ON LIE IDEALS OF PRIME RINGS WITH GENERALIZED JORDAN DERIVATION

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ABSTRACT. The purpose of this paper is to show that every generalized Jordan derivation of prime ring with characteristic not two is a generalized derivation on a nonzero Lie ideal $U$ of $R$ such that $u^2 \in U$ for all $u \in U$ which is a generalization of the well-known result of I. N. Herstein.

1. Introduction

Let $R$ be a prime ring with characteristic different from two. An additive mapping $d : R \rightarrow R$ is called Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. An additive subgroup $U$ of $R$ is said to be a Lie ideal of $R$ if $[u, r] \in U$, for all $u \in U, r \in R$. The notion of generalized derivation of prime ring $R$ was introduced by B. Hvala in [1]. An additive map $f$ of an associative ring $R$ is called a generalized derivation if there is a derivation $d$ of $R$ such that

$$f(xy) = f(x)y + xd(y), \text{ for all } x, y \in R.$$ 

A classical result of I. N. Herstein states that every Jordan derivation of prime ring with characteristic not two is a derivation in [2]. A brief proof of this theorem can be found in [4]. Latter on, this result was generalized on Lie ideals of $R$ such that $u^2 \in U$ for all $u \in U$ in [7] and generalized derivations of prime ring $R$ in [6]. We shown that
this result holds for generalized derivation on a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$.

Throughout this paper, let $R$ be a prime ring with its characteristic not two, $U$ a nonzero Lie ideal of $R$ with $u^2 \in U$ for all $u \in U$ and let $f : R \to R$ be a generalized Jordan derivation of $R$ associated with a derivation $d$ of $R$ such that $f(x^2) = f(x)x + xd(x)$, for all $x \in R$.

In view of the hypothesis the $u^2 \in U$ for all $u \in U$, we get $(u + v)^2 \in U$ and so $(u + v)^2 - u^2 - v^2 = uv + vu \in U$ for all $u, v \in U$. Also $vu - uv \in U$, for all $u, v \in U$. Hence we find that $2vu \in U$, for all $u, v \in U$.

**Lemma 1.** For all $u, v, w \in U$,

1. $f(uv + vu) = f(u)v + ud(v) + f(v)u + vd(u)$
2. $f(uvu) = f(u)vu + ud(v)u + uvd(u)$
3. $f(uvw + wvu) = f(u)vw + ud(v)w + uvd(w) + f(w)vu + vd(v)u + uvd(u)$.

**Proof.** i) Linearizing, we get

$$f((u + v)^2) = f((u + v)(u + v)) = f(u^2 + uv + vu + v^2) = f(u^2) + f(uv + vu) + f(v^2)$$

$$= f(u)u + ud(u) + f(uv + vu) + f(v)v + vd(v)$$

for all $u, v \in U$.

On the other hand,

$$f((u + v)^2) = f((u + v)u + v) + (u + v)d(u + v)$$

$$= f(u)u + f(v)u + f(u)v + f(v)v + ud(u) + ud(v) + vd(u) + vd(v)$$

for all $u, v \in U$.

Comparing (1.1) and (1.2), we have

$$f(uv + vu) = f(u)v + ud(v) + f(v)u + vd(u)$$

for all $u, v \in U$.

ii) Replacing $v$ by $uv + vu$ in (i), we get
\begin{equation}
\begin{aligned}
f(u(uv + vu) + (uv + vu)u) &= f(u^2v + uvu + uvu + vu^2) \\
&= f(u^2v + vu^2) + 2f(uvu) = f(u^2)v + u^2d(v) + f(v)u^2 \\
&\quad + vd(u^2) + 2f(uvu) = f(u)uv + ud(u)v + u^2d(v) \\
&\quad + f(v)u^2 + vd(u)u + vud(u) + 2f(uvu) \text{ for all } u, v \in U.
\end{aligned}
\tag{1.3}
\end{equation}

On the other hand, we have
\begin{equation}
\begin{aligned}
f(u(uv + vu) + (uv + vu)u) &= f(u)(uv + vu) + ud(uv + vu) \\
&\quad + f(uv + vu)u + (uv + vu)d(u) = f(u)uv + f(u)vu + ud(uu) \\
&\quad + ud(vu) + f(u)vu + ud(v)u + f(v)u^2 + vd(u)u \\
&\quad + uvd(u) + vud(u) = f(u)uv + f(u)vu + ud(u)v + u^2d(v) \\
&\quad + ud(v)u + uvd(u) + f(u)vu + ud(v)u + f(v)u^2 + vd(u)u \\
&\quad + uvd(u) + vud(u) \text{ for all } u, v \in U.
\end{aligned}
\tag{1.4}
\end{equation}

Comparing (1.3) and (1.4), using \(\text{char}R \neq 2\), we get the required result.

iii) Linearizing (ii) on \(u\), we get
\begin{equation}
\begin{aligned}
f((u + w)v(u + w)) &= f(uwu + uwu + wvu + wvu) \\
&= f(uwu) + f(uvw + wvu) + f(wvw) = f(u)vu + ud(v)u + uvd(u) \\
&\quad + f(uvw + wvu) + f(w)vw + wd(v)v + wvd(u) \text{ for all } u, v, w \in U.
\end{aligned}
\tag{1.5}
\end{equation}

Now compute \(f((u + w)v(u + w))\) in other way, we get
\begin{equation}
\begin{aligned}
f((u + w)v(u + w)) &= f(u + w)(vu + vw) + (u + w)d(v)(u + w) \\
&\quad + (uv + vw)d(u + w) = f(u)vu + f(u)vw + f(w)vu + f(w)vw \\
&\quad + ud(v)u + ud(v)w + wd(v)v + wvd(u) \\
&\quad + wvd(u) + wvd(w) + wvd(u) + wvd(w) \text{ for all } u, v, w \in U.
\end{aligned}
\tag{1.6}
\end{equation}

Comparing (1.5) and (1.6), we have
\[f(uvw + wvu) = f(u)vu + ud(v)w + wvd(w) + f(w)vu \\
\quad + wd(v)u + wvd(u) \text{ for all } u, v, w \in U. \]

\[\square\]

\textbf{Remark 1.} We introduce abbreviation
\[ u^v = f(uv) - f(u)v - ud(v) \text{ for all } u, v \in U. \]

Observe also by Lemma 1 (i), we have
\[ f(uv + vu) = f(u)v + ud(v) + f(v)u + vd(u) \]
and so,
\[ f(uv) - f(u)v - ud(v) = -(f(vu) - f(v)u - vd(u)). \]

That is,
\[
(1.7) \quad u^v = -v^u \text{ for all } u, v \in U.
\]

**Lemma 2.** For all \( u, v \in U \), \( u^v[u,v] = 0 \).

**Proof.** Replace \( w \) by \( uv \) in Lemma 1 (iii) and using the fact that \( \text{char} R \neq 2 \), we get
\[
(1.8) \quad f((uv)^2 + uv^2 u) = f(uv)uv + uv^2 d(u) + f(u)v^2 u + ud(v)u + u^2 d(u) \text{ for all } u, v \in U.
\]

On the other hand, we get
\[
(1.9) \quad f(uv(uv) + (uv)vu) = f(u)vuv + ud(v)uv + uv^2 d(u) + f(vu)uv + ud(v)uv + u^2 d(u) \text{ for all } u, v \in U.
\]

Comparing this two equations, we have
\[
\begin{align*}
  f(uv)uv + f(v)u^2 u + ud(v)vu &= f(u)vuv + ud(v)uv + f(u)uvu.
\end{align*}
\]

That is,
\[
(u^v)[u,v] = 0 \text{ for all } u, v \in U. \]

**Theorem 1.** Let \( R \) be a non-commutative prime ring with characteristic not two, \( U \) a noncentral Lie ideal of \( R \) such that \( u^2 \in U \) for all \( u \in U \). If \( f \) be a generalized Jordan derivation on \( U \) then \( f \) is a generalized derivation on \( U \).
Proof. From Lemma 1 (iii), we have

\[ f(uwv + vwu) = f(u)wv + ud(w)v + uw(u) \]
\[ + f(v)wu + vd(w)u + vwd(u) \]
for all \( u, v, w \in U \).

Replacing \( u \) by \( uv \) and \( v \) by \( vu \) in (1.10), we get

\[ f((uv)w(vu) + (vu)w(uv)) = f(uvw)w + uwd(vu) + uvw(u) \]
\[ + f(v)uwv + vd(w)uv + vwd(vu) \]
\[ = f(uvw)w + uwd(vu) + uvw(u) \]
\[ + f(v)uwv + vd(w)uv + vwd(vu) \]
\[ = uvwd(v) + vuwd(v) \]
for all \( u, v, w \in U \).

Comparing equations (1.11) and (1.12), we obtain

\[ \{ f(u) - f(v) - ud(u) \} uvw \]
\[ + \{ f(u) - f(v) - ud(v) \} wvu = 0 \]

and hence

\[ u^6wuw + w^6wvu = 0 \]
for all \( u, v, w \in U \).

Using the \( u^6 = -u^6 \), we get

\[ u^6U[v, u] = 0 \]
for all \( u, v \in U \).

Since \( U \not\subseteq Z \), by [3, Lemma 4] we obtain for each pair \( u, v \in U \)
either \( u^6 = 0 \) or \( [u, v] = 0 \). Notice that the mappings \((u, v) \rightarrow u^6 \) and \((u, v) \rightarrow [u, v] \) satisfy the requirements of the [4, Lemma 4]. Hence \( u^6 = 0 \) for all \( u, v \in U \) or \( [u, v]^2 = 0 \) for all \( u, v \in U \). If \( [u, v]^2 = 0 \) for
all \( u, v \in U \) then for each \( u \in U \), \( I_u(v)^2 = 0 \), for all \( v \in U \), where \( I_u \)

is the inner derivation. Hence we get \( I_u(U) = 0 \) by [3, Theorem 1].

This yields that \( U \subset Z \), a contradiction. Thus, we have \( u^v = 0 \) for all \( u, v \in U \). This completes the proof. \( \square \)

REFERENCES


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