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Vibration-based Identification of Directional Damages in a Cylindrical Shell

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Abstract This paper introduces a structural damage identification method to identify the multiple directional damages generated within a cylindrical shell by using the measured frequency response function (FRF). The equations of motion for a damaged cylindrical shell are derived. by using a theory of continuum damage mechanics in which a small material volume containing a directional damage is represented by the effective orthotropic elastic stiffness. In contrast with most existing vibration-based structural damage identification methods which require the modal parameters measured in both intact and damaged states, the present method requires only the FRF-data measured at damaged state. Numerically simulated damage identification tests are conducted to verify the feasibility of the proposed structural damage identification method.

Keywords: damage identification, directional damage, cylindrical shell, frequency response function, Continuum damage model

1. Introduction

The cylindrical shell structures such as the oil or gas tanks, compressor shells, boilers and airplane fuselages should be free from disastrous structural failures due to (structural) damages. Therefore it is very important to detect all significant damages in the very early stage of damage progression. As the structural damages change the vibration characteristics of a structure, the damage-induced changes in vibration characteristics can be used to detect and identify the existing damages. In most existing vibration-based structural damage identification methods (SDIMs), the modal parameters such as the natural frequencies, modal damping, mode shapes and FRF-data have been widely used (e.g., Adams et al., 1978; Banks et al., 1996; Thyagarajan et al., 1998; Lee and Shin, 2002).

The SDIMs for cylindrical shells have been introduced by some researchers. Srinivasan and Kot (1998) proposed to use a damage index method for locating damage in circular cylindrical shells, which is basically based on the damage-induced change in the modal strain energy. Royston et al. (2000) proposed a flaw detection method by using the slit-mode phenomena that may occur when the degenerate vibratory modes of the axisymmetric structure become non-generate due to the flaws. Ip and Tse (2002) presented a feasibility study on locating damage in circular cylindrical fiber-reinforced composite shells based on the frequency sensitivities and mode shape information at specific locations. Very recently Kim et al. (2004) developed an FRF-data based SDIM to identify the locations and magnitudes of multiple damages within a cylindrical shell. As in most existing SDIMs, they intrinsically assumed that damages were isotropic damages

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such as the circular holes which have no directivity and thus they did not take into account the directivity of damage, *i.e.*, the orientation of damage with respect to the global coordinates.

The failure of most structural members involves general degradation of elastic properties due to the localized nucleation and growth of damages (i.e., voids, cavities, or cracks of the size of crystal grains) and their ultimate coalescences into the larger size of material fracture. This implies that the directivities of damages may control the direction of crack propagation within a structure. Because the damage directivity plays a very important role to determine the failure pattern and the remaining life of a structure, it will be very important to identify the directivities of all damages as well, in addition to identifying their locations and severities (or magnitudes). Thus it is mandatory to develop a SDIM which has the capability of simultaneously identifying the directivities of all damages: this motivates this work.

Thus, the purposes of the present paper are to propose an FRF-based SDIM for the cylindrical shells containing multiple directional damages and to conduct the numerical feasibility tests to evaluate the proposed SDIM.

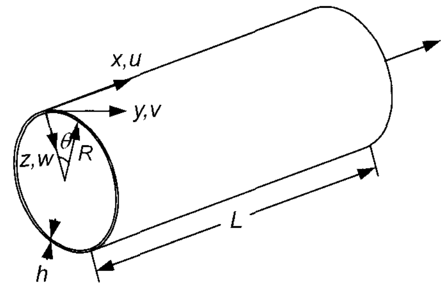
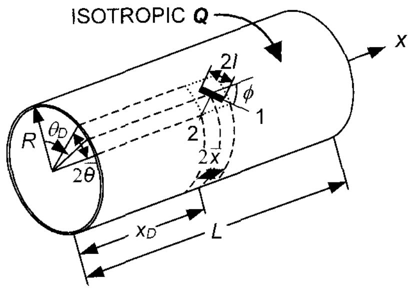
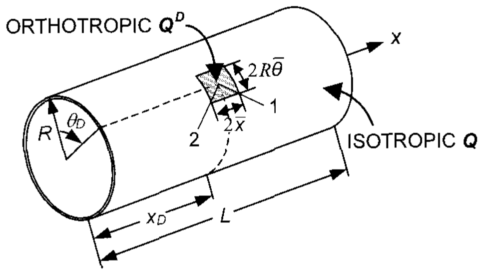


Fig. 1 Geometry of a cylindrical shell



(a)



(b)

Fig. 2 (a) An isotropic cylindrical shell with a line through-crack and (b) its equivalent continuum damage model represented by effective orthotropic elastic stiffness.

2. Equations of motion

Consider an elastic, thin circular cylindrical shell as shown in Fig. 1. The shell has the radius R, the length L, and the thickness h. At intact state, the shell is isotropic and it has Young's modulus E and Poisson's ratio v. The x-axis is directed along the symmetry axis of the median shell surface, the y-axis in the circumferential direction, and the z-direction along the interior normal of the meridian surface. Define the displacements in the longitudinal, circumferential and radial directions by $u(x, \theta, t)$, $v(x, \theta, t)$ and $w(x, \theta, t)$, respectively. Similarly, define the external load normal to the shell surface by $pz(x, \theta, t)$.

Assume that there is a line crack-like damage of length $2l$, centered at (x_D, θ_D) and aligned with the crack coordinate '1' which is oriented ϕ degrees with respect to the global coordinate x, as shown in Fig. 2(a). In the following, the crack coordinates will be represented by the subscripts 1 and 2, if not mentioned otherwise.

The (*effective*) orthotropic elastic stiffness Q^D for the SMV (small material volume) containing a line crack-like damage can be expressed as (Lee et al., 1997)

$$\boldsymbol{Q}^D = \boldsymbol{Q} - \Delta\boldsymbol{Q} \tag{1}$$

where

$$\boldsymbol{Q}^D = [Q_{ij}^D], \quad \boldsymbol{Q} = [Q_{ij}]$$
$$\Delta\boldsymbol{Q} = [Q_{ij} e_{ij}]D = \hat{\boldsymbol{Q}}D \qquad (i, j = 1, 2, 6; \text{no sum}) \tag{2}$$

In above equations, $\boldsymbol{Q}$ is the reduced elastic stiffness of the intact isotropic material under the plane stress state (Whitney, 1996) and e_{ij} are the (*effective*) material directivity parameters defined in Lee et al. (1997). In (2), $0 \leq D \leq 1$ is the damage variable defined by the ratio of the whole volume of a SMV (*i.e.*, $4\bar{x}R\bar{\theta}h$) and the effective damaged volume (*i.e.*, $\pi hl2$) determined by crack size 2l. Thus, D represents the severity of damage within an SMV, which is called herein (*effective*) damage magnitude.

Since the orthotropic elastic stiffness $\boldsymbol{Q}^D$ given by (1) is measured with respect to the crack coordinates (1, 2), the elastic stiffness with respect to the global coordinates (x, θ) can be obtained by using the coordinates transformation as follows:

$$\overline{\boldsymbol{Q}}^D = \boldsymbol{T}(\phi)^{\mathrm{T}} \boldsymbol{Q}^D \, \boldsymbol{T}(\phi) \tag{3}$$

where $\boldsymbol{T}(\phi)$ is the coordinates transformation matrix (Whitney, 1996) and ϕ denotes the angle of crack orientation (degrees) with respect to the global coordinate x. The superscript T denotes the transpose of a matrix. By using eqns. (1) and (2), eqn. (3) can be rewritten as

$$\overline{\boldsymbol{Q}}^D = \boldsymbol{Q} - \Delta\overline{\boldsymbol{Q}} \tag{4}$$

where

$$\Delta\overline{\mathbf{Q}} = \boldsymbol{T}(\phi)^{\mathrm{T}} \hat{\boldsymbol{Q}} \boldsymbol{T}(\phi) D = \mathbf{Q}^*(\phi) D \tag{5}$$

with

$$\mathbf{Q}^*(\phi) = \frac{E}{1-\nu^2} \begin{bmatrix} \frac{1+\nu^2}{1-\nu^2} - \cos 2\phi & \frac{2\nu}{1-\nu^2} & -\frac{1}{2}\sin 2\phi \\ \frac{2\nu}{1-\nu^2} & \frac{1+\nu^2}{1-\nu^2} + \cos 2\phi & -\frac{1}{2}\sin 2\phi \\ -\frac{1}{2}\sin 2\phi & -\frac{1}{2}\sin 2\phi & \frac{1-\nu}{2(1+\nu)} \end{bmatrix} \tag{6}$$

In the following, the damaged SMV will be represented by the transformed elastic stiffness $\overline{\boldsymbol{Q}}^D$ of eqn. (4), which is determined by the intact isotropic elastic stiffness $\boldsymbol{Q}$, the material directivity parameters e_{ij}, the damage orientation angle ϕ and the damage magnitude D.

The equations of motion for a thin cylindrical shell subject to a small amplitude vibration are given by (Ugural, 1999)

$$\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{1}{R}\frac{\partial N_{x\theta}}{\partial \theta} = \rho h \ddot{u} \\ &\frac{\partial N_{x\theta}}{\partial x} + \frac{1}{R}\frac{\partial N_\theta}{\partial \theta} = \rho h \ddot{v} \\ &\frac{\partial^2 M_x}{\partial x^2} + \frac{2}{R}\frac{\partial^2 M_{x\theta}}{\partial x \partial \theta} + \frac{1}{R^2}\frac{\partial^2 M_\theta}{\partial \theta^2} + \frac{1}{R}N_\theta + p_z = \rho h \ddot{w} \end{aligned} \tag{7}$$

where the dot ($\dot{}$) indicates the derivative with respect to the time t. The force resultants (N_x, N_θ, $N_{x\theta}$) and the moment resultants (M_x, M_θ, $M_{x\theta}$) are defined by

$$\begin{aligned} \begin{Bmatrix} N_x \\ N_\theta \\ N_{x\theta} \end{Bmatrix} &= \begin{bmatrix} \overline{K}_{11}^D & \overline{K}_{12}^D & \overline{K}_{16}^D \\ & \overline{K}_{22}^D & \overline{K}_{26}^D \\ symm & & \overline{K}_{66}^D \end{bmatrix} \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{\theta 0} \\ \varepsilon_{x\theta} \end{Bmatrix} \\ \begin{Bmatrix} M_x \\ M_\theta \\ M_{x\theta} \end{Bmatrix} &= -\frac{h^2}{12} \begin{bmatrix} \overline{K}_{11}^D & \overline{K}_{12}^D & \overline{K}_{16}^D \\ & \overline{K}_{22}^D & \overline{K}_{26}^D \\ symm & & \overline{K}_{66}^D \end{bmatrix} \begin{Bmatrix} \chi_x \\ \chi_\theta \\ 2\chi_{x\theta} \end{Bmatrix} \end{aligned} \tag{8}$$

where ε_{x0}, $\varepsilon_{\theta 0}$ and $\varepsilon_{x\theta}$ are the membrane strains, χ_x, χ_θ and $\chi_{x\theta}$ are the changes in curvatures, and $\overline{\boldsymbol{K}}^D = [\overline{K}_{ij}^D]$ $(i, j = 1, 2, 6)$ is the membrane stiffness for damaged shells defined by

$$\overline{\boldsymbol{K}}^D = h\overline{\boldsymbol{Q}}^D \tag{9}$$

The membrane strains and the changes in curvatures are related to the displacement fields as (Markuš, 1988)

$$\begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{\theta 0} \\ \varepsilon_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{1}{R}\frac{\partial v}{\partial \theta} - \frac{w}{R} \\ \frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \chi_x \\ \chi_\theta \\ \chi_{x\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2} \\ \frac{1}{R}\frac{\partial^2 w}{\partial x \partial \theta} \end{Bmatrix} \tag{10}$$

Substituting (4) into (9) gives

$$\bar{\boldsymbol{K}}^D = \boldsymbol{K} - \Delta\bar{\boldsymbol{K}} \qquad (11)$$

where

$$\Delta\bar{\boldsymbol{K}} = h\Delta\bar{\boldsymbol{Q}} = h\boldsymbol{Q}^*(\phi)D \qquad (12)$$

The perturbed membrane stiffness $\Delta\bar{\boldsymbol{K}}$ represents the *effective* degradation of the membrane stiffness due to the presence of a damage of magnitude D. Because the cylindrical shell shown in Fig. 2(b) consists of the intact zone (outside of SMV, isotropic) and the damaged zone (inside of SMV, orthotropic), $\bar{\boldsymbol{K}}^D$ may have the values as follows:

$$\bar{\boldsymbol{K}}^D = \begin{cases} \bar{\boldsymbol{K}}^D(D=0) = \boldsymbol{K} & \text{(outside of SMV)} \\ \bar{\boldsymbol{K}}^D(D\neq 0) = \bar{\boldsymbol{K}}^D & \text{(inside of SMV)} \end{cases} \qquad (13)$$

The intact membrane stiffnesses (outside of SMV) are given by

$$K_{11} = K_{22} = K, \quad K_{12} = K_{21} = \nu K$$
$$K_{16} = K_{61} = K_{26} = K_{62} = 0, \quad K_{66} = \left(\frac{1-\nu}{2}\right)K \qquad (14)$$

where

$$K = \frac{Eh}{1-\nu^2} \qquad (15)$$

Based on eqn. (13), the distribution of membrane stiffness over the cylindrical shell can be expressed by the function as

$$\bar{\boldsymbol{K}}^D(x,\theta) = \boldsymbol{K} - \Delta\bar{\boldsymbol{K}}(x,\theta) \quad (i,\ j = 1,\ 2,\ 6) \qquad (16)$$

By substituting eqn. (16) into eqn. (8) and substituting the result into eqn. (7), one may obtain the equations of motion for the damaged cylindrical shell in the form as

$$[\bar{\boldsymbol{L}}]\{\boldsymbol{u}(x,\theta,t)\} + \{\boldsymbol{f}(x,\theta,t)\} = \rho h\{\ddot{\boldsymbol{u}}(x,\theta,t)\} \qquad (17)$$

where

$$\{\boldsymbol{u}(x,\theta,t)\} = \{u(x,\theta,t) \quad v(x,\theta,t) \quad w(x,\theta,t)\}^T$$
$$\{\boldsymbol{f}(x,\theta,t)\} = \{0 \quad 0 \quad p_z(x,\theta,t)\}^T \qquad (18)$$

In eqn. (17), $[\bar{\boldsymbol{L}}] = [\boldsymbol{L}] + [\boldsymbol{L}_D]$ represents the matrix of differential operators for the damaged shell, where $[\boldsymbol{L}]$ is the matrix of differential operators for the intact shell defined in Markuš (1988) and $[\boldsymbol{L}_D]$ is the perturbed matrix of differential operators given by

$$[\boldsymbol{L}_D] = -\begin{bmatrix} L_{D11} & L_{D12} & L_{D13} \\ L_{D21} & L_{D22} & L_{D23} \\ L_{D31} & L_{D32} & L_{D33} \end{bmatrix} \qquad (19)$$

where

$$\begin{aligned}
L_{D11} &= \frac{\partial}{\partial x}\left(\Delta\bar{K}_{11}\frac{\partial}{\partial x}\right) + \frac{1}{R}\frac{\partial}{\partial x}\left(\Delta\bar{K}_{16}\frac{\partial}{\partial\theta}\right) + \frac{1}{R}\frac{\partial}{\partial\theta}\left(\Delta\bar{K}_{16}\frac{\partial}{\partial x}\right) + \frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\Delta\bar{K}_{26}\frac{\partial}{\partial\theta}\right) \\
L_{D12} &= \frac{\partial}{\partial x}\left(\Delta\bar{K}_{16}\frac{\partial}{\partial x}\right) + \frac{1}{R}\frac{\partial}{\partial x}\left(\Delta\bar{K}_{12}\frac{\partial}{\partial\theta}\right) + \frac{1}{R}\frac{\partial}{\partial\theta}\left(\Delta\bar{K}_{66}\frac{\partial}{\partial x}\right) + \frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\Delta\bar{K}_{26}\frac{\partial}{\partial\theta}\right) \\
L_{D13} &= -\frac{1}{R}\left(\frac{\partial(\Delta\bar{K}_{12})}{\partial x} + \Delta\bar{K}_{12}\frac{\partial}{\partial x}\right) - \frac{1}{R^2}\left(\frac{\partial(\Delta\bar{K}_{26})}{\partial\theta} + \Delta\bar{K}_{26}\frac{\partial}{\partial\theta}\right) \\
L_{D21} &= \frac{\partial}{\partial x}\left(\Delta\bar{K}_{16}\frac{\partial}{\partial x}\right) + \frac{1}{R}\frac{\partial}{\partial x}\left(\Delta\bar{K}_{66}\frac{\partial}{\partial\theta}\right) + \frac{1}{R}\frac{\partial}{\partial\theta}\left(\Delta\bar{K}_{12}\frac{\partial}{\partial x}\right) + \frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\Delta\bar{K}_{26}\frac{\partial}{\partial\theta}\right) \\
L_{D22} &= \frac{\partial}{\partial x}\left(\Delta\bar{K}_{66}\frac{\partial}{\partial x}\right) + \frac{1}{R}\frac{\partial}{\partial x}\left(\Delta\bar{K}_{26}\frac{\partial}{\partial\theta}\right) + \frac{1}{R}\frac{\partial}{\partial\theta}\left(\Delta\bar{K}_{26}\frac{\partial}{\partial x}\right) + \frac{1}{R^2}\frac{\partial}{\partial\theta}\left(\Delta\bar{K}_{22}\frac{\partial}{\partial\theta}\right) \\
L_{D23} &= -\frac{1}{R}\left(\frac{\partial(\Delta\bar{K}_{26})}{\partial x} + \Delta\bar{K}_{26}\frac{\partial}{\partial x}\right) - \frac{1}{R^2}\left(\frac{\partial(\Delta\bar{K}_{22})}{\partial\theta} + \Delta\bar{K}_{22}\frac{\partial}{\partial\theta}\right) \\
L_{D31} &= \frac{1}{R}\left(\Delta\bar{K}_{12}\frac{\partial}{\partial x}\right) + \frac{1}{R^2}\left(\Delta\bar{K}_{26}\frac{\partial}{\partial\theta}\right) \\
L_{D32} &= \frac{1}{R}\left(\Delta\bar{K}_{26}\frac{\partial}{\partial x}\right) + \frac{1}{R^2}\left(\Delta\bar{K}_{22}\frac{\partial}{\partial\theta}\right) \\
L_{D33} &= -\frac{h^2}{12}\left[\frac{\partial^2}{\partial x^2}\left(\Delta\bar{K}_{11}\frac{\partial^2}{\partial x^2}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial x^2}\left(\Delta\bar{K}_{12}\frac{\partial^2}{\partial\theta^2}\right) + \frac{2}{R}\frac{\partial^2}{\partial x^2}\left(\Delta\bar{K}_{16}\frac{\partial^2}{\partial x\partial\theta}\right) + \frac{2}{R}\frac{\partial^2}{\partial x\partial\theta}\left(\Delta\bar{K}_{16}\frac{\partial^2}{\partial x^2}\right)\right. \\
&\quad + \frac{2}{R^3}\frac{\partial^2}{\partial x\partial\theta}\left(\Delta\bar{K}_{26}\frac{\partial^2}{\partial\theta^2}\right) + \frac{4}{R^2}\frac{\partial^2}{\partial x\partial\theta}\left(\Delta\bar{K}_{66}\frac{\partial^2}{\partial x\partial\theta}\right) + \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\left(\Delta\bar{K}_{12}\frac{\partial^2}{\partial x^2}\right) + \frac{1}{R^4}\frac{\partial^2}{\partial\theta^2}\left(\Delta\bar{K}_{22}\frac{\partial^2}{\partial\theta^2}\right) \\
&\quad \left. + \frac{2}{R^3}\frac{\partial^2}{\partial\theta^2}\left(\Delta\bar{K}_{26}\frac{\partial^2}{\partial x\partial\theta}\right)\right] - \frac{\Delta\bar{K}_{22}}{R^2}
\end{aligned} \qquad (20)$$

Notice that $[\boldsymbol{L}_D]$ completely vanishes for the intact cylindrical shell because $\Delta\bar{\boldsymbol{K}}(x,\theta)$ is zero when there are no damages within the cylindrical shell.

3. Forced vibration responses of a damaged cylindrical shell

3.1. Forced vibration responses

Assume that a harmonic point load is applied at (x_F, θ_F), only in the direction normal to the surface of the cylindrical shell:

$$p_z(x_F,\theta_F,t)=F_0\delta(x-x_F)\,\delta(\theta-\theta_F)e^{i\omega t} \quad (21)$$

where F_o is the amplitude of the harmonic point load and ω is the excitation circular frequency. In eqn. (21), the function $\delta(\cdot)$ represents the Dirac delta function.

The forced vibration responses of a damaged shell can be assumed in the form

$$\{u(x,\theta,t)\}=\sum_{I=1}^{M}\{U_I(x,\theta)\}q_I(t) \quad (22)$$

where $\{U_I\}$ = $\{U_I \quad V_I \quad W_I\}^T$ (I = 1, 2, 3, ..., M) are the normal modes of the intact shell, q_I (I = 1, 2, 3, ..., M) are the modal coordinates, and M is the total number of normal modes to be superposed in the analysis. The normal modes $\{U_I\}$ should satisfy the eigenvalue problem for the intact shell

$$[L]\{U_I\}=-\rho h\Omega_I^2\{U_I\} \quad (no\ sum\ on\ I) \quad (23)$$

and the orthogonality properties

$$\int_A \rho h\{U_I\}^T\{U_J\}\,dxd\theta=\delta_{IJ} \quad (24)$$

$$\int_A \{U_I\}^T[L]\{U_J\}\,dxd\theta=-\Omega_I^2\delta_{IJ} \quad (no\ sum\ on\ I) \quad (25)$$

where Ω_I are the natural frequencies of the intact shell and δ_{IJ} is the Kronecker symbol.

Substituting eqns. (21) and (22) into eqn. (17) and applying eqns. (23) through (25) yields a set of coupled modal equations as

$$\ddot{q}_I+\Omega_I^2 q_I-\sum_{J=1}^{M}\lambda_{IJ}q_J=f_I(t) \quad (no\ sum\ on\ I) \quad (26)$$

where f_I(I = 1, 2, 3, ..., M) are the modal forces defined by

$$\begin{aligned} f_I(t)&=\int_0^L\int_0^{2\pi}p_z(x_F,\theta_F,t)W_I\,dxd\theta \\ &=W_I(x_F,\theta_F)F_0e^{i\omega t} \end{aligned} \quad (27)$$

and λ_{IJ} are defined by

$$\lambda_{IJ}=\int_A\{U_I\}^T[L_D]\{U_J\}dxd\theta \quad (28)$$

The matrix $\lambda_{IJ}=[\lambda_{IJ}]$ is the damage influence matrix (DIM) which reflects the influence of damages. Equation (26) implies that, since the DIM is not a diagonal matrix in general, the damages tend to induce the coupling between modal coordinates. The natural frequencies of damaged shell ($\overline{\Omega}_I$) can be readily computed from

$$\det\left[(\Omega_I^2-\overline{\Omega}_I^2)\delta_{IJ}-\lambda_{IJ}\right]=0 \quad (no\ sum\ on\ I) \quad (29)$$

Solving (26) for q_I and substituting the results into eqn. (22) will give the forced vibration responses for a damaged shell as follows:

$$\begin{aligned} w(x_M,\theta_M,t)=& \\ &\left(\sum_{I=1}^{M}\frac{W_I(x_M,\theta_M)W_I(x_F,\theta_F)}{\Omega_I^2-\omega^2}\right. \\ &\left.+\sum_{I=1}^{M}\sum_{J=1}^{M}\lambda_{IJ}\frac{W_I(x_M,\theta_M)W_J(x_F,\theta_F)}{(\Omega_I^2-\omega^2)(\Omega_J^2-\omega^2)}\right)F_0e^{i\omega t} \end{aligned} \quad (30)$$

where (x_M, θ_M) represents the response measurement point. The effects of structural damping can be readily taken into account in eqn. (30), if needed, by simply replacing the natural frequency Ω_I with $\Omega_I\,(1+i\eta_I)^{1/2}$, where η_I represents the modal loss factor.

3.2. Computation of damage influence matrix λ

As an example, consider a cylindrical shell which is simply-supported at both ends. In principle, the cylindrical shell may vibrate in rotationally asymmetric vibration modes or in rotationally symmetric (axisymmetric) vibration modes. By using the asymmetric and symmetric normal modes available from Markuš (1988) and Soedel (1993), the damage influence matrix λ of eqn. (28) can be readily rewritten in the form as

$$\lambda = [\lambda_{IJ}] = \int_A \boldsymbol{M}_{IJ}(x,\theta)\Lambda(x,\theta)\, dx d\theta \qquad (31)$$

where $\boldsymbol{M}_{IJ}$ (x, θ) is the one by six matrix defined by

$$\boldsymbol{M}_{IJ}(x,\theta) = [M_{IJ}^{(1)}\ M_{IJ}^{(2)}\ M_{IJ}^{(3)}\ M_{IJ}^{(4)}\ M_{IJ}^{(5)}\ M_{IJ}^{(6)}] \qquad (32)$$

and Λ(x, θ) is the six by one vector defined by

$$\Lambda(x,\theta) = \{\Delta\overline{K}_{11}\ \Delta\overline{K}_{12}\ \Delta\overline{K}_{16}\ \Delta\overline{K}_{22}\ \Delta\overline{K}_{26}\ \Delta\overline{K}_{66}\}^{T} \qquad (33)$$

The contracted subscripts I and J represent the mode numbers mni and rsj, respectively. For instance, the components of M_{IJ} can be derived, when n $\geq$ 1 and $s \geq 1$, as follows:

$$M_{IJ}^{(1)} = \left(P_{1I}P_{1J} + \frac{h^2}{12}\right)\left(\frac{m\pi}{L}\right)^2\left(\frac{r\pi}{L}\right)^2 W_I W_J$$
$$M_{IJ}^{(2)} = \left[\left(P_{1I}P_{2J}\frac{1}{R} + \frac{h^2}{12R^2}\right)\left(\frac{m\pi}{L}\right)^2 s^2 + P_{1I}\frac{1}{R}\left(\frac{m\pi}{L}\right)^2 + \left(P_{2I}P_{1J}\frac{1}{R} + \frac{h^2}{12R^2}\right)n^2\left(\frac{r\pi}{L}\right)^2 + P_{1J}\frac{1}{R}\left(\frac{r\pi}{L}\right)^2\right]W_I W_J \qquad (34)$$
$$\vdots$$
$$M_{IJ}^{(6)} = \left(P_{1I}P_{1J}\frac{1}{R^2} + P_{1I}P_{2J}\frac{1}{R} + P_{2I}P_{1J}\frac{1}{R} + P_{2I}P_{2J} + \frac{h^2}{3R^2}\right)\frac{\partial^2 W_I}{\partial x\partial\theta}\frac{\partial^2 W_J}{\partial x\partial\theta}$$

The perturbed membrane stiffness $\Delta\overline{K}$ may vary depending on the damage distribution over the cylindrical shell. In order to express it as a function of (x, θ), assume that there exists a damage at (x_D, θ_D) and consider a rectangular-shaped SMV containing the damage inside of it. As shown in Fig. 2, the dimensions of SMV in the axial and circumferential directions are $2\overline{x}$ and $2R\overline{\theta}$, respectively. The orientation angle of the damage is represented by ϕ and the damage magnitude within the SMV by D. Then the perturbed membrane stiffness $\Delta\overline{K}$ over the cylindrical shell can be expressed, by using eqn. (12), as follows:

$$\Delta\overline{\boldsymbol{K}}(x,\theta) = h\boldsymbol{Q}^*(\phi)Dd(x,\theta) \qquad (35)$$

where $\boldsymbol{Q}^*(\phi)$ is defined by eqn. (6) and $d(x, \theta)$ is the damage distribution function defined by

$$d(x,\theta) = [H\{x-(x_D-\overline{x})\} - H\{x-(x_D+\overline{x})\}] \times [H\{\theta-(\theta_D-\overline{\theta})\} - H\{\theta-(\theta_D+\overline{\theta})\}] \qquad (36)$$

where $H(\cdot)$ is the Heaviside's unit function. The expression of eqn. (35) is for a single damage, but it can be readily generalized for multiple damages (say N damages) as follows:

$$\Delta\overline{\boldsymbol{K}}(x,\theta) = h\mathbf{Q}^*(\phi_1)D_1 d_1(x,\theta) + h\mathbf{Q}^*(\phi_2)D_2 d_2(x,\theta) + \cdots + h\mathbf{Q}^*(\phi_N)D_N d_N(x,\theta) = \sum_{l=1}^{N} h\mathbf{Q}^*(\phi_l)D_l d_l(x,\theta) \qquad (37)$$

where the subscript l denotes the lth damage and d_l(x, θ) is now defined by

$$d_l(x,\theta) = [H\{x-(x_{Dl}-\overline{x}_l)\} - H\{x-(x_{Dl}+\overline{x}_l)\}] \times [H\{\theta-(\theta_{Dl}-\overline{\theta}_l)\} - H\{\theta-(\theta_{Dl}+\overline{\theta}_l)\}] \qquad (38)$$

where, as shown in Fig. 3, (x_{Dl}, θ_{Dl}) represents the location of the lth damage, and $2\overline{x}_l$ and $2\overline{\theta}_l$ represent the dimensions of the SMV containing the lth damage in the axial and circumferential directions, respectively.

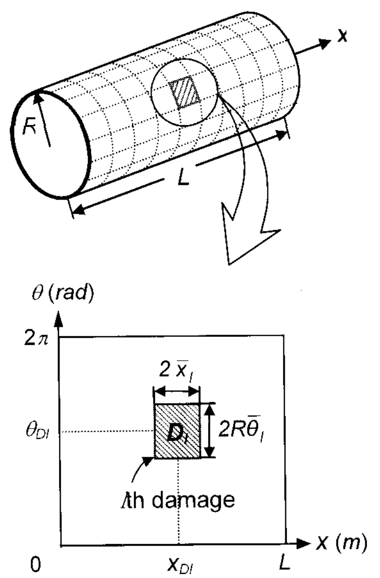


Fig. 3 Dimensions of the small material volume (SMV) containing the lth local damage.

Substituting the components of $\Delta\overline{K}(x,\theta)$ derived from eqn. (38) and Mij from eqn. (32) into eqn. (31) gives

$$\lambda = \sum_{l=1}^{N}(\alpha^l + \beta^l \cos 2\phi_l + \gamma^l \sin 2\phi_l)D_l \quad (39)$$

where $\alpha^l = [\alpha_{IJ}^l]$, $\beta^l = [\beta_{IJ}^l]$ and $\gamma^l = [\gamma_{IJ}^l]$ are defined by

$$\alpha_{IJ}^l = K\int_{\theta_{Dl}-\bar{\theta}_l}^{\theta_{Dl}+\bar{\theta}_l}\int_{x_{Dl}-\bar{x}_l}^{x_{Dl}+\bar{x}_l}\left[\left(\frac{1+\nu^2}{1-\nu^2}\right)M_{IJ}^{(1)}+\left(\frac{2\nu}{1-\nu^2}\right)M_{IJ}^{(2)} + \left(\frac{1+\nu^2}{1-\nu^2}\right)M_{IJ}^{(4)}+\left(\frac{1-\nu}{2(1+\nu)}\right)M_{IJ}^{(6)}\right]dxd\theta$$

$$\beta_{IJ}^l = K\int_{\theta_{Dl}-\bar{\theta}_l}^{\theta_{Dl}-\bar{\theta}_l}\int_{x_{Dl}-\bar{x}_l}^{x_{Dl}+\bar{x}_l}\left(-M_{IJ}^{(1)}+M_{IJ}^{(4)}\right)dxd\theta$$

$$\gamma_{IJ}^l = -\frac{1}{2}K\int_{\theta_{Dl}-\bar{\theta}_l}^{\theta_{Dl}+\bar{\theta}_l}\int_{x_{Dl}-\bar{x}_l}^{x_{Dl}+\bar{x}_l}\left(M_{IJ}^{(3)}+M_{IJ}^{(5)}\right)dxd\theta \quad (40)$$

The damage influence matrix λ depends on the normal modes of the intact shell {UI}, the damage orientation angles ϕ_l, and the damage magnitudes D_l.

4. Damage identification method

4.1. Damage identification algorithm

In general, it will be easier to measure the radial displacement $w(x, \theta, t)$ than to measure the other displacement components. Thus the inertance FRF of the radial displacement measured from a damaged shell, will be used for damage identification. The inertance FRF $\mathcal{A}$ is defined as the ratio of the acceleration to the applied force as

$$\mathcal{A}(\omega;x_M,\theta_M) = \frac{\ddot{w}(x_M,\theta_M,t)}{p_z(x_F,\theta_F,t)} = -\omega^2 W(x_M,\theta_M) \quad (41)$$

where $\ddot{w}(x_M,\theta_M,t)$ is the radial acceleration measured at (x_M, θ_M) and $p_z(x_F, \theta_F, t)$ is the harmonic point load externally applied at (x_F, θ_F). Applying the external load $p_z(x_F, \theta_F, t)$ defined by eqn. (21) and the radial displacement $w(x_M, \theta_M, t)$ from eqn. (30) into eqn. (41) yields

$$\mathcal{A}^D(\omega;x_M,\theta_M) = \mathcal{A}(\omega;x_M,\theta_M) + \Delta\mathcal{A}(\omega;x_M,\theta_M) \quad (42)$$

where $\mathcal{A}$ is the intact inertance FRF measured from the intact shell and $\Delta\mathcal{A}$ is the perturbed inertance FRF due to the presence of damages. They are given by

$$\mathcal{A}(\omega;x_M,\theta_M) = -\omega^2\boldsymbol{\Psi}_M^T\, diag[\Omega^2-\omega^2]\boldsymbol{\Psi}_F \quad (43)$$

$$\Delta\mathcal{A}(\omega;x_M,\theta_M) = -\omega^2\boldsymbol{\Psi}_M^T\boldsymbol{\lambda}\boldsymbol{\Psi}_F \quad (44)$$

where

$$\boldsymbol{\Psi}_M = \begin{Bmatrix}\vdots\\ \dfrac{W_I(x_M,\theta_M)}{\Omega_I^2-\omega^2}\\ \vdots\end{Bmatrix},\quad \boldsymbol{\Psi}_F = \begin{Bmatrix}\vdots\\ \dfrac{W_I(x_F,\theta_F)}{\Omega_I^2-\omega^2}\\ \vdots\end{Bmatrix}$$

$$diag[\Omega^2-\omega^2] = \begin{bmatrix}\ddots & & \\ & \Omega_I^2-\omega^2 & \\ & & \ddots\end{bmatrix} \quad (45)$$

The symbol *diag*[·] indicates the diagonal matrix. Equation (42) shows that the effect of damages appears only in the perturbed inertance FRF $\Delta\mathcal{A}$, which is determined by the damage influence matrix λ.

Substituting eqn. (39) into eqn. (44) gives

$$\sum_{l=1}^{N}\left[a^l(\omega;x_M,\theta_M)+b^l(\omega;x_M,\theta_M)\cos 2\phi_l + c^l(\omega;x_M,\theta_M)\sin 2\phi_l\right]D_l = \Delta\mathcal{A}(\omega;x_M,\theta_M) \quad (46)$$

where

$$a^l(\omega;x_M,\theta_M) = -\omega^2\boldsymbol{\Psi}_M^T(\omega;x_M,\theta_M)\boldsymbol{\alpha}^l\boldsymbol{\Psi}_F(\omega)$$
$$b^l(\omega;x_M,\theta_M) = -\omega^2\boldsymbol{\Psi}_M^T(\omega;x_M,\theta_M)\boldsymbol{\beta}^l\boldsymbol{\Psi}_F(\omega) \quad (47)$$
$$c^l(\omega;x_M,\theta_M) = -\omega^2\boldsymbol{\Psi}_M^T(\omega;x_M,\theta_M)\boldsymbol{\gamma}^l\boldsymbol{\Psi}_F(\omega)$$

Equation (46) provides the relationship between the damage information (*i.e.*, damage magnitudes D_l and damage orientation angles ϕ_l) and the damage-induced change in FRF $\Delta\mathcal{A}$. Thus, once $\Delta\mathcal{A}$ is experimentally measured from the damaged shell, eqn. (46) can be used to identify the unknown damage information.

For a chosen set of excitation frequency (ω) and response measurement point (x_M, θ_M), eqn. (46) provides an algebraic equation for unknown damage magnitudes D_l and damage orientation angles ϕ_l. Thus, by properly choosing as many different sets of (ω ; x_M, θ_M) as required, $2N$ for instance, a set of simultaneous algebraic equations may be obtained in the form as

$$\boldsymbol{X}(\boldsymbol{\Phi})\boldsymbol{D} = \Delta\boldsymbol{\mathcal{A}} \tag{48}$$

where

$$\begin{aligned} \boldsymbol{D} &= \{D_1 \quad D_2 \quad \cdots \quad D_N\}^{\mathrm{T}} \\ \boldsymbol{\Phi} &= \{\phi_1 \quad \phi_2 \quad \cdots \quad \phi_N\}^{\mathrm{T}} \\ \Delta\boldsymbol{\mathcal{A}} &= \{\Delta\mathcal{A}_1 \quad \Delta\mathcal{A}_2 \quad \cdots \quad \Delta\mathcal{A}_{2N}\}^{\mathrm{T}} \end{aligned} \tag{49}$$

and

$$\begin{aligned} &\boldsymbol{X}(\boldsymbol{\Phi}) = \boldsymbol{A} + \boldsymbol{B}\,diag[\cos 2\phi] + \boldsymbol{C}\,diag[\sin 2\phi] \\ &\boldsymbol{A} = [a_{kl}] = a^l(\omega; x_M, \theta_M)_k \qquad (k = 1, 2, ..., 2N) \\ &\boldsymbol{B} = [b_{kl}] = b^l(\omega; x_M, \theta_M)_k \qquad (l = 1, 2, ..., N) \\ &\boldsymbol{C} = [c_{kl}] = c^l(\omega; x_M, \theta_M)_k \end{aligned} \tag{50}$$

Equation (48) represents the structural damage identification algorithm to be used in the present paper to locate multiple directional damages and also to identify their severities (*i.e.*, damage magnitudes) and orientation angles with respect to the reference coordinates.

For the case of isotropic damages, the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ will vanish and eqn. (48) is reduced to the algorithm developed for isotropic damages in the previous study (Kim et al., 2004):

$$\boldsymbol{A}\boldsymbol{D} = \Delta\boldsymbol{\mathcal{A}} \tag{51}$$

where the matrix $\boldsymbol{A}$ is computed from eqn. (50). By properly choosing N different sets of (ω ; x_M, θ_M), the number of simultaneous algebraic equations can be made equal to N which is same as the number of SMV or the number of damage magnitudes D_l to be determined. Then, one may apply the direct matrix inversion method to solve eqn. (51) for $\boldsymbol{D}$.

4.2. Iterative approach of damage identification analysis

Since eqn. (48) is nonlinear in the unknown damage magnitudes D and damage orientation angles Φ, one may need to use a proper iterative approach to solve eqn. (48) for $\boldsymbol{D}$ and Φ. This solution process implies the identification of multiple directional damages. In this study, the Newton-Raphson method is used. Equation (48) can be rewritten as

$$\Delta\boldsymbol{\mathcal{A}} = \boldsymbol{A}\{D_l\} + \boldsymbol{B}\{D_l \cos 2\phi_l\} + \boldsymbol{C}\{D_l \sin 2\phi_l\} \tag{52}$$

From eqn. (52), one can define a vector as

$$\boldsymbol{g} = -\Delta\boldsymbol{\mathcal{A}} + \boldsymbol{A}\{D_l\} + \boldsymbol{B}\{D_l \cos 2\phi_l\} + \boldsymbol{C}\{D_l \sin 2\phi_l\} \tag{53}$$

where

$$\begin{aligned} &\boldsymbol{g} = \{g_1 \; g_2 \cdots g_{2N}\}^{\mathrm{T}} \text{ where } g_k = g_k(\boldsymbol{u}) \\ &\boldsymbol{u} = \{u_1 \; u_2 \cdots u_{2N}\}^{\mathrm{T}} \text{ where } u_l = D_l,\; u_{N+l} = \phi_l \\ &(k = 1, 2, \cdots, 2N),\; (l = 1, 2, \cdots, N) \end{aligned} \tag{54}$$

Then, the unknown variables $u_i (i = 1, 2, ..., 2N)$ can be iteratively computed by using the Newton-Raphson method as follows (Chapra and Canale, 1998):

$$\boldsymbol{u}^{(j+1)} = \boldsymbol{u}^{(j)} - \boldsymbol{J}^{(j)^{-1}} \boldsymbol{g}^{(j)} \tag{55}$$

where $\boldsymbol{J}^{(j)}$ is the Jacobian matrix.

The details of an iterative approach used in this paper can be summarized as follows:

A. Start-up:
   Guess the initial values of damage orientation angles and denote it by $\boldsymbol{\Phi}^{(1)}$. Then compute the damage magnitudes from eqn. (48) by using the first half N by N matrix, and consider the results as the initial values of damage magnitudes and denote it by $\boldsymbol{D}^{(1)}$. Set $\boldsymbol{D}^{(j)} = \boldsymbol{D}^{(1)}$ and $\boldsymbol{\Phi}^{(j)} = \boldsymbol{\Phi}^{(1)}$ to begin the iterative process. The initial values $\boldsymbol{\Phi}^{(1)}$ and $\boldsymbol{D}^{(1)}$ may be guessed or estimated by using other appropriate methods instead of using the approach described above. In the present numerical studies, we put $\boldsymbol{\Phi}^{(1)} = \boldsymbol{0}$.

B. Iterative step:
   Step 1: Compute $\boldsymbol{g}^{(j)}$ from eqn. (53) and $\boldsymbol{J}^{(j)}$ by using $\boldsymbol{D}^{(j)}$ and $\boldsymbol{\Phi}^{(j)}$.

Step 2: Compute the updated values of $\boldsymbol{D}$ and $\boldsymbol{\Phi}$ from eqn. (55) by using $g^{(j)}$ and $J^{(j)}$ computed at Step 1 and denote them by $\boldsymbol{D}^{(j+1)}$ and $\boldsymbol{\Phi}^{(j+1)}$.

Step 3: Consider $\boldsymbol{D}^{(j+1)}$ and $\boldsymbol{\Phi}^{(j+1)}$ computed at Step 2 as the previous values. That is, set $\boldsymbol{D}^{(j)} = \boldsymbol{D}^{(j+1)}$ and $\boldsymbol{\Phi}^{(j)} = \boldsymbol{\Phi}^{(j+1)}$.

Step 4: Continue the next iteration which starts again from Step 1 and repeat the iteration process until sufficiently converged results are obtained within a pre-specified accuracy or convert-gence limit.

5. Numerical illustrations and discussions

As an illustrative example, consider a cylindrical shell which is simply-supported at both ends. The cylindrical shell has the radius $R = 0.125m$, length $L = 0.3m$, thickness $h = 0.003m$, Young's Modulus $E = 206GPa$, Poisson's ratio $\nu = 0.33$, and the mass density $\rho = 7850kg/m^3$.

First, assume that the cylindrical shell has a line crack and investigate its effect on the natural frequencies of the cylindrical shell. The line crack is 0.015m long and it is centered at $(x_D, \theta_D) = (0.135m, 0.9\pi)$. To compute the effective orthotropic elastic stiffness Q_{ij}^D for the SMV containing the line crack, the dimensions of the SMV are chosen as $2\bar{x} = 0.3m$ and $2R\bar{\theta} = 0.025\pi m$ so that the effective damage magnitude becomes $D = 0.3$. Table 1 compares the natural frequencies of the example damaged shell, with varying the orientation angle of the line crack, with those of the intact cylindrical shell. Table 1 shows that in general the natural frequencies are reduced in magnitude due to the presence of damage and they are certainly dependent of damage direction.

Next, the numerically simulated damage identification tests are conducted to validate the present SDIM. An example problem shown in Fig. 4 is considered: the cylindrical shell with

Table 1 Natural frequencies (Hz) of a simply supported cylindrical shell at the intact and damaged states

Mode number (m,n,i)	Intact	Damaged $\phi = 0°$	Damaged $\phi = 30°$	Damaged $\phi = 45°$
(1,1,1)	3269.2	3262.4	3234.1	3226.9
(1,3,1)	1095.9	1094.7	1053.7	1043.7
(1,5,1)	1317.2	1309.3	1277.5	1266.6
(1,7,1)	2411.9	2393.3	2376.3	2373.7
(1,9,1)	3936.7	3907.2	3929.8	3932.7
(2,1,1)	5444.1	5436.0	5440.9	5442.5
(2,3,1)	2789.2	2784.7	2789.2	2788.5
(2,5,1)	2027.8	2025.6	2022.8	2022.0
(2,7,1)	2759.5	2756.6	2752.4	2750.8
(2,9,1)	4207.8	4204.1	4205.1	4205.8

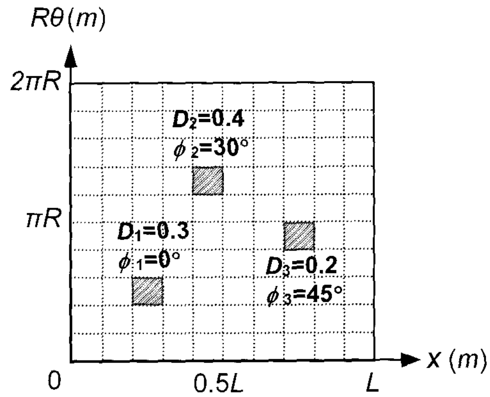


Fig. 4 An example problem considered for the numerically simulated damage identification tests

three line crack-like damages. The details of the line crack-like damages considered for two example problems are given in Table 2. As shown in Fig. 4, the cylindrical shells are divided into 100 equal-sized finite segments, and the damage identification analyses are conducted to determine the effective damage magnitudes and orientations within all finite segments. A harmonic point load is applied at $(x_F = 0.15m, \theta_F = \pi)$, and the inertance FRFs are analytically computed at each center of the finite segments.

Figure 5 shows the damage identification results. The result shown in Fig. 5 is the sufficiently converged result obtained after 25 iterations. Figure 5 clearly shows that the present SDIM certainly has the capability of identifying the directivities of multiple directional damages, in addition to the capability of identifying their locations and severities.

The result shown in Fig. 5 is obtained without taking into account any possible measurement noises in FRF-data. However, in practice, the experimentally measured FRF-data is liable to be contaminated by some sorts of measurement noises. Therefore, to investigate the effects of any possible measurement noises in FRF-data on the reliability of the present SDIM, e% random noises are added to the inertance FRFs analytically computed from eqn. (42) by following the approach used by Thyagarajan et al. (1998).

The effects of the random noises in FRF-data on the damage identification results are shown in Fig. 6 for the case of three-damage problem. As expected, increasing the level of random noises in FRF-data indeed degrades the damage identification results. However, for the example problems considered in this study, Fig. 6 shows that the present SDIM provides quite satisfactory damage identification results up to about 7% random noises in FRF-data.

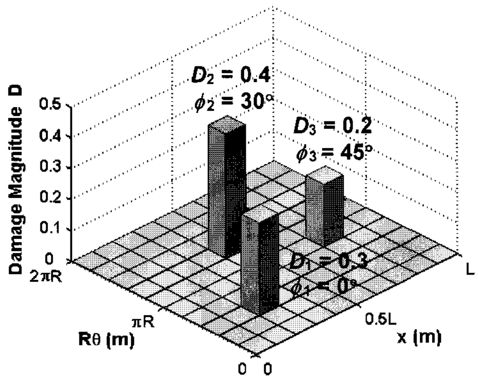
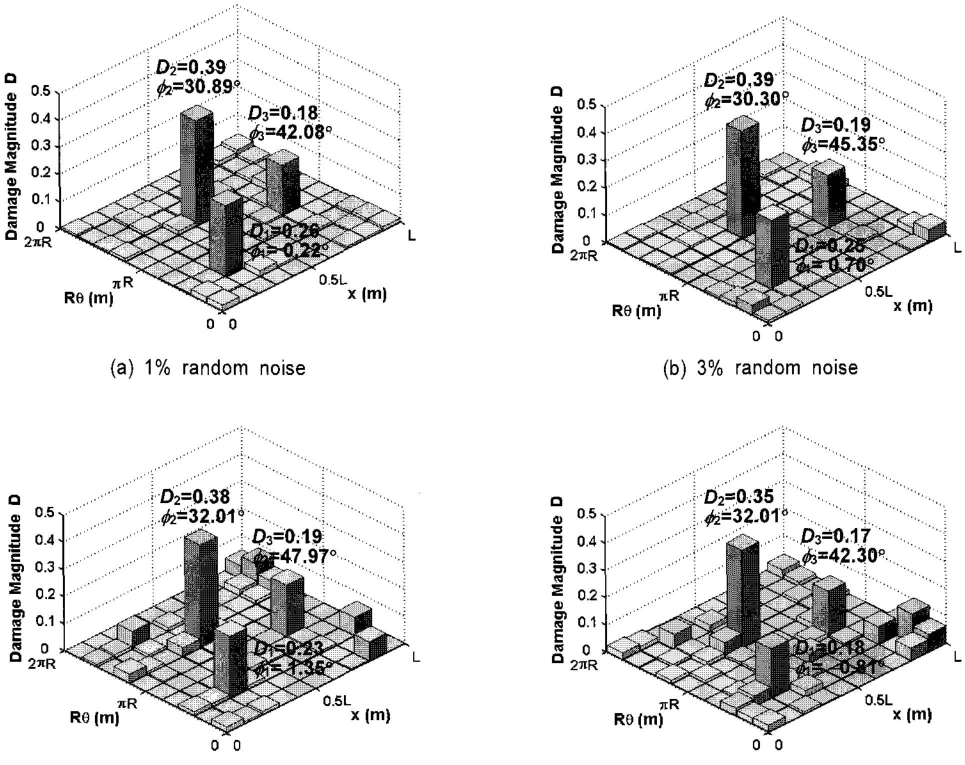


Fig. 5 Damage identification results for the three problem



(a) 1% random noise

(b) 3% random noise

(c) 5% random noise

(d) 7% random noise

Fig. 6 Effects of the measurement noises in FRF data on the damage identification results.

6. Conclusion

An FRF-based SDIM is proposed in this paper for the cylindrical shell with multiple directional damages. To develop the SDIM, first the equations of motion are derived for a damaged cylindrical shell. A structural damage identification algorithm is then derived by using the FRF solved from the equations of motion. The proposed SDIM has the capability of simultaneously identifying the directivities of multiple damages, in addition to the capability of identifying the locations and severities of the damages. The numerically simulated damage identification tests are conducted to evaluate the proposed SDIM. It is shown that the proposed SDIM provides quite reliable damage identification results up to about 7% measurement noises in FRF-data for the example problems considered in this paper.

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