Lp NORM INEQUALITIES FOR AREA FUNCTIONS WITH APPROACH REGIONS

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Abstract. In this paper we first introduce a space of homogeneous type $X$, and then consider a kind of generalized upper half-space $X \times (0, \infty)$. We are mainly considered with inequalities for the $L^p$ norms of area functions associated with approach regions in $X \times (0, \infty)$.

1. Introduction

Recently enormous progress in harmonic analysis has been made. This paper will announce a problem related to harmonic analysis: In this paper we first introduce a space of homogeneous type $X$, which is a more general setting than $\mathbb{R}^n$, and we also consider a kind of generalized upper half-space $X \times (0, \infty)$. Suppose that for each boundary point $x \in X$ we are given an approach region $\Gamma_\alpha(x) \subset X \times (0, \infty)$. Let $f$ be a measurable function defined on $X \times (0, \infty)$. Then we define an area function $A^{(\alpha)}(f)$ associated with $\Gamma_\alpha(x)$.

The purpose of this paper is to study inequalities for the $L^p$ norms of area functions $A^{(\alpha)}(f)$ and $A^{(1)}(f)$ for $\alpha > 1$.

2. Preliminaries and Notations

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We begin by introducing the notion of a space of homogeneous type (see Coifman and Weiss [2]): Let $X$ be a topological space endowed with Borel measure $\mu$. Assume that $d$ is a pseudo-metric on $X$, that is, a non-negative function on $X \times X$ satisfying

(i) $d(x, x) = 0$; $d(x, y) > 0$ if $x \neq y$,
(ii) $d(x, y) = d(y, x)$, and
(iii) $d(x, z) \leq K [d(x, y) + d(y, z)]$, where $K$ is some fixed constant.

Assume further that

(a) the balls $B(x, \rho) = \{ y \in X : d(x, y) < \rho \}$, $\rho > 0$, form a basis of open neighborhoods at $x \in X$, and that $\mu$ satisfies the doubling property:

(b) $0 < \mu(B(x, 2\rho)) \leq A \mu(B(x, \rho)) < \infty$, where $A$ is some fixed constant.

Then we call $(X, d, \mu)$ a space of homogeneous type.

Property (iii) will be referred to as the "triangle inequality."

We consider the space $X \times (0, \infty)$, which is a kind of generalized upper half-space over $X$. We then introduce the analogue of non-tangential or conical regions. For $x \in X$ and $\alpha > 0$, set

$$\Gamma_\alpha(x) = \{ (y, t) \in X \times (0, \infty) : x \in B(y, \alpha t) \}.$$ 

Throughout this paper, we put $\Gamma(x) = \Gamma_1(x)$ for simplicity.

For any closed subset $F \subset X$ and $\alpha > 0$, $\mathcal{R}^{(\alpha)}(F)$ will be the union of approach regions with boundary points in $F$, that is,

$$\mathcal{R}^{(\alpha)}(F) = \bigcup_{x \in F} \Gamma_\alpha(x).$$

We now introduce an area function associated with an approach region. Let $f$ be a measurable function defined on $X \times (0, \infty)$. For $x \in X$ and $\alpha > 0$, we define an area function $A^{(\alpha)}(f)$ by

$$A^{(\alpha)}(f)(x) = \left\{ \int_{\Gamma_\alpha(x)} |f(y, t)|^2 \frac{d\mu(y)dt}{t^{\sigma+1}} \right\}^{1/2}, \quad \sigma \in \mathbb{R}$$ (1)
whenever the integral exists.

Let \( f \) be a locally integrable function on \( X \). Then for \( x \in X \), we define
\[
M(f)(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y),
\]
where the supremum is taken over all balls \( B \) containing \( x \). Then we call \( M(f) \) the Hardy-Littlewood maximal function of \( f \).

We need the notion of points of density. Suppose \( F \) is a closed subset of \( X \) whose complement has finite measure. Let \( \gamma \) be a fixed parameter, \( 0 < \gamma < 1 \). Then we say that a point \( x \in X \) has global \( \gamma \)-density with respect to \( F \), if
\[
\frac{\mu(F \cap B)}{\mu(B)} \geq \gamma
\]
for all balls \( B \) centered at \( x \) in \( X \). Let \( F^* \) be the set of points of global \( \gamma \)-density with respect to \( F \); then \( F^* \) is closed, \( F^* \subset F \), and
\[
\mathring{c}F^* = \{ x \in X : M(\chi_{cF})(x) > 1 - \gamma \},
\]
where \( \chi_{cF} \) is the characteristic function of the open set \( \mathring{c}F \).

3. \( L^p \) estimate for \( A^{(\alpha)}(f) \) and \( A^{(1)}(f) \), \( \alpha > 1 \)

We state the four lemmas we need:

**Lemma 1.** Let \((X, d, \mu)\) be a space of homogeneous type. Assume \( F \) is a closed subset of \( X \). Then there is a constant \( C_\gamma \) such that
\[
\mu(\mathring{c}F^*) \leq C_\gamma \mu(\mathring{c}F),
\]
where \( F^* \) is the set of points of global \( \gamma \)-density with respect to \( F \).

**Proof.** See Suh [6].
Lemma 2. Let \((X, d, \mu)\) be a space of homogeneous type. Suppose \(\alpha > 0\) is given. Then there is a constant \(C_\alpha\) so that whenever \(F\) is a closed subset of \(X\) and \(S(y, t)\) is a non-negative measurable function on \(X \times (0, \infty)\), then

\[
\int_F \left\{ \int_{\Gamma_\alpha(x)} S(y, t) d\mu(y) dt \right\} d\mu(x) \leq C_\alpha \int_{\mathcal{R}(\alpha) (F)} S(y, t) t^\alpha d\mu(y) dt,
\]

where \(\sigma\) is given as (1).

Proof. Fubini’s theorem gives

\[
\int_F \left\{ \int_{\Gamma_\alpha(x)} S(y, t) d\mu(y) dt \right\} d\mu(x) = \int_{X \times (0, \infty)} S(y, t) \left\{ \int_F \chi_{B(y, at)}(x) d\mu(x) \right\} d\mu(y) dt,
\]

where \(\chi_{B(y, at)}\) is the characteristic function of the ball \(B(y, at)\). Thus for given \((y, t) \in \mathcal{R}(\alpha) (F)\), it will suffice to show that there is a constant \(C_\alpha\) so that

\[
\int_F \chi_{B(y, at)}(x) d\mu(x) \leq C_\alpha t^\alpha.
\]

In fact, let \((y, t) \in \mathcal{R}(\alpha) (F)\). Then

\[
\int_F \chi_{B(y, at)}(x) d\mu(x) \leq \int_X \chi_{B(y, at)}(x) d\mu(x) = C_\alpha t^\alpha,
\]

as desired. The proof is therefore complete.

\(\square\)
Lemma 3. Let $(X,d,\mu)$ be a space of homogeneous type. Suppose $\alpha > 0$ is given. Then there are constants $C_{\alpha,\gamma}$ and $\gamma$, $0 < \gamma < 1$, sufficiently close to 1, so that whenever $F$ is a closed subset of $X$ whose complement has finite measure and $S(y,t)$ is a non-negative measurable function on $X \times (0,\infty)$, then

$$\int_{\mathcal{R}(\infty)(F^*)} S(y,t)\sigma d\mu(y) dt \leq C_{\alpha,\gamma} \int_{F} \left\{ \int_{\mathcal{P}(x)} S(y,t) d\mu(y) dt \right\} d\mu(x),$$

where $F^*$ is the set of points of global $\gamma$-density with respect to $F$, and $\sigma$ is given as in (1).

Proof. See Suh [6].

Lemma 4. Let $(X,d,\mu)$ be a space of homogeneous type. If $f$ is a non-negative function defined on $X$, and $M(f)$ is the Hardy-Littlewood maximal function of $f$. Suppose

$$\Phi_t(f)(x) = \frac{1}{t^\sigma} \int_X \chi_{B(x,t)}(y) f(y) d\mu(y), \quad t > 0,$$

where $\chi_{B(x,t)}$ is the characteristic function of the ball $B(x,t)$, and $\sigma$ is given as in (1). Then for $\alpha > 0$, there is a constant $C_{\alpha}$ such that

$$\Phi_{\alpha t}(f) \leq C_{\alpha} \Phi_t(M(f)).$$

Proof. We observe that if $f$ is a non-negative function on $X$, then

$$\Phi_{\alpha t}(f) \leq C_{\alpha} \Phi_t(\Phi_{\alpha t}(f)).$$

Because $\Phi_{\alpha t}(f) \leq C M(f)$, we get (2). The proof is therefore complete.

The main result of this paper is now the following:
Theorem 5. Let \((X, d, \mu)\) be a space of homogeneous type. Suppose \(0 < p < \infty\) and \(\alpha > 1\). Then there is a constant \(C_{\alpha, p}\) such that
\[
\|A^{(\alpha)}(f)\|_{L^p(d\mu)} \leq C_{\alpha, p}\|A^{(1)}(f)\|_{L^p(d\mu)}.
\]

Proof. Assume first that \(0 < p < 2\). For each \(\lambda > 0\), we define the open set \(O\) by
\[
O = \{x \in X : A^{(1)}(f)(x) > \lambda\}.
\]
Then we take \(F^* \subset F\) to be the set of points of global \(\gamma\)-density with respect to \(F\), with \(O^* = \cup F^*\). Apply Lemma 2 with \(S(y, t) = |f(y, t)|^2 t^{-\sigma-1}\) (and \(F^*\) in place of \(F\)), and we obtain
\[
(3) \quad \int_{F^*} (A^{(\alpha)}(f)(x))^2 d\mu(x) \leq C_{\alpha} \int_{R^{(\alpha)}(F^*)} |f(y, t)|^2 \frac{d\mu(y)dt}{t}.
\]
Next apply Lemma 3, again with \(S(y, t) = |f(y, t)|^2 t^{-\sigma-1}\), and we obtain
\[
(4) \quad \int_{R^{(\alpha)}(F^*)} |f(y, t)|^2 \frac{d\mu(y)dt}{t} \leq C'_{\alpha, \gamma} \int_{F} \left\{ \int_{F(x)} |f(y, t)|^2 \frac{d\mu(y)dt}{t^{\sigma+1}} \right\} d\mu(x).
\]
Then (3) and (4) imply that
\[
(5) \quad \int_{F^*} (A^{(\alpha)}(f)(x))^2 d\mu(x) \leq C'_{\alpha, \gamma} \int_{F} (A^{(1)}(f)(x))^2 d\mu(x).
\]
Thus it follows from Lemma 1 and (5) that
\[
(6) \quad \mu(\{x \in X : A^{(\alpha)}(f)(x) > \lambda\})
\]
\[
\leq \mu(O^*) + \frac{C'_{\alpha, \gamma}}{\lambda^2} \int_{F} (A^{(1)}(f)(x))^2 d\mu(x)
\]
\[
\leq C'_{\alpha} \left( \mu(O) + \frac{1}{\lambda^2} \int_{F} (A^{(1)}(f)(x))^2 d\mu(x) \right)
\]
\[
= C'_{\alpha} \left( \mu(\{x \in X : A^{(1)}(f)(x) > \lambda\}) + \frac{1}{\lambda^2} \int_{F} (A^{(1)}(f)(x))^2 d\mu(x) \right).
\]
If we multiply both sides of (6) by $\lambda^{p-1}$ and integrate, then we get that

$$||A^{(\alpha)}(f)||_{L^p(d\mu)} \leq C_{\alpha,p}||A^{(1)}(f)||_{L^p(d\mu)}$$

for $0 < p < 2$.

Assume second that $2 \leq p < \infty$. Then observe that

$$||A^{(\alpha)}(f)||^2_{L^p(d\mu)} = \sup_{\psi} \int_X (A^{(\alpha)}(f)(x))^2 \psi(x)d\mu(x),$$

where the supremum is taken over all non-negative $\psi$ which belong to the space $L^q(X)$ with $q$ dual to $p/2$, and $||\psi||_{L^q(d\mu)} \leq 1$. Then it follows from Lemma 4 that

$$\int_X (A^{(\alpha)}(f)(x))^2 \psi(x)d\mu(x)$$

$$= \int_{X \times (0,\infty)} |f(y,t)|^2 \left\{ \int_X \chi_{B(y,\alpha t)}(x) \psi(x)d\mu(x) \right\} \frac{d\mu(y)dt}{t^{\sigma+1}}$$

$$= \alpha^\sigma \int_{X \times (0,\infty)} |f(y,t)|^2 \Phi_t(\psi)(y) \frac{d\mu(y)dt}{t}$$

$$\leq C_{\alpha}'' \int_X (A^{(1)}(f)(x))^2 M(\psi)(x)d\mu(x)$$

$$\leq C_{\alpha}'' ||A^{(1)}(f)||^2_{L^p(d\mu)} ||M(\psi)||_{L^{q'}(d\mu)}$$

$$\leq C_{\alpha}'' ||A^{(1)}(f)||^2_{L^p(d\mu)} ||\psi||_{L^{q'}(d\mu)}$$

$$\leq C_{\alpha}'' ||A^{(1)}(f)||^2_{L^p(d\mu)}.$$

Taking the supremum over all allowable $\psi$ in (7) gives us then

$$||A^{(\alpha)}(f)||_{L^p(d\mu)} \leq C_{\alpha,p}||A^{(1)}(f)||_{L^p(d\mu)}$$

for $2 \leq p < \infty$. The proof of the theorem is therefore complete. □

**Remark.** In the above proof, the limitation $p < \infty$ arises since the maximal inequality $||M(\psi)||_{L^q(d\mu)} \leq C_q ||\psi||_{L^{q'}(d\mu)}$ requires $q > 1$. 
REFERENCES


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