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FUZZY MULTIPLICATION RINGS

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Abstract. We will introduce the notion of fuzzy multiplication ring using fuzzy ideal. In this paper we will show that a fuzzy ideal I is primary if $radI$ is prime. And we will investigate some properties related the theorem.

1. Introduction

The theory of fuzzy ideals in rings is a generalization of theory of ideals in rings. The notion of fuzzy ideals in rings was first introduced by Liu[5]. The theory of fuzzy ideals has been futher developed by several mathematicians such as Kumbhojkar,Malik and Mordeson.

P.Das[1] introduced the notion of fuzzy multiplication semigroups induced by multiplication semigroups and studied some properties of fuzzy multiplication semigroups.

In this background we will introduce the notion of fuzzy miltiplication rings and study some properties.

2. Preliminaries

Throughout this paper, R stands for a commutative ring with identity.

Definition 2.1. *A nonempty fuzzy subset* $I : R \to [0,1]$ *is called a fuzzy ideal of* R *if*

(i) $I(x-y) \geq \min\{I(x), I(y)\}$, *and*

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(ii) $I(xy) \geq I(x) \vee I(y)$ *for all* $x, y \in R$.

We easily know that if I is a fuzzy ideal of R then $I(0) \geq I(x) \geq I(1)$ for all $x \in R$. Two fuzzy ideals I and J of R are called equivalent if and only if $I(x) > I(y) \Leftrightarrow J(x) > J(y)$ for all $x, y \in R$.

It is clear that two equivalent fuzzy ideals have the same properties as fuzzy ideals. So in this paper we assume that every fuzzy ideals I has the property such that $I(0) = 1$.

DEFINITION 2.2. *Let A and B fuzzy subset of a ring R. We define some operations between A and B. For every $x \in R$*

(i) $(A+B)(x) = \sup\limits_{x=\alpha+\beta} \{\min\{A(\alpha), B(\beta)\}\}$

(ii) $(A*B)(x) = \sup\limits_{x=\sum_{i=1}^{m} \alpha_i\beta_i} \{\min\{A(\alpha_1), \cdots, A(\alpha_m), B(\beta_1), \cdots, B(\beta_m)\}\}$

(iii) $(AB)(x) = \sup\limits_{x=\alpha\beta} \{\min\{A(\alpha), B(\beta)\}\}$

PROPOSITION 2.3. *Let I and J be fuzzy ideals of R.*

(i) *$I + J$ is a fuzzy ideal of R*

(ii) *$I * J$ is a fuzzy ideal of R.*

Proof. (ii)

$$
\begin{aligned}
(I*J)(x-y) &= \sup_{x-y=\sum_{i=1}^{t} a_i b_i} \{\min\{I(a_1), \cdots, I(a_t), J(b_1), \cdots J(b_t)\}\} \\
&\geq \sup_{x=\sum_{j=1}^{m}\alpha_j\beta_j, -y=\sum_{k=1}^{n}\gamma_k\delta_k} \{\min\{I(\alpha_1), \cdots I(\alpha_m), I(\gamma_1), \\
&\quad \cdots, I(\gamma_n), J(\beta_1), \cdots, J(\beta_m), J(\delta_1), \cdots, J(\delta_n)\}\} \\
&\geq \sup\{\min\{I(\alpha_1), \cdots I(\alpha_m), J(\beta_1) \cdots J(\beta_m)\}\} \\
&\quad \wedge \sup\{\min\{I(\gamma_1), \cdots, I(\gamma_n), J(\delta_1), \cdots, J(\delta_n)\}\} \\
&= I*J(x) \wedge I*J(y)
\end{aligned}
$$

$$
\begin{aligned}
(I*J)(xy) &= \sup_{xy=\sum_{i=1}^{n}\alpha_i\beta_i} \{\min\{I(\alpha_1), \cdots, I(\alpha_n), J(\beta_1), \cdots J(\beta_n)\}\} \\
&\geq \sup_{x=\sum_{i=1}^{n} a_i b_i, xy=\sum a_i(b_i y)} \{\min\{I(a_1), \cdots I(a_n), J(b_1 y) \cdots J(b_n y)\}\} \\
&\geq \sup\{\min\{I(a_1), \cdots, I(a_n), J(b_l), \cdots J(b_n)\}\} \\
&= I*J(x)
\end{aligned}
$$

Similarly $(I * J)(xy) \geq (I * J)(y)$. Therefore $I * J$ is a fuzzy ideal of R. □

A fuzzy ideal I contains a fuzzy ideal J(denoted : $J \subseteq I$) if $J(x) \leq I(x)$ for all $x \in R$.

The following example show that $I + J$ may not contains I and J where I and J are fuzzy ideals.

EXAMPLE 2.4. *Consider the fuzzy subsets I, J of R defined by*

$$I(a) = \begin{cases} \frac{1}{2} & \text{if } a \in 3Z \\ 0 & \text{if } a \notin 3Z \end{cases} \quad \text{and } J(b) = \begin{cases} 1 & \text{if } b \in 2Z \\ 0 & \text{if } b \notin 2Z \end{cases}$$

Then I and J are fuzzy ideal of R. Now

$$\begin{aligned} I + J(2) &= \sup_{2=a+b} \{\min\{I(a), J(b)\}\} \\ &\leq \frac{1}{2} \ (\because I(a) \leq \frac{1}{2}). \end{aligned}$$

But $J(2) = 1$. Therefore $I + J$ may not contain J.

But the following Lemma shows that $I + J$ always contain I and J where I and J are fuzzy ideals and $I(0) = J(0)$.

LEMMA 2.5. *If I and J are fuzzy ideals satisfying $I(0) = J(0)$, then $I \cup J \subseteq I + J$.*

Proof. $I \cup J$ is defined by $I \cup J(x) = I(x) \vee J(x)$ for all $x \in R$. For all $x \in R$

$$\begin{aligned} (I + J)(x) &= \sup_{x=\alpha+\beta} \{\min\{I(\alpha), J(\beta)\}\} \\ &\geq \min\{I(0), J(x)\} \geq J(x) \end{aligned}$$

and

$$\begin{aligned} (I + J)(x) &= \sup_{x=\alpha+\beta} \{\min\{I(\alpha), J(\beta)\}\} \\ &\geq \min\{I(x), J(0)\} \geq I(x) \end{aligned}$$

Thus $(I + J) \supseteq I \cup J$. □

DEFINITION 2.6. *A fuzzy ideal I of R is called prime ideal if for any two fuzzy ideals J, K of R, $J * K \subseteq I$, implies that $J \subseteq I$ or $K \subseteq I$.*

DEFINITION 2.7. *Let I be a fuzzy ideal of R. The fuzzy radical of I, denoted by $radI$, is defined by $(radI)(x) = \sup\{I(x^n) \mid n \in N\}$ for all $x \in R$.*

It is well-known that $radI$ is a fuzzy ideal of R if I is a fuzzy ideal.

DEFINITION 2.8. *A fuzzy ideal I of R is called primary ideal if for any two fuzzy ideals J, K of R, $J * K \subseteq I$, implies that $J \subseteq I$ or $K \subseteq radI$.*

DEFINITION 2.9. *Let x_r be a fuzzy point of R (i.e. $x_r(x) = r$ and $x_r(y) = 0$ if $y \neq x$). The principal fuzzy ideal generated by x_r denoted by $< x_r >$ is defined by*

$$< x_r > (y) = \begin{cases} 1 & \text{if } y = 0 \\ r & \text{if } y \in xR \backslash \{0\} \\ 0 & \text{if } y \in R \backslash xR \ . \end{cases}$$

The following Propositions are well-known and proofs of those are shown in [2],[3].

PROPOSITION 2.10. *A fuzzy ideal I of R is called prime ideal if and only if for any two fuzzy points x_r, y_s of R,$x_r y_s \in I$, implies that $x_r \in I$ or $y_s \in I$ for some $n \geq 0$.*

PROPOSITION 2.11. *A fuzzy ideal I of R is called primary ideal if and only if for any two fuzzy points x_r, y_s of R,$x_r y_s \in I$, implies that $x_r \in I$ or ${y_s}^n \in I$ for some $n \geq 0$ where ${y_s}^2 = y_s y_s$.*

PROPOSITION 2.12. *If I is a prime fuzzy ideal then $I = radI$.*

PROPOSITION 2.13. *For any fuzzy ideal I of R,*

$$radI = \cap\{J \mid J \text{ is a prime fuzzy ideal of } R \text{ such that } I \subseteq J\}.$$

PROPOSITION 2.14. *If I is a primary fuzzy ideal of R, then $radI$ is a prime fuzzy ideal of R.*

3. Fuzzy Multiplication Ring

DEFINITION 3.1. *R is called a fuzzy mutiplication ring if for any two fuzzy ideals I and J of R, satisfying $I \subseteq J$, there exists a fuzzy ideal K of R such that $I = J * K$.*

THEOREM 3.2. *If I be a prime fuzzy ideal of S and J is any fuzzy ideal of R such that $I \subset J$, then $I = I * J$ and $I = J^\omega$ or $I = I * J^\omega$, where $J^\omega = \cap\{J^i \mid i > 0\}$ and $J^2 = J * J$.*

Proof. Since $I \subset J$, and R is a fuzzy multiplication ring, there exist a fuzzy ideal K of R such that $I = J * K$. By primeness of I, we know that $K \subseteq I$. But $I = J * K \subseteq K$ implies $I = K$. Thus $I = J * I$. Thus $I = J^\omega * I$ or $I = J^\omega$. □

LEMMA 3.3. [4] *Let I, J, K be fuzzy ideals of R. Then*

(i) $(I * J) * K = I * (J * K)$
(ii) $I * (J + K) = (I * J) + (I * K)$

LEMMA 3.4. $[2, P.80]$ *Let $x, y \in R$ and $\alpha, \beta \in [0, 1]$. Then*

$$< x_\alpha > * < y_\beta > = < x_\alpha >< y_\beta > = < x_\alpha y_\beta > .$$

THEOREM 3.5. *Let I be a fuzzy ideal satisfying the sup properties. If $radI$ is a prime, then I is primary.*

Proof. It is easily prove that $J = radI$ has the sup property if I has the sup property. Let $J = radI$.Now, if $J = \chi_R$, then clearly I is primary. Next,let $J \neq \chi_R$. Assume that I is not primary. Then there exist fuzzy points x_r, y_t suth that $x_r y_t \in I, x_r \in J$, but $x_r \notin I$ and $y_t^n \notin I$ for all $n > 0$. Let $U = I + J* < x_r >$. Clearly, U is a fuzzy ideal of R. If possible let $x_r \in U$. Since $(J* < x_r >)(a) \leq < x_r > (a) \leq r$ for all $a \in R$,

$$\begin{aligned} r = x_r(x) \leq U(x) &= [I + J* < x_r >](x) \\ &= \sup_{x=a+b} \min\{I(a), J* < x_r > (b)\} \leq r \end{aligned}$$

Thus there exist some $a, b \in R$ such that $x = a + b$ and $J* < x_r > (b) = r \leq I(a)$. In this case $b = \sum \alpha_i \beta_i$ such that $\beta_i \in < x >$ and

$J(\alpha_1) \wedge \cdots \wedge J(\alpha_k) \geq r$. Thus $b = \sum \alpha_i \beta_i = \sum \alpha_i (x s_i) = (\sum \alpha_i s_i) x = lx$ where $l = \sum \alpha_i s_i$.

Thus $x = a + b = a + xl = a + (a + xl)l = \cdots = a(1 + l + l^2 + \cdots + l^{n-1}) + xl^n$ for every positive integer n. Since

$$\begin{aligned} J(l) = radI(l) &= \sup\{I(l^m) \mid m \in N\} \\ &= J(\alpha_1 s_1 + \cdots + \alpha_k s_k) \\ &\geq min\{J(\alpha_1 s_1), \cdots, J(\alpha_k s_k)\} \\ &\geq min\{J(\alpha_1), \cdots, J(\alpha_k)\} \geq r, \end{aligned}$$

there exist n such that $I(l^n) \geq r$ by the sup property of I. Thus

$$\begin{aligned} I(x) &= I(a(1 + l + l^2 + \cdots + l^{n-1}) + xl^n) \\ &\geq min\{I(a(1 + l + l^2 + \cdots + l^{n-1})), I(xl^n)\} \\ &\geq \min\{I(a), I(l^n)\} \\ &\geq r = x_r(x) \end{aligned}$$

Hence $x_r \in I$. This is a contradiction. Thus, $x_r \notin U$. Now $I + < x_r > \subseteq J$ since $I \subseteq J$ and $x_r \in J$. Then there exist fuzzy ideal K of R. such that $I + < x_r > = J * K$. Since ${y_t}^n \notin I$, $J \subset J * < y_t >$. Then by theorem 3.2 $J = J * (J + < y_t >)$.

Now

$$\begin{aligned} x_r \in I + < x_r > &= J * K \\ &= (J * (J + < y_t >)) * K \\ &= J * ((J + < y_t >) * K) \\ &= J * (K * (J + < y_t >)) \\ &= (J * K) * (J + < y_t >) \\ &= (I + < x_r >) * (J + < y_t >) \\ &\subseteq I * J + + < x_r > * J + I * < y_t > + < x_r > * < y_t > \\ &\subseteq I + < x_r > * J = U \ (\because < x_r > * < y_t > = < x_r y_t > \subseteq I). \end{aligned}$$

Thus $x_r \in U$, a contradiction by $x_r \notin U$. Hence I is primary □

Corollary 3.6. *Let P be prime. Then for all positive integer n, P^n is primary and its fuzzy radical is P*

Proof. We first prove that $radP^n = P$ for all $n > 0$. If $n = 1$, the result is obvious. Let $n > 1$. Now $radP^n(x) = \sup\{P^n(x^m) \mid m > 0\} \geq P^n(x^n) \geq P(x)$ Since P is prime, $P(x) = radP(x) = \sup\{P(x^m) \mid m > 0\} \geq \sup\{P^n(x^m) \mid m > 0\} = radP^n(x)$ for all $x \in R$. Hence $radP^n = P$. Now the result follows from Theorem 3.3. □

THEOREM 3.7. *Let P be prime and $P^n \neq P^{n+1}$ for all $n > 0$. Then P^ω is a prime.*

Proof. Let x_l, y_m be fuzzy points such that $x_l \notin P^\omega$ and $y_m \notin P^\omega$. We shall show that $x_l y_m \notin P^\omega$ If $x_l \notin P, y_m \notin P$ then since P is prime, $x_l y_m \notin P$ and so $x_l y_m \notin P^\omega$. Next, let $x_l \in P, y_m \notin P$, since $x_l \notin P^\omega$, there exist a positive inter p such that $x_l \in P^p, x_l \notin P^{p+1}$. Since, by Corollary 3.3, P^{p+1} is primary fuzzy ideal with fuzzy radical $P, x_l y_m \notin P^{p+1}$ and so $x_l y_m \notin P^\omega$. The case when $x_l \notin P, y_m \in P$ may be similarly disposed of.

Finally, Let $x_l, y_m \in P$. Then there exist positive integer q, r such that $x_l \in P^q, x_l \notin P^{q+1}$ and $y_m \in P^r, y_m \notin P^{r+1}$. Then, $< x_l > \subseteq P^q, < y_m > \subseteq P^r$. Since R is a fuzzy multiplication ring, there exist fuzzy ideal I, J of S such that $< x_l >= P^q * I$, $< y_m >= P^r * J$, $J, I \nsubseteq P$. Now, if $x_l y_m \in P^{q+r+1}$ then $P^{q+r} * J * I = (P^q * I) * (P^r * J) =< x_l > * < y_m > \subseteq P^{q+r+1}$. Since P^{q+r+1} is primary fuzzy ideal with fuzzy radical P and since P is prime, $J * I \nsubseteq P$ so $P^{q+r} \subseteq P^{q+r+1}$. Also $P^{q+r} \geq P^{q+r+1}$ So $P^{q+r} = P^{q+r+1}$, a contradiction. Hence, $x_l y_m \notin P^{q+r+1}$,i.e. $x_l y_m \notin P^\omega$. Thus, P^ω is prime ideal. □

DEFINITION 3.8. *A ring R is said to be a fuzzy principal ideal ring if every fuzzy ideal of R is a principal fuzzy ideal.*

THEOREM 3.9. *If R is regular or a fuzzy principal ideal ring, then R is a fuzzy multiplication ring.*

Proof. Let R be a regular ring. Let I and J be any two fuzzy ideals of R such that $I \subseteq J$. Then $I * J \subseteq I$. Next, let $x \in R$. Then there exists $a \in R$ such that $xax = x$. So

$$\begin{aligned} I * J(x) &= \sup_{x=\sum_{i=1}^{n} x_i y_i} \{\min\{I(x_1), \cdots, I(x_n), J(y_1), \cdots, I(y_n)\}\} \\ &\geq \min\{I(x), J(ax)\}(\because x = xax) \\ &\geq \min\{I(x), J(x)\}(\because J \text{ is a fuzzy ideal }) \\ &= I(x)(\because I \subseteq J) \end{aligned}$$

So $I \subseteq I * J$ and thus $I * J = I$. Hence R is a fuzzy multiplication ring. The case when R is a fuzzy principal ideal ring is obvious. □

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