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ON THE CLASS OF S_3-ALGEBRAS

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Abstract. In this paper we investigate some more properties of of S_3-algebras. We also prove that the class of S_3-algebras is contained in the class of commutative BCI-algebras.

Introduction

In [6], K. Iseki gave the concept of BCI-algebras.In [1], S.A. Bhatti, M.A. Chaudhry and B. Ahmad classified BCI-algebras into S_i-algebras, i=1, 2, 3, 4 and investigated some properties of these algebras.
In this paper we investigate some more properties of of S_3-algebras. We also prove tahat the class of S_3-algebras is contained in the class of commutative BCI-algebras.

1. Preliminaries

Definition 1.1. [6] A BCI-algebra X is an abstract algebra $(X, *, o)$ of type (2, 0), where * is a binary operation, o is a constant which is the smallest element in X, satisfying the following conditions; for all $x, y, z \in X$,

1.1 $((x * y) * (x * z)) * (z * y) = o$
1.2 $(x * (x * y)) * y = o$
1.3 $x * x = o$
1.4 $x * y = o = y * x \Rightarrow x = y$
1.5 $x * o = o \Rightarrow x = o$

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where $x * y = o \Leftrightarrow x \leq y$
If $o * x = o$ holds for all $x \in X$, then X is a BCK-algebra.[4, 5]
Moreover, the following properties hold in every BCK/BCI-algebra ([6]):

1.6 $x * o = x$
1.7 $(x * y) * z = (x * z) * y$
1.8 Let X be a BCI-algebra with M as its BCK-part. For $m \in M$, $x \in X - M$, $m * x, x * m \in X - M$. [6]
1.9 $x * (x * (x * y)) = x * y$
1.10 $o * (x * y)) = (o * x) * (o * y)$

We prove (1.10) as follows:

$$\begin{aligned} & o * (x * y) = ((o * y) * (o * y)) * (x * y) && (Because\ of\ 1.3) \\ \Rightarrow\ & o * (x * y) = ((o * y) * (x * y)) * (o * y) && (Because\ of\ 1.7) \\ \Rightarrow\ & o * (x * y) = (((x * x) * y) * (x * y)) * (o * y) && (Because\ of\ 1.3) \\ \Rightarrow\ & o * (x * y) = (((x * y) * x) * (x * y)) * (o * y) && (Because\ of\ 1.7) \\ \Rightarrow\ & o * (x * y) = (((x * y) * (x * y)) * x) * (o * y) && (Because\ of\ 1.7) \\ \Rightarrow\ & o * (x * y) = (o * x) * (o * y) && (Because\ of\ 1.3) \end{aligned}$$

1.11 Let X be a BCI-algebra. If $M = o$, then X is called a p-semisimple BCI-algebra.[7]
1.12 Let X be a p-semisimple BCI-algebra. The following properties are equivalent:
(i) X be a p-semisimple.
(ii) $o * (o * x) = x$.[7]

Definition 1.2. [6] A nonempty subset S of a BCI-algebra X is known as a subalgebra of X, if

$$x, y \in S \Rightarrow x * y \in X$$

Definition 1.3. [4] A BCK-algerba X is said to be commutative if $y * (y * x) = x * (x * y)$ holds for all $x, y \in X$.

Definition 1.4. [4] A BCK-algerba X is said to be implicative if $x * (y * x) = x$ holds for all $x, y \in X$.

1.13 An implicative BCK-algebra is commutative and positive implicative. [4]

THEOREM 1. [2] *A BCI-algebra $(X, *, o)$ is commutative if and only if it satisfies the condition for all $x, y \in X$,*

$$x * (x * y) = y * (y * (x * (x * y))$$

DEFINITION 1.5. [3] A BCI-algebra $(X, *, o)$ is said to be positive implicative if it satisfies the condition for all $x, y \in X$,

$$(x * (x * y)) * (y * x) = x * (x * (y * (y * x))$$

DEFINITION 1.6. [1] Let X be a BCI-algebra, for $x, y \in X$, x, y are said to be comparable if $x \leq y$ or $y \leq x$.
Similarly in BCK-algebras, if $x * y = o$ or $y * x = o$, then x and y are comparable.

DEFINITION 1.7. [1] Let X be a BCI-algerba. We choose an element $x_o \in X$ such that there does not exist any $y \neq x_o$, satisfying $y * x_o = o$ and define

$$A(x_o) = \{x \in X : x_o * x = o\}$$

$A(x_o)$ is known as the branch of X determined by x_o. Let I_x denote the set of all initial elements of X. We call it the center of X. The reason for calling this subset as the center of X is that each branch originates from a unique point of this subset. Note that each branch $A(x_o)$ is nonempty, because of (1.3), $x_o * x_o = o \Rightarrow x_o \in A(x_o)$. Also note that the BCK-part M of the BCI-algebra X is equal to $A(o)$ because

$$M = \{x \in X; o * x = o\} = A(o)$$

If $A(x_o) = \{x_o\}$, then $A(x_o)$ the branch determined by x_o is known as a uniary comparable.

DEFINITION 1.8. [1] A proper BCI-algerba X with $M \neq o$ is S_3-algebra if each $A(x_o)$ in $X - M$ is uniary comparable i.e for all $x \in X - M$, $A(x) = \{x\}$.

THEOREM 2. [1] *Let X be a S_3-algebra with M as its BCK-part. Then $G = \{o\} \bigcup (X - M)$ is a subalgebra.*

THEOREM 3. [1] *Let X be a S_3-algebra with M as its BCK-part. Then the following hold:*

(1) $x * (x * y) = y$, *for all $x \in X$, $y \in X - M$.*

(2) $y * x = y$, *for all* $x \in M$, $y \in X - M$.
(3) $x * y = o * y$, *for all* $x \in M$, $y \in X - M$.
(4) $o * (y * x) = x * y$, *for all* $x \in M$, $y \in X - M$.
(5) $x * (o * y) = y$, *for all* $x \in X$, $y \in X - M$.

LEMMA 1. *Let* X *be a* S_3*-algebra. Then for all* $x \in X - M$, $o * (o * x) = x$

Proof. Let X be a S_3-algebra with M as its BCK-part. Because of (1.8), for $o \in M$, $x \in X - M$, $o * x \in X - M$. Again by (1.8), $o \in M$, $o * x \in X - M$, $o * (o * x) \in X - M$. Since X is a S_3-algebra, therefore by (1.2),

$$o * (o * x) \leq x \tag{1}$$

As $x \in X - M$, so $A(x) = \{x\}$. Thus inequality (1) becomes

$$o * (o * x) = x$$

This gives the proof. □

LEMMA 2. *Let* X *be a* S_3*-algebra with* M *as its BCK-part. Then* $G = \{o\} \bigcup (X - M)$ *is p-semisimple.*

Proof. By theorem 2[1], $G = \{o\} \bigcup (X - M)$ is a subalgebra of X. According to above lemma 1, for all $x \in X - M \subset G$, $o * (o * x) = x$. Further for $o \in G = \{o\} \bigcup (X - M)$, $o * (o * o) = o$. Thus for all $x \in G = \{o\} \bigcup (X - M)$, $o * (o * x) = x$. Hence because of (1.12), part (ii), G is p-semisimple. □

Since every p-semisimple algebra is a S_4-algebra (see [1, theorem 6]), the p-semisimple algebra $G = \{o\} \bigcup (X - M)$ described in lemma 2 is a S_4-algebra.

EXAMPLE 1. Let $X = \{o, a, b, c, d, e, f\}$ be a S_3-algebra in which * is defined as follows:

Table 1

*	o	a	b	c	d	e	f
o	o	o	o	f	e	d	c
a	a	o	o	f	e	d	c
b	b	a	o	f	e	d	c
c	c	c	c	o	f	e	d
d	d	d	d	c	o	f	e
e	e	e	e	d	c	o	f
f	f	f	f	e	d	c	o

Note that BCK-part $M = A(o) = \{o, a, b\}$ and BCI-part $X - M = \{c, d, e, f\}$. Since X is a S_3-algebra, therefore $A(c) = \{c\}$, $A(d) = \{d\}$, $A(e) = \{e\}$ and $A(f) = \{f\}$. So, $G = \{o\} \bigcup (X - M) = \{o, c, d, e, f\}$. Note that G is a p-semisimple BCI-algebra. Also note that for all $x \in X - M$, $o * (o * x) = x$.

EXAMPLE 2. Let $X = \{o, a, b, c, d, e, f\}$ be a S_3-algebra in which * is defined as follows:

Table 2

*	o	a	b	c	d	e	f
o	o	o	o	o	d	e	f
a	a	o	a	a	d	e	f
b	b	b	o	b	d	e	f
c	c	c	c	o	d	e	f
d	d	d	d	d	o	f	e
e	e	e	e	e	f	o	d
f	f	f	f	f	e	d	o

Note that BCK-part $M = A(o) = \{o, a, b, c\}$ and BCI-part $X - M = \{d, e, f\}$. Since X is a S_3-algebra, therefore $A(d) = \{d\}$, $A(e) = \{e\}$ and $A(f) = \{f\}$. So, $G = \{o\} \bigcup (X - M) = \{o, d, e, f\}$. Note that G is a p-semisimple BCI-algebra. Also note that for all $x \in X - M$, $o * (o * x) = x$.

THEOREM 4. *Let X be a S_3-algebra with M as its BCK-part. Then the following hold:*
For $x, y \in M$, $z \in X - M$,

(i) $o * (x * y) = y$, $x \in M$, $y \in X - M$

(ii) $y * (y * x) = o$, $x \in M$, $y \in X - M$
(iii) $x * z = y * z$, $x, y \in M$, $z \in X - M$

Proof. (i) Since S_3-algebra is a BCI-algebra, therefore by (1.10), for all $x, y \in X$,

$$o * (x * y) = (o * x) * (o * y) \tag{1}$$

Suppose that $x \in M$ and $y \in X - M$. Since M is a BCK-algebra, therefore $o * x = o$, so equation (1) becomes

$$o * (x * y) = o * (o * y)$$

Because $y \in X - M$, therefore by lemma 1, $o * (o * y) = y$. Thus above equation becomes

$$o * (x * y) = y$$

(ii) Because of theorem 3, part (2), for $x \in M$, $y \in X - M$,

$$\begin{aligned} & y * x = y \\ \Rightarrow \quad & y * (y * x) = y * y \\ \Rightarrow \quad & y * (y * x) = o \end{aligned}$$

(iii) Because of part (i), for $x \in M$, $z \in X - M$,

$$\begin{aligned} & o * (x * z) = z \\ \Rightarrow \quad & o * (o * (x * z)) = o * z \end{aligned}$$

For $x \in M$, $z \in X - M$, by (1.8), $x * z \in X - M$. So, by lemma 1, above equation implies that

$$x * z = o * z \tag{2}$$

Similarly, for $y \in M$, $z \in X - M$,

$$y * z = o * z \tag{3}$$

From equations (2) and (3) it follows that

$$x * z = y * z$$

This givesthe proof.

□

THEOREM 5. *Let X be a S_3-algebra with commutative BCK-part M. Then X is a commutative BCI-algebra.*

Proof. Let X be a S_3-algebra with commutative BCK-part M. For distinct $x, y \in X$, we have the following three possibilities:

(1) $x, y \in M$
(2) $x, y \in X - M$
(3) $x \in M$, $y \in X - M$

Case (1): Let $x, y \in M$. Because M is a commutative BCK-algebra, therefore

$$\begin{aligned} & x*(x*y) = y*(y*x) \\ \Rightarrow\ & y*(x*(x*y)) = y*(y*(y*x)) \\ \Rightarrow\ & y*(x*(x*y)) = y*x \qquad (using\ (1.9)) \\ \Rightarrow\ & y*(y*(x*(x*y))) = y*(y*x) \end{aligned}$$

In this case M is commutative , therefore we replace $y*(y*x)$ by $x*(x*y)$ and get

$$y*(y*(x*(x*y))) = x*(x*y) \qquad (A)$$

Case (2): Let $x, y \in X - M$. Then for $x \in X - M \subset X$, $y \in X - M$, by theorem 3, part (1),

$$\begin{aligned} & x*(x*y) = y \\ \Rightarrow\ & y*(x*(x*y)) = y*y = o \qquad (using(1.3)) \\ \Rightarrow\ & y*(y*(x*(x*y))) = y*o = y \quad (using(1.6)) \end{aligned}$$

But in this case $y = x*(x*y)$, so replacing y by $x*(x*y)$, above equation becomes

$$y*(y*(x*(x*y))) = x*(x*y) \qquad (B)$$

Case (3): Let $x \in M$, $y \in X - M$. Then by theorem 4, part (2),

$$\begin{aligned} & y*(y*x) = o \\ \Rightarrow\ & x*(y*(y*x)) = x*o = x \\ \Rightarrow\ & x*(x*(y*(y*x))) = x*x = o \end{aligned}$$

But in this case $o = y*(y*x)$, so replacing o by $y*(y*x)$, above equation becomes

$$x * (x * (y * (y * x))) = y * (y * x) \qquad (C)$$

Further for $x \in M \subset X$, $y \in X - M$, by theorem 3, part (1),

$$\begin{aligned} & x * (x * y) = y \\ \Rightarrow \quad & y * (x * (x * y)) = y * y = o \\ \Rightarrow \quad & y * (y * (x * (x * y))) = y * o = y \end{aligned}$$

But in this case $y = x * (x * y)$, so replacing y by $x * (x * y)$, above equation becomes

$$y * y * (x * (x * y)) = x * (x * y) \qquad (D)$$

Hence from (A), (B), (C) and (D)

$$x * (x * y) = y * (y * (x * (x * y))$$

which implies that X is a commutative BCI-algebra. □

THEOREM 6. *If the BCK-part of a S_3-algebra implicative then X is a positive implicative BCI-algebra.*

Proof. Let X be a S_3-algebra with implicative BCK-part M. For distinct $x, y \in X$, we have the following three possibilities:

(1) $x, y \in M$
(2) $x, y \in X - M$
(3) $x \in M$, $y \in X - M$

Case (1): Let $x, y \in M$. Because M is an implicative BCK-algebra, therefore

$$\begin{aligned} & x * (y * x) = x \\ \Rightarrow \quad & (x * (y * x)) * (x * y) = x * (x * y) \\ \Rightarrow \quad & (x * (x * y)) * (y * x) = x * (x * y) \quad (1)(using(1.7)) \end{aligned}$$

Now,

$$\begin{aligned} & (x * (x * (y * (y * x))) * (x * (x * y)) \\ & \quad = (x * (x * (x * y)) * (x * (y * (y * x)) \quad (using(1.7)) \\ & \quad = (x * y) * (x * (y * (y * x)) \quad (using(1.9)) \\ & \quad \leq (y * (y * x)) * y \quad (using(1.1)) \\ & \quad = (y * y) * (y * x) \quad (using(1.7)) \end{aligned}$$

Because $x, y \in M$, therefore $y * x \in M$, so $o * (y * x) = o$, so we get

$$(x * (x * (y * (y * x))) * (x * (x * y)) = o \tag{2}$$

Further,

$$\begin{aligned}(x*(x * y)) * (x * (x * (y * (y * x))) \\ \leq \quad (x * (y * (y * x)) * (x * y) \\ = \quad (x * (x * y)) * (y * (y * x)) \qquad (using(1.7))\end{aligned}$$

Because of (1.13), an implicative BCK-algebra is commutative, so $x * (x * y) = y * (y * x)$. Thus

$$(x * (x * y)) * (x * (x * (y * (y * x))) \leq (x * (x * y)) * (y * (y * x)) = o$$

$$\Rightarrow (x * (x * y)) * (x * (x * (y * (y * x))) = o \tag{3}$$

Because of (1.4), from equations (2) and (3) it follows that

$$x * (x * y) = x * (x * (y * (y * x))$$

Thus equation (1) becomes

$$(x * (x * y)) * (y * x) = x * (x * (y * (y * x)) \tag{A}$$

Case (2): Let $x, y \in X - M$. Then for $x \in X - M \subset X$, $y \in X - M$, by theorem 3, part (1),

$$x * (x * y) = y$$

$$\Rightarrow (x * (x * y)) * (y * x) = y * (y * x) \tag{4}$$

Because of (1.12), an implicative BCK-algebra is commutative, it means X is a S_3-algebra with commutative BCK-part M. So by above theorem 5, X is a commutative BCI-algebra. Thus $y * (y * x) = x * (x * (y * (y * x))$. Hence equation (4) becomes,

$$(x * (x * y)) * (y * x) = x * (x * (y * (y * x)) \tag{B}$$

Case (3): Let $x \in M$, $y \in X - M$. Then by theorem 4, part (2),

$$y * (y * x) = o \tag{5}$$

and for $x \in M \subset X$,$y \in X - M$ by theorem 3, part (1),

$$x * (x * y) = y \tag{6}$$

Now equation (5) implies

$$(y * (y * x)) * (x * y) = o * (x * y)$$

Because of theorem 4,part (1), above equation becomes

$$(y * (y * x)) * (x * y) = y \tag{7}$$

Now

$$\begin{aligned}
&(y*(y * (x * (x * y)))) * y \\
&\quad = \quad (y * y) * (y * (x * (x * y)) && (using\ (1.7)) \\
&\quad = \quad o * (y * (x * (x * y)) && (using\ (1.3)) \\
&\quad = \quad o * (y * y) = o * o = o \qquad (8) && (using\ equations (6) and (1.3))
\end{aligned}$$

Further,

$$\begin{aligned}
y * (y * (y * (x * (x * y)))) &= y * (y * (y * y)) \quad (using\ equation(6)) \\
&= y * (y * o) = y * y = o \qquad (9)
\end{aligned}$$

So by above theorem 5, X is a commutative BCI-algebra. Thus because of (1.4), equations (8) and (9) imply

$$y = y * (y * (x * (x * y)))$$

Hence equation (7) becomes

$$(y * (y * x)) * (x * y) = y * (y * (x * (x * y)) \tag{C}$$

Further from equation (6)

$$(x * (x * y)) * (y * x) = y * (y * x) \tag{10}$$

Because of (1.12), an implicative BCK-algebra is commutative, it means that X is a S_3-algebra with commutative BCK-part M. So by above theorem 5, X is a commutative BCI-algebra. Thus $y * (y * x) = x * (x * (y * (y * x))$ holds. Hence equation (10) becomes,

$$(x * (x * y)) * (y * x) = x * (x * (y * (y * x)) \tag{D}$$

Hence from (A), (B), (C) and (D) it follows that for all $x, y \in X$,

$$(x * (x * y)) * (y * x) = x * (x * (y * (y * x))$$

Which shows that X is a positive implicative BCI-algebra.
From theorem 5 and theorem 6 it follows that the class of positive implicative S_3-algebras is contained in the class of commutative S_3-algebras. □

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