*

4. $\mathcal{I}$-locally closed subsets

A subset A of a space (X,τ) is *locally closed* ([10]) if A is the intersection of an open set and a closed set. Locally closed sets are further investigated by Ganster and Reilly in [6]. In 1999, Dontchev ([2]) introduced *$\mathcal{I}$-locally closed* subsets in an ideal space. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is called $\mathcal{I}$-*locally closed* if $A = U \cap V$, where $U \in \tau$ and V is $\star$-perfect. Clearly, in the case of the minimal ideal, the concepts $\mathcal{I}$-locally closed and locally closed are equivalent. It is clear that open as well as closed sets need not be $\mathcal{I}$-locally closed. If $(X,\tau,\mathcal{I})$ is an ideal space where $\mathcal{I}$-is not codense, then X is not $\mathcal{I}$-locally closed. But if $\mathcal{I}$ is codense, then every open set is $\mathcal{I}$-locally closed [2, Proposition 4.1] and from Lemma 1.1, it is clear that every regularclosed set is $\mathcal{I}$-locally closed. The following theorem gives characterizations of $\mathcal{I}$-locally closed subsets.

Theorem 4.1. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. Then the following are equivalent.*

(a) *A is $\mathcal{I}$-locally closed.*

(b) *$A = G \cap A^\star$ for some open set G.*

(c) *$A \subset A^\star$ and $A^\star - A$ is closed.*

(d) *$A \subset A^\star$ and $A \cup (X - A^\star)$ is open.*

(e) *$A \subset A^\star$ and $A \subset int(A \cup (X - A^\star))$.*

Proof. (a) $\Rightarrow$ (b). Suppose $A = U \cap V$ where $U \in \tau$ and $V = V^\star$. Then $A \subset V$ and so $A^\star \subset V^\star = V$. Also, $A^\star = (U \cap V)^\star \supset U \cap V^\star = U \cap V = A$ and so $A = A \cap A^\star = (U \cap V) \cap A^\star = U \cap (V \cap A^\star) = U \cap A^\star$.

(b) $\Rightarrow$ (c). Suppose $A = G \cap A^\star$ for some open set G. Clearly, $A \subset A^\star$ and $A^\star - A = A^\star \cap (X - A) = A^\star \cap (X - (G \cap A^\star)) = A^\star \cap (X - G)$. Since $A^\star$ is closed, $A^\star - A$ is closed.

(c) $\Rightarrow$ (d). $A^\star - A$ is closed $\Rightarrow A^\star \cap (X - A)$ is closed $\Rightarrow X - (A^\star \cap (X - A))$ is open $\Rightarrow A \cup (X - A^\star)$ is open.

(d) $\Rightarrow$ (e) is clear.

(e) $\Rightarrow$ (a). $X - A^\star = int(X - A^\star) \subset int(A \cup (X - A^\star))$ and so $A \cup (X - A^\star)$ is open, by hypothesis. Since $A = (A \cup (X - A^\star)) \cap A^\star$, A is $\mathcal{I}$-locally closed. □

Corollary 4.2. *If $(X,\tau,\mathcal{I})$ is an ideal space and $A \subset X$ is $\mathcal{I}$-locally closed, then*

(a) *A is $\star$-dense in itself*

(b) *$A^\star$ is $\star$-perfect, and*

(c) *$cl(A) = cl^\star(A) = A^\star$.*

Proof. (a) follows from Theorem 4.1, (b) follows from (a) and the fact that $(A^\star)^\star \subset A^\star$ ([8]) and (c) follows from Lemma 1.2. □

In the following theorems, we give the relation of $\star$-perfect, locally closed and $\star$-dense in itself subsets with $\mathcal{I}$-locally closed subsets.

Theorem 4.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A \subset X$. If A is $\star$-perfect, then*

A is $\mathcal{I}$-locally closed. The converse is true, if A is $\tau^\star$-closed.

Proof. If A is $\star$-perfect, then $A = A^\star$ and so $A = X \cap A = X \cap A^\star$ which implies that A is $\mathcal{I}$-locally closed. Conversely, if A is $\mathcal{I}$-locally closed, then $A \subset A^\star$. A is $\tau^\star$-closed implies that $A^\star \subset A$. Hence $A = A^\star$. □

Theorem 4.4. *Let $(X, \tau, \mathcal{I})$ be an ideal space and $A \subset X$. Then A is $\mathcal{I}$-locally closed if and only if A is locally closed and A is $\star$-dense in itself.*

Proof. If A is $\mathcal{I}$-locally closed, then A is $\star$-dense in itself and $A = G \cap A^\star$ for some $G \in \tau$. Since $A^\star$ is closed, A is locally closed. Conversely, if A is locally closed, then $A = G \cap F$ where $G \in \tau$ and F is closed. $A \subset F \Rightarrow A^\star \subset F \Rightarrow A^\star \cap F = A^\star$. Now A is $\star$-dense in itself implies that $A \subset A^\star$ and so $A = A \cap A^\star = (G \cap F) \cap A^\star = G \cap (F \cap A^\star) = G \cap A^\star$. Therefore, A is $\mathcal{I}$-locally closed. □

Theorem 4.5. *Let $(X, \tau, \mathcal{I})$ be a T_1 ideal space and $A \subset X$. If A is discrete and $\star$-dense in itself, then A is $\mathcal{I}$-locally closed.*

Proof. A is locally closed by Proposition 2(i) of [6] and so A is $\mathcal{I}$-locally closed by Theorem 4.4. □

The following example shows that $\mathcal{I}$-locally closed set need not be $\star$-perfect and locally closed set need not be $\mathcal{I}$-locally closed. Also, it shows that the complement of an $\mathcal{I}$-locally closed set need not be $\mathcal{I}$-locally closed.

Example 4.6. Consider the ideal space $(X, \tau, \mathcal{I})$ of Example 2.7. If $A = \{c\}$, then A is $\mathcal{I}$-locally closed but not $\star$-perfect. If $B = \{a, b, d\}$, then B is locally closed but not $\mathcal{I}$-locally closed. Since $X - A = B$, the complement of an $\mathcal{I}$-locally closed set need not be $\mathcal{I}$-locally closed.

The following example shows that $\star$-dense in itself set need not be $\mathcal{I}$-locally closed.

Example 4.7. Let $X = \{a, b, c, d\}$, $\tau = \{\phi, \{d\}, \{a, c\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. If $A = \{a\}$, then A is $\star$-dense in itself but not $\mathcal{I}$-locally closed.

Recall that a dense subset of a topological space is open if and only if it is locally closed. As a generalization of this result to $\mathcal{I}$-locally closed sets, we have the following theorem.

Theorem 4.8. *Let $(X, \tau, \mathcal{I})$ be an ideal space and A be an $\mathcal{I}$-dense subset of X. Then A is open if and only if A is $\mathcal{I}$-locally closed.*

Proof. Suppose A is $\mathcal{I}$-dense and open. Then $A = A \cap X = A \cap A^\star$ and so A is $\mathcal{I}$-locally closed. Conversely, if A is $\mathcal{I}$-locally closed and $\mathcal{I}$-dense, then $A = G \cap A^\star$ where $G \in \tau$ which implies $A = G \cap X = G$ and so A is open. □

Acknowledgement. The authors sincerely thank Prof. M. Ganster for sending some of his reprints which are helpful for the preparation of this paper.

References

[1] Biswanath Garai and Chhanda Bandyopadhyay, *$\mathcal{I}$-dense sets and $\mathcal{I}$-o.h.i. spaces*, Kyungpook Math. J., **41**(2001), 351-359.

[2] J. Dontchev, Idealization of Ganster Reilly decomposition theorems, Math. GN/9901017, 5, Jan. 1999(Internet).

[3] J. Dontchev and M. Ganster, *On compactness with respect to countable extensions of ideals and the generalized Banach category Theorem*, Acta. Math. Hung., **88**(2000), 53-58.

[4] J. Dontchev, M. Ganster and D. Rose, *Ideal resolvability*, Topology and its Applications, **93**(1999), 1-16.

[5] M. Ganster, *Preopen sets and resolvable spaces*, Kyungpook Math. J., **27(2)**(1987), 135-143.

[6] M. Ganster and I. L. Reilly, *Locally closed sets and LC-continuous functions*, Internat., J. Math. and Math. Sci., **12**(1989), 417-424.

[7] E. Hewitt, *A problem of set theoretic topology*, Duke Math. J., **10**(1943), 309-333.

[8] D. Jankovic and T. R. Hamlett, *New Topologies from old via Ideals*, Amer. Math. Monthly, **97(4)**(1990), 295-310.

[9] D. Jankovic and T. R. Hamlett, *Compatible Extensions of Ideals*, Boll. U.M.I., **(7)6-B**(1992), 453-465.

[10] K. Kuratowski, Topology, Academic Press, New York, **I**(1966).

[11] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, **53**(1982), 47-53.

[12] O. Njastad, On some Classes of Nearly Open Sets, Pacific J. Math., **15**(1965), 961-970.

[13] D. A. Rose and T. R. Hamlett, *On one-point $\mathcal{I}$-compactification and Local $\mathcal{I}$-compactness*, Math. Slovaca, **42**(1992), 359-369.

[14] V. Renuka Devi, D. Sivaraj and T. Tamizh Chelvam, *Codense and completely codense ideals*, Acta Math. Hungar., **108(4)** (To appear).

[15] R. Vaidyanathaswamy, *The Localization Theory in Set Topology*, Proc. Indian Acad. Sci. Math. Sci., **20**(1945), 51-61.

[16] R. Vaidyanathaswamy, Set Topology, Chelsea Publishing Company, 1960.