INTUITIONISTIC FUZZY $\sigma$-SUBALGEBRAS OF BCK-ALGEBRAS WITH CONDITION (S)

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ABSTRACT. In this paper, some properties of intuitionistic fuzzy $\sigma$-subalgebras of BCK-algebra with condition (S) are investigated.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [12]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups and rings, etc. In 1991, Xi [11] applied the concept of fuzzy sets to BCK-algebras which are introduced by Y. Imai and K. Iséki in 1966 [6]. K. Iséki [5] introduced the notion of BCK-algebra with condition (S) and several researchers considered the fuzzification of it. Recently, Y. B. Jun et al. [7] introduced the notion of fuzzy $\sigma$-subalgebras in BCK-algebras with condition (S). In 1986, K. T. Atanassov [1] introduced the notion of intuitionistic fuzzy set which is a generalization of the notion of fuzzy set. S. M. Hong et al. [4], using the Atanassov’s idea, introduced the concept of intuitionistic fuzzy subalgebras in BCK-algebras and S. M. Hong and H. G. Kim [3] studied the Cartesian product of fuzzy $\sigma$-subalgebras in BCK-algebras with condition (S). In this paper, we introduce the notion of intuitionistic fuzzy $\sigma$-subalgebra in BCK-algebras with condition (S) and investigated some of their properties.

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2. Preliminaries

**Definition 2.1.** An algebra \((X, *, 0)\) of type \((2,0)\) is called a BCK-algebra if for all \(x, y, z \in X\) the following conditions hold:

(a) \(((x * y) * (x * z)) * (z * y) = 0\)
(b) \((x * (x * y)) * y = 0\)
(c) \(x * x = 0\)
(d) \(0 * x = 0\)
(e) \(x * y = 0\) and \(y * x = 0\) imply \(x = y\).

For any BCK-algebra \(X\), the relation \(\leq\) defined by \(x \leq y\) if and only if \(x * y = 0\) is a partial order on \(X\).

**Definition 2.2.** A BCK-algebra \(X\) is said to be with condition \((S)\) if for all \(x, y \in X\), the set \(\{z \in X | z * x \leq y\}\) has a greatest element, written \(x \circ y\).

In any BCK-algebra \(X\) with condition \((S)\), the following holds: for all \(x, y \in X\)

1. \(x \leq x \circ y, y \leq x \circ y\),
2. \(x \circ 0 = 0 \circ x = x\),
3. \(x \circ y = y \circ x\).

**Definition 2.3.** [3] Let \(X\) be a BCK-algebra with condition \((S)\) and let \(S\) be a nonempty subset of \(X\). Then \(S\) is called a \(\circ\)-subalgebra of \(X\) if, for any \(x, y \in S\), \(x \circ y \in S\).

**Definition 2.4.** [3] A map \(f : X \rightarrow Y\) of BCK-algebras with condition \((S)\) is called a \(*\)-homomorphism (resp. \(\circ\)-homomorphism) if \(f(x * y) = f(x) * f(y)\) (resp. \(f(x \circ y) = f(x) \circ f(y)\)) for all \(x, y \in X\). If \(f\) is both a \(*\)-homomorphism and a \(\circ\)-homomorphism of \(X\), we say that \(f\) is a homomorphism.

We now review some fuzzy logic concepts. Let \(X\) be a set. By a fuzzy set \(\mu\) in \(X\) we mean a function \(\mu : X \rightarrow [0,1]\), and the complement of \(\mu\), denoted by \(\overline{\mu}\), is the fuzzy set in \(X\) given by \(\overline{\mu}(x) = 1 - \mu(x)\) for all \(x \in X\).

**Definition 2.5.** [11] Let \(X\) be a BCK-algebra. A fuzzy subset \(\mu\) of \(X\) is called a fuzzy \(*\)-subalgebra of \(X\) if for all \(x, y \in X\), \(\mu(x * y) \geq \min\{\mu(x), \mu(y)\}\).
DEFINITION 2.6. [7] Let $X$ be a BCK-algebra with condition (S). A fuzzy subset $\mu$ of $X$ is called a fuzzy $\circ$-subalgebra of $X$ if for all $x, y \in X$, $\mu(x \circ y) \geq \min\{\mu(x), \mu(y)\}$.

THEOREM 2.7. [3] A fuzzy subset $\mu$ of a BCK-algebra $X$ with condition (S) is a fuzzy $\circ$-subalgebra of $X$ if and only if, for every $t \in [0, 1]$, $\mu_t$ is either $\emptyset$ or a $\circ$-subalgebra of $X$.

DEFINITION 2.8. [11] Let $\mu$ be a fuzzy subset of a set $S$. For $t \in [0, 1]$, the set

$$\mu_t = \{x \in S | \mu_t(x) \geq t\}$$

is called a level subset of $\mu$.

DEFINITION 2.9. [3] Let $X$ be a BCK-algebra with condition (S) and let $\mu$ be a fuzzy $\circ$-subalgebra of $X$. Then $\circ$-subalgebra $\mu_t$, $t \in [0, 1]$ are called level $\circ$-subalgebras of $\mu$.

DEFINITION 2.10. [1] Let $X$ be a nonempty fixed set. An intuitionistic fuzzy set (IFS for short) $A$ is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$$

where the function $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$.

DEFINITION 2.11. [4] An IFS $A = (\mu_A, \gamma_A)$ in a BCK-algebra $X$ is called an intuitionistic fuzzy subalgebra of $X$ if for all $x, y \in X$,

\begin{itemize}
  \item[(11)] $\mu_A(x \circ y) \geq \min\{\mu_A(x), \mu_A(y)\}$,
  \item[(12)] $\gamma_A(x \circ y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$.
\end{itemize}

PROPOSITION 2.12. [4] Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy subalgebra of BCK-algebra $X$. Then $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$ for all $x \in X$. 
3. Intuitionistic fuzzy $\circ$-subalgebras

In what follows, let $X$ denote a BCK-algebra with condition (S) unless otherwise specified.

**Definition 3.1.** An IFS $A = (\mu_A, \gamma_A)$ of $X$ is called an *intuitionistic fuzzy $\circ$-subalgebra* of $X$ if for all $x, y \in X$,

1. $\mu_A(x \circ y) \geq \min\{\mu_A(x), \mu_A(y)\}$,
2. $\gamma_A(x \circ y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$.

**Example 3.2.** Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by:

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Then $(X; *, 0)$ is a BCK-algebra with condition (S) and we can find the following $\circ$-table

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Let $s, t \in [0, 1]$ be such that $s + t \leq 1$. Define an IFS $A = (\mu_A, \gamma_A)$ in $X$ as follows:

- $\mu_A(0) = 1, \mu_A(1) = \mu_A(2) = s, \mu_A(3) = 0$,
- $\gamma_A(0) = 0, \gamma_A(1) = \gamma_A(2) = t, \gamma_A(3) = 1$.

By routine calculation we know that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $\circ$-subalgebra of $X$.

**Lemma 3.3.** An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $\circ$-subalgebra of $X$ if and only if the fuzzy sets $\mu_A$ and $\gamma_A$ are fuzzy $\circ$-subalgebras of $X$. 
**Proof.** Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy $\alpha$-subalgebra of $X$. Then $\mu_A$ is a fuzzy $\alpha$-subalgebra of $X$. Now, for every $x, y \in X$, we have

\[
\overline{\gamma}_A(x \circ y) = 1 - \gamma_A(x \circ y) \\
\geq 1 - \max\{\gamma_A(x), \gamma_A(y)\} \\
= \min\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\
= \min\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\}
\]

Hence $\overline{\gamma}_A(x)$ is a fuzzy $\alpha$-subalgebra of $X$.

Conversely, assume that both $\mu_A$ and $\overline{\gamma}_A$ are fuzzy $\alpha$-subalgebras of $X$. For every $x, y \in X$, we have $\mu_A(x \circ y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and

\[
1 - \gamma_A(x \circ y) = \overline{\gamma}_A(x \circ y) \\
\geq \min\{\overline{\gamma}_A(x), \overline{\gamma}_A(y)\} \\
= \min\{1 - \gamma_A(x), 1 - \gamma_A(y)\} \\
= 1 - \max\{\gamma_A(x), \gamma_A(y)\}.
\]

It follows that $\gamma_A(x \circ y) \leq \max\{\gamma_A(x), \gamma_A(y)\}$. Thus, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $\alpha$-subalgebra of $X$. $\square$

**Theorem 3.4.** Let $A = (\mu_A, \gamma_A)$ be an IFS in $X$. Then it is an intuitionistic fuzzy $\alpha$-subalgebra of $X$ if and only if $\overline{\gamma}_A := (\mu_A, \overline{\gamma}_A)$ and $\check{\Diamond} A := (\overline{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy $\alpha$-subalgebras of $X$.

**Proof.** If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $\alpha$-subalgebra of $X$, then $\mu_A = \overline{\mu}_A$ and $\gamma_A$ are fuzzy $\alpha$-subalgebras from Lemma 3.3.

Conversely, if $\overline{\gamma}_A = (\mu_A, \overline{\gamma}_A)$ and $\check{\Diamond} A = (\overline{\gamma}_A, \gamma_A)$ are intuitionistic fuzzy $\alpha$-subalgebras of $X$, then the fuzzy sets $\mu_A$ and $\overline{\gamma}_A$ are fuzzy $\alpha$-subalgebras of $X$. Thus, $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $\alpha$-subalgebra of $X$. $\square$

**Definition 3.5.** Let $A = (\mu_A, \gamma_A)$ be an IFS in $X$ and let $t \in [0, 1]$. Then the set $U(\mu_A; t) = \{x \in X \mid \mu_A(x) \geq t\}$ (resp. $L(\gamma_A; t) = \{x \in X \mid \gamma_A(x) \leq t\}$) is called upper $t$-level cut (resp. lower $t$-level cut) of $A$.

**Theorem 3.6.** If an IFS $A = (\mu_A, \gamma_A)$ in $X$ is an intuitionistic fuzzy $\alpha$-subalgebra of $X$, the upper $t$-level cut and lower $t$-level cut of $A$ are $\alpha$-subalgebras of $X$ for every $t \in [0, 1]$ such that $t \in \text{Im}(\mu_A) \cap \text{Im}(\gamma_A)$. 
which are called an upper level subalgebra and a lower level subalgebra respectively.

Proof. If \( x, y \in U(\mu_A; t) \), then \( \mu_A(x) \geq t \) and \( \mu_A(y) \geq t \). Hence we have \( \mu_A(x \circ y) \geq \min\{\mu_A(x), \mu_A(y)\} \geq t \). It follows that \( x \circ y \in U(\mu_A; t) \). Thus, \( U(\mu_A; t) \) is a \( \alpha \)-subalgebra of \( X \). Now let \( x, y \in L(\gamma_A; t) \). Then \( \gamma_A(x \circ y) \leq \max\{\gamma_A(x), \gamma_A(y)\} \leq t \) and hence \( x \circ y \in L(\gamma_A; t) \). Thus, \( L(\gamma_A; t) \) is a \( \alpha \)-subalgebra of \( X \). \( \square \)

**Theorem 3.7.** Let \( A = (\mu_A, \gamma_A) \) be an IFS in \( X \) such that the nonempty sets \( U(\mu_A; t) \) and \( L(\gamma_A; t) \) are \( \alpha \)-subalgebras of \( X \) for every \( t \in [0, 1] \). Then \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy \( \alpha \)-subalgebra of \( X \).

Proof. We need to prove that \( A = (\mu_A, \gamma_A) \) satisfies the conditions (IF1) and (IF2). First, if the condition (IF1) does not hold, then there exist \( x_0, y_0 \in X \) such that \( \mu_A(x_0 \circ y_0) < \min\{\mu_A(x_0), \mu_A(y_0)\} \). Let

\[
t_0 = \frac{1}{2}[\mu_A(x_0 \circ y_0) + \min\{\mu_A(x_0), \mu_A(y_0)\}].
\]

Then \( \mu_A(x_0 \circ y_0) < t_0 < \min\{\mu_A(x_0), \mu_A(y_0)\} \) and hence, \( x_0 \circ y_0 \notin U(\mu_A; t_0) \), but \( x_0, y_0 \in U(\mu_A; t_0) \). This is a contradiction.

Second, if the condition (IF2) does not hold, then

\[
\gamma_A(x_0 \circ y_0) > \max\{\gamma_A(x_0), \gamma_A(y_0)\},
\]

for some \( x_0, y_0 \in X \). Let

\[
s_0 = \frac{1}{2}[\gamma_A(x_0 \circ y_0) + \max\{\gamma_A(x_0), \gamma_A(y_0)\}].
\]

Then \( \max\{\gamma_A(x_0), \gamma_A(y_0)\} < s_0 < \gamma_A(x_0 \circ y_0) \). It follows that \( x_0, y_0 \in L(\gamma_A; s_0) \) and \( x_0 \circ y_0 \notin L(\gamma_A; s_0) \), which is a contradiction. This completes the proof. \( \square \)

**Theorem 3.8.** Any \( \alpha \)-subalgebra of \( X \) can be realized as both an upper level subalgebra and a lower level subalgebra of some intuitionistic fuzzy \( \alpha \)-subalgebra of \( X \).

Proof. Let \( S \) be a \( \alpha \)-subalgebra of \( X \) and let \( \mu_A \) and \( \gamma_A \) be fuzzy sets of \( X \) defined by

\[
\mu_A(x) = \begin{cases} 
\alpha, & \text{if } x \in S, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
\gamma_A(x) = \begin{cases} 
\alpha, & \text{if } x \in S, \\
0, & \text{otherwise},
\end{cases}
\]
and

$$\gamma_A(x) = \begin{cases} \beta, & \text{if } x \in S, \\ 1, & \text{otherwise,} \end{cases}$$

for all $x \in X$, where $\alpha$ and $\beta$ are fixed numbers in $(0, 1)$ such that $\alpha + \beta < 1$. Let $x, y \in X$. If $x, y \in S$, then $x \circ y \in S$. Thus $\mu_A(x \circ y) = \min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(x \circ y) = \max\{\gamma_A(x), \gamma_A(y)\}$. If at least one of $x$ and $y$ does not belong to $S$, then at least one of $\mu_A(x)$ and $\mu_A(y)$ is equal to 0, and at least one of $\gamma_A(x)$ and $\gamma_A(y)$ is equal to 1. It follows that $\mu_A(x \circ y) \geq 0 = \min\{\mu_A(x), \mu_A(y)\}$ and $\gamma_A(x \circ y) \leq 1 = \max\{\gamma_A(x), \gamma_A(y)\}$. Thus $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $o$-subalgebra of $X$. Clearly, we have $U(\mu_A; o) = S = L(\gamma_A; o)$. This completes the proof.

Let $f$ be a function from a set $X$ to a set $Y$. If $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are IFSs in $X$ and $Y$ respectively, then the preimage of $B$ under $f$, denoted by $f^{-1}(B)$, is an IFS in $X$ defined by

$$f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)),$$

and the image of $A$ under $f$, denoted by $f(A)$, is an IFS of $Y$ defined by

$$f(A) = (f_s(\mu_A), f_i(\gamma_A)),$$

where

$$f_s(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_i(\gamma_A)(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \gamma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for each $y \in Y$ ([2]).

**Theorem 3.9.** Let $f : X \to Y$ be a $o$-homomorphism of BCK-algebras with condition (S). If $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy $o$-subalgebra of $Y$, then the preimage $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ of $B$ under $f$ is an intuitionistic fuzzy $o$-subalgebra of $X$. 
Proof. Suppose that $B = (\mu_B, \gamma_B)$ is an intuitionistic fuzzy $\circ$-subalgebra of $Y$. Let $x_1, x_2 \in X$. Then

$$f^{-1}(\mu_B)(x_1 \circ x_2) = \mu_B(f(x_1 \circ x_2)) = \mu_B(f(x_1) \circ f(x_2)) \geq \min\{\mu_B(f(x_1)), \mu_B(f(x_2))\} \geq \min\{f^{-1}(\mu_B)(x_1), f^{-1}(\mu_B)(x_2)\}$$

and

$$f^{-1}(\gamma_B)(x_1 \circ x_2) = \gamma_B(f(x_1 \circ x_2)) = \gamma_B(f(x_1) \circ f(x_2)) \leq \max\{\gamma_B(f(x_1)), \gamma_B(f(x_2))\} \leq \max\{f^{-1}(\gamma_B)(x_1), f^{-1}(\gamma_B)(x_2)\}.$$ 

Thus, $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ is an intuitionistic fuzzy $\circ$-subalgebra of $X$. 

**Theorem 3.10.** Let $f : X \to Y$ be an onto $\circ$-homomorphism of BCK-algebras with condition (S). If $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy $\circ$-subalgebra of $X$, then the image $f(A) = (f_\circ(\mu_A), f_\circ(\gamma_A))$ of $A$ under $f$ is an intuitionistic fuzzy $\circ$-subalgebra of $Y$.

**Proof.** Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy $\circ$-subalgebra of $X$ and let $y_1, y_2 \in Y$. Observing that $\{x_1 \circ x_2 \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \subseteq \{x \in X \mid x \in f^{-1}(y_1 \circ y_2)\}$. We have

$$f_\circ(\mu_A)(y_1 \circ y_2) = \sup\{\mu_A(x) \mid x \in f^{-1}(y_1 \circ y_2)\} \geq \sup\{\mu_A(x_1 \circ x_2) \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \geq \sup\{\min\{\mu_A(x_1), \mu_A(x_2)\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \geq \min\{\sup\{\mu_A(x_1) \mid x_1 \in f^{-1}(y_1)\}, \sup\{\mu_A(x_2) \mid x_2 \in f^{-1}(y_2)\}\} = \min\{f_\circ(\mu_A)(y_1), f_\circ(\mu_A)(y_2)\}.$$
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and

\[
\begin{align*}
f_i(\gamma_A)(y_1 \circ y_2) &= \inf\{\gamma_A(x) \mid x \in f^{-1}(y_1 \circ y_2)\} \\
&\leq \inf\{\gamma_A(x_1 \circ x_2) \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\
&\leq \inf\{\max\{\gamma_A(x_1), \gamma_A(x_2)\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\} \\
&= \max\{\inf\{\mu_A(x_1) \mid x_1 \in f^{-1}(y_1)\}, \inf\{\mu_A(x_2) \mid x_2 \in f^{-1}(y_2)\}\} \\
&= \max\{f_i(\gamma_A)(y_1), f_i(\gamma_A)(y_2)\}.
\end{align*}
\]

Thus, \( f(A) = (f_i(\mu_A), f_i(\gamma_A)) \) is an intuitionistic fuzzy \( \alpha \)-subalgebra of \( Y \). \( \square \)

Let \( f : X \rightarrow Y \) be a \( \alpha \)-homomorphism of BCK-algebras with condition \((S)\). For any IFS \( A = (\mu_A, \gamma_A) \) in \( Y \), we define an IFS \( A' = (\mu_A', \gamma_A') \) in \( X \) by

\[
\mu_A'(x) := \mu_A(f(x)), \quad \gamma_A'(x) := \gamma_A(f(x)),
\]

for all \( x \in X \).

**Theorem 3.11.** Let \( f : X \rightarrow Y \) be a \( \alpha \)-homomorphism of BCK-algebras with condition \((S)\). If an IFS \( A = (\mu_A, \gamma_A) \) in \( Y \) is an intuitionistic fuzzy \( \alpha \)-subalgebra of \( Y \), then the IFS \( A' = (\mu_A', \gamma_A') \) in \( X \) is an intuitionistic fuzzy \( \alpha \)-subalgebra of \( X \).

**Proof.** Let \( x, y \in X \). Then

\[
\begin{align*}
\mu_A'(x \circ y) &= \mu_A(f(x \circ y)) \\
&= \mu_A(f(x) \circ f(y)) \\
&\geq \min\{\mu_A(f(x)), \mu_A(f(y))\} \\
&= \min\{\mu_A'(x), \mu_A'(y)\}
\end{align*}
\]

and

\[
\begin{align*}
\gamma_A'(x \circ y) &= \gamma_A(f(x \circ y)) \\
&= \gamma_A(f(x) \circ f(y)) \\
&\leq \max\{\gamma_A(f(x)), \gamma_A(f(y))\} \\
&= \max\{\gamma_A'(x), \gamma_A'(y)\}.
\end{align*}
\]
Hence, \( A^f = (\mu_A^f, \gamma_A^f) \) is an intuitionistic fuzzy \( 	riangleright \)-subalgebra of \( X \). This completes the proof. \( \square \)

**Theorem 3.12.** Let \( f : X \rightarrow Y \) be an epimorphism of \( \text{BCK}\)-algebras with condition \((S)\) and let \( A = (\mu_A, \gamma_A) \) be an IFS in \( Y \). If \( A^f = (\mu_A^f, \gamma_A^f) \) is an intuitionistic fuzzy \( 	riangleright \)-subalgebra of \( X \), then \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy \( 	riangleright \)-subalgebra of \( Y \).

**Proof.** Let \( y_1, y_2 \in Y \). Then there exist \( x_1, x_2 \in X \) such that \( f(x_i) = y_i \), for \( i = 1, 2 \). Then

\[
\mu_A(y_1 \circ y_2) = \mu_A(f(x_1) \circ f(x_2)) = \mu_A(f(x_1 \circ x_2)) = \mu_A^f(x_1 \circ x_2) \geq \min\{\mu_A^f(x_1), \mu_A^f(x_2)\} = \min\{\mu_A(f(x_1)), \mu_A(f(x_2))\} = \min\{\mu_A(y_1), \mu_A(y_2)\}
\]

and

\[
\gamma_A(y_1 \circ y_2) = \gamma_A(f(x_1) \circ f(x_2)) = \gamma_A(f(x_1 \circ x_2)) = \mu_A^f(x_1 \circ x_2) \leq \max\{\gamma_A^f(x_1), \gamma_A^f(x_2)\} = \max\{\gamma_A(f(x_1)), \gamma_A(f(x_2))\} = \max\{\gamma_A(y_1), \gamma_A(y_2)\}.
\]

Thus, \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy \( 	riangleright \)-subalgebra of \( Y \). This completes the proof. \( \square \)

**References**

FUZZY O-SUBALGEBRAS OF BCK-ALGEBRAS


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