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A KL-PRODUCT OF FINITE BCI-ALGEBRAS

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Abstract. We have proved that a finite BCI-algebra $(X, *, 0)$ is a KL-product if and only if for any subset I of X such that $I * X \subseteq I$ the cardinality of $0 * X$ divides the cardinality of I.

1. Introduction

The notion of BCK-algebras was proposed by Y. Iami and K. Iséki in 1966. In the same year K. Iséki [5] introduced the notion of BCI-algebras, which are a generalization of BCK-algebras. After than many mathematical papers have been published investigating some algebraic properties of the BCK/BCI-algebras and their relationship with other universal structures including lattices and Boolean algebras.

2. Basic definitions and results

Definition 2.1. A nonempty set X with a binary operation * and a distinguished element 0 is called a *BCI-algebra* if the following axioms

(*i*) $((x * y) * (x * z)) * (z * y) = 0$,
(*ii*) $(x * (x * y)) * y = 0$,
(*iii*) $x * x = 0$,
(*iv*) $x * y = y * x = 0 \longrightarrow x = y$

are satisfied for every $x, y, z \in X$.

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A BCI-algebra satisfying the identity $0 * x = 0$ is called a BCK-*algebra.*

In any BCI-algebra we can define a natural order $\leq$ putting

$$x \leq y \longleftrightarrow x * y = 0.$$

An element a of a BCI-algebra is called an *atom* if $x \leq a$ implies $x = a$. The set of all atoms of a BCI-algebra X will be denoted by $L(X)$. It is always nonempty because it contains at least 0.

A BCI-algebra X satisfying the identity $0 * (0 * x) = x$ is called p-*semisimple.* In such BCI-algebra we have $x*(x*y) = y$ for all $x, y \in X$ (cf. [2]). Moreover such BCI-algebra is medial and can be uniquely described by some group [1]. All elements of such BCI-algebra are atoms [3].

Lemma 2.2. [9] *An element a of a BCI-algebra X is an atom if and only if $x * (x * a) = a$ for every $x \in X$.*

Lemma 2.3. [9] *In any BCI-algebra X we have $L(X) = 0 * X$.*

In [8]J. Meng and X. L. Xin introduced the following notion of KL-product BCI-algebras.

Definition 2.4. A BCI-algebra X is called a *KL-product BCI-algebras,* if there exists a BCK-algebra Y and a p-semisimple BCI-algebra Z such that $X \approx Y \times Z$.

Theorem 2.5. [9] *A BCI-algebra X is a KL-product if and only if for every $x \in X$ and $e \in L(X)$ the following equality is satisfied*

$$x = (x * e) * (0 * e).$$

Proposition 2.6. [3], [6] *In any BCI-algebra X the following conditions are satisfied for every $x, y, z \in X$*

(1) $x * 0 = x$,
(2) $x * (x * (x * y)) = x * y$,
(3) $(x * y) * z = (x * z) * y$,
(4) $0 * (x * y) = (0 * x) * (0 * y)$,
(5) $x \leq y \longrightarrow x * z \leq y * z$ *and* $z * y \leq z * x$,
(6) $0 * (x * y) = 0 \longleftrightarrow 0 * x = 0 * y$.

3. Main results

In this section we describe properties of some special subsets of a BCI-algebra X. At first we consider the subset

$$T_a = \{x \in X | a * (a * x) = x\}.$$

Note that similar subsets are studied in [4].

LEMMA 3.1. *In any BCI-algebra X for an arbitrary $a \in X$ we have $0, a \in T_a$ and $L(X) = T_0$.*

Proof. Since the first condition is obvious, we prove only the second. Let $x \in T_0$. Then $0 * (0 * x) = x$, i.e. $x \in 0 * X$. Thus $T_0 \subset 0 * X$.

Conversely, if $x \in 0 * X$, then $x = 0 * y$ for some $y \in X$. Hence $0 * (0 * x) = 0 * (0 * (0 * y)) = 0 * y = x$, i.e. $x \in T_0$, which implies $0 * X \subset T_0$. Therefore $T_0 = 0 * X = L(X)$. □

PROPOSITION 3.2. *In any BCI-algebra X the following hold:*

(1) $T_a = a * X = \{a * x | x \in X\}$,
(2) $T_a * a = L(X) = T_0$,
(3) $T_{a*x} \subset T_a$,
(4) $T_0 \subset T_a$,
(5) $T_a * X = T_a$,
(6) $x \in T_a \longrightarrow T_x \subset T_a$,
(7) $T_0 = T_a \longleftrightarrow a$ *is an atom.*

Proof. (1) For $y \in T_a$ we have $y = a * (a * y)$, which gives $y \in a * X$. Thus $T_a \subset a * X$. Conversely, for any $y \in a * X$ there exists $x \in X$ such that $y = a * x$. Hence $a * (a * y) = a * (a * (a * x)) = a * x = y$, i.e. $y \in T_a$, whence $a * X \subset T_a$. This completes the proof of (1).

(2) For every $a \in X$ we have $(a * x) * a = (a * a) * x = 0 * x$, so,

$$T_a * a = \{(a * x) * a | x \in X\} = \{0 * x | x \in X\} = 0 * X = L(X) = T_0.$$

(3) Let $y \in T_{a*x}$. Then

$$y = (a * x) * ((a * x) * y) = (a * x) * ((a * y) * x).$$

But

$$((a*x)*((a*y)*x))*(a*(a*y)) = ((a*x)*((a*(a*y))*((a*y)*x) = 0.$$

So, $y*(a*(a*y))=0$. On the other hand, $(a*(a*y))*y=0$, which, together with the previous equality, implies $a*(a*y)=y$. Therefore $y\in T_a$.

(4) It follows from (3).

(5) Since for any $z\in T_a$ there exists $x\in X$ such that $z=a*x$, for $y\in X$ we have $T_{z*y}=T_{(a*x)*y}\subset T_{a*x}\subset T_a$. So, $z*y\in T_a$. Thus $T_a*X\subset T_a$. This completes the proof, because $T_a=a*X\subset T_a*X$, by just proved first condition.

(6) It is a simple consequence of previous conditions.

(7) If a is an atom, then $a\in L(X)=T_0$ implies $T_a\subset T_0$. But $T_0\subset T_a$ by (4), so $T_0=T_a$. The converse is obvious. □

Now we consider the set

$$S_a=\{x\in X|x*(x*a)=a\}.$$

PROPOSITION 3.3. *Let X be a BCI-algebra, then for any elements $a,b\in X$ we have*

(1) $a\in S_a$,
(2) $x\in S_a\longrightarrow S_x\subset S_a$.
(3) $S_a\subset S_b\longleftrightarrow T_b\subset T_a$,
(4) $S_a\subset S_{a*x}$ *for any* $x\in X$,
(5) $S_0=X$,
(6) $S_a=X$, *if a is an atom in X.*
(7) $(X\setminus S_a)*X=X\setminus S_a$, *if a is not an atom.*

Proof. (1) We have $a*(a*a)=a*0=a$, which gives $a\in S_a$.

(2) Consider $x\in S_a$ and $y\in S_x$. Then $x*(x*a)=a$ and $y*(y*x)=x$, which imply

$$(y*(y*x))*((y*(y*x))*a)=a$$

and

$$(y*(y*x))*((y*a)*(y*x))=a.$$

But

$$((y*(y*x))*((y*a)*(y*x)))*(y*(y*a))$$
$$=((y*(y*x))*(y*(y*a)))*((y*a)*(y*x)))=0.$$

So, $a*(y*(y*a))=0$ and $(y*(y*a))*a=0$, which implies $y*(y*a)=a$. Thus $y\in S_a$, i.e. $S_x\subset S_a$.

(3) Suppose that $S_a \subset S_b$ and $x \in T_b$, then $b * (b * x) = x$, i.e. $b \in S_x$. Therefore $S_b \subset S_x$, and in the consequence, $S_a \subset S_x$. So, $a \in S_x$, whence $x \in T_a$. This proves the inclusion $T_b \subset T_a$.

In a similar way we can prove that $T_b \subset T_a$ implies $S_a \subset S_b$.

(4) We know that $T_{a*x} \subset T_a$, therefore $S_a \subset S_{a*x}$.

(5) Obvious.

(6) It follows from the fact that $x * (x * a) = a$ for every $x \in X$ if and only if a is an atom.

(7) If a is not an atom, then $S_a \neq X$. So, if $x \in X \setminus S_a$, $y \in X$, and $x * y \in S_a$, then $S_{x*y} \subset S_a$. But $S_x \subset S_{x*y}$ implies $x \in S_a$, which is a contradiction. Therefore must be $x \in X \setminus S_a$. □

COROLLARY 3.4. *In a BCI-algebra X, the following properties are equivalent:*

(1) $S_a = S_b$,
(2) $T_a = T_b$,
(3) $b \in S_a \cap T_a$,
(4) $a \in S_b \cap T_b$.

On a BCI-algebra X we define a binary relation $\sim$ putting

$$x \sim y \longleftrightarrow T_x = T_y.$$

It is clear that it is an equivalence relation. By the above corollary, an equivalence class containing an element $a \in X$ coincides with the set $S_a \cap T_a$.

THEOREM 3.5. *In any finite BCI-algebra the following two conditions are equivalent:*

(1) $(x * e) * (0 * e) = x$ *for all* $x \in X$ *and* $e \in 0 * X$,
(2) $Card\,(0 * X)$ *divides* $Card\, I$ *for any* $I \subset X$ *such that* $I * X \subset I$.

Proof. Consider the function $\varphi_a : S_a \cap T_a \to S_0 \cap T_0$ such that $\varphi_a(x) = a * x$.

It is well defined because for $x \in S_a \cap T_a$ we have

$$\varphi_a(x) = a * x = (x * (x * a)) * x = (x * x) * (x * a) = 0 * (x * a),$$

which gives $\varphi_a(x) \in 0 * X = T_0 = X \cap T_0 = S_0 \cap T_0$.

If $\varphi_a(x_1) = \varphi_a(x_2)$ for some $x_1, x_2 \in S_a \cap T_a$, then $a * x_1 = a * x_2$, which implies

$$x_1 = a * (a * x_1) = a * (a * x_2) = x_2.$$

This means that φ_a is injective. Hence

$$Card\,(S_a \cap T_a) \leq Card\,(S_0 \cap T_0).$$

(1) → (2) Let $(x * e) * (0 * e) = x$ for all $x \in X$ and $e \in 0 * X$. Then

$$y \in S_0 \cap T_0 = T_0 = 0 * X = L(X) \longrightarrow y \in L(X),$$

i.e. $a * (a * y) = y$.

If $x = a * y$, then obviously $x \in T_a$. Moreover,

$$x*(x*a) = (a*y)*((a*y)*a) = (a*y)*((a*a)*y) = (a*y)*(0*y) = a,$$

which means that $x \in S_a$. In the consequence, $x \in S_a \cap T_a$. But $y = a * x$ is equivalent to $y = \varphi_a(x)$. In this way, we have proved that φ_a is surjective. φ_a is bijective because it is injective by the first part of the theorem. Consequently

$$Card\,(S_0 \cap T_0) = Card\,(S_a \cap T_a).$$

Now let I be an arbitrary subset of X such that $I * X \subset I$. Then I is a union of separated subsets of the form $S_a \cap T_a$. Indeed, for any $a \in I$ we have $T_a = a * X \subset I * X \subset I$. But $S_a \cap T_a \subset T_a$, which implies $S_a \cap T_a \subset I$. So, $I = \cup_{x \in C}(S_x \cap T_x)$, where $C \subset I$. The subsets $S_x \cap T_x$, $x \in C$, as equivalence classes of the equivalence defined above, are obviously separated. Therefore

$$Card\,I = \textstyle\sum_{x \in C} Card\,(S_x \cap T_x) = \sum_{x \in C} Card\,(S_0 \cap T_0)$$

$$= \textstyle\sum_{x \in C} Card\,(0 * X) = Card\,C \times Card\,(0 * X).$$

This proves that $Card\,(0 * X)$ divides $Card\,I$.

(2) → (1) If for all $I \subset X$ such that $I * X \subset I$ $Card\,(0 * X)$ divides $Card\,I$, then for any $a \in X$ we have

$$T_a = X \cap T_a = (S_a \cup (X \setminus S_a)) \cap T_a = (S_a \cap T_a) \cup ((X \setminus S_a) \cap T_a).$$

This means that $Card\,T_a = Card\,(S_a \cap T_a) + Card\,((X \setminus S_a) \cap T_a)$.

If a is not an atom then $(X \setminus S_a) * X = (X \setminus S_a)$, by Proposition 3.3, and $T_a * X = T_a$, by Proposition 3.2. So,

$$((X \setminus S_a) \cap T_a) * X \subset (X \setminus S_a) \cap T_a.$$

From the above, according to (2), we conclude that $Card\,(0 * X)$ divides $Card\ T_a$ and $Card\,((X \setminus S_a) \cap T_a$. Therefore $Card\,(0 * X)$ divides

$$Card\, T_a - Card\,((X \setminus S_a) \cap T_a) = Card\,(S_a \cap T_a).$$

Hence $Card\,(S_0 \cap T_0)$ divides $Card\,(S_a \cap T_a)$, because $0 * X = L(X) = T_0 = S_0 \cap T_0$ by Lemma 2.3 and Lemma 3.1. Thus

$$Card\,(S_0 \cap T_0) \leq Card\,(S_a \cap T_a).$$

But, as it was proved in the first part of this proof, $Card\,(S_a \cap T_a) \leq Card\,(S_0 \cap T_0)$. So, $Card\,(S_a \cap T_a) = Card\,(S_0 \cap T_0)$, which means that the map φ_a is surjective.

If a is an atom, then $S_a = X$, by Proposition 3.3, and $T_a = T_0$, by Proposition 3.2. Thus $S_a \cap T_a = S_0 \cap T_0$, i.e. φ_a is surjective. So, φ_a is surjective in any case.

Now let $e \in L(X) = 0 * X = T_0 = S_0 \cap T_0$. Then there exists an element $x \in (S_a \cap T_a)$ such that $\varphi_a(x) = a * x = e$. Hence $a * (a * x) = a * e$. But $a * (a * x) = x$ because $x \in T_a$, whence $x = a * e$ and $x \in S_a$. Therefore

$$a = (a * e) * ((a * e) * a) = (a * e) * ((a * a) * e) = (a * e) * (0 * e).$$

This proves (1) and completes our proof. □

As a simple consequence of the above theorem and Theorem 2.5 we obtain

COROLLARY 3.6. *A finite BCI-algebra is a KL-product if and only if for every* $I \subset X$ *such that* $I * X \subset I$ $Card\,(0 * X)$ *divides* $Card\, I$.

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