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ON FUZZY DIMENSION OF N-GROUPS WITH DCC ON IDEALS

Satyanarayana Bhavanari, Syam Prasad Kuncham,
and Venkata Pradeep Kumar Tumurukota

Abstract. In this paper we consider the fuzzy ideals of N-group G where N is a near-ring. We introduce the concepts: minimal elements, fuzzy linearly independent elements, and fuzzy basis of an N-group G and obtained fundamental related results.

1. Introduction

We first recall some basic concepts for the sake of completeness. A non-empty set N with two binary operations + and . is called a *near-ring* if it satisfies the following axioms.

(i) (N, +) is a group (not necessarily abelian)

(ii) (N, $\cdot$) is a semi-group;

(iii) $(a + b)\cdot c = a\cdot c + b\cdot c$ for all a, b, c $\in$ N.

Precisely speaking, it is a right near-ring because it satisfies the right distributive law. We denote ac instead of a.c. Moreover, a near-ring N is said to be *zero-symmetric* if $n0 = 0$ for all $n \in N$, where 0 is the additive identity in N. By an N-group, we mean an additively written group G (but not necessarily abelian), together with a mapping $N \times G \to G$ (the image of (n, g) denoted by $n\cdot g$) satisfying the following conditions:

1. $(n_1 + n_2)\cdot g = n_1\cdot g + n_2\cdot g$ and
2. $n_1\cdot(n_2\cdot g) = (n_1\cdot n_2)\cdot g$ for all $g \in G$, and $n_1, n_2 \in N$.

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Throughout, by a near-ring, we mean a zero-symmetric right near-ring. N stands for a near-ring and G stands for an N-group. <X> denotes the ideal generated by X for a given subset X of G and <a> denotes <{a}>.

Now we collect the necessary literature.

An ideal A of G is said to be *essential* in an ideal B of G (denote as, A $\leq_e$ B) if I is an ideal of G contained in B and A $\cap$ I = (0) imply I = (0). In [14], it is observed that (i) intersection of finite number of essential ideals is essential; (ii) For any ideals I, J, K of G such that I $\leq_e$ J, and J $\leq_e$ K, then I $\leq_e$ K; and (iii) If I $\subseteq$ J, then I $\leq_e$ J implies that (I $\cap$ K) $\leq_e$ (J $\cap$ K). An ideal A of G is said to be *uniform* if every non-zero ideal of G, which is contained in A, is essential in A. An element 0 $\neq$ u $\in$ G is said to be uniform element (or u-element) if <u> is a uniform ideal of G. The concept of finite Goldie Dimension in N-Groups was introduced by Reddy & Satyanarayana [5]. An ideal H of G is said to have *finite Goldie dimension* (abbr. FGD) if H does not contain an infinite number of non-zero ideals of G whose sum is direct. G has FGD if G does not contain a direct sum of infinite number of non-zero ideals. Equivalently, G has FGD if for any strictly increasing sequence $H_0 \subset H_1 \subset H_2 \subset \ldots$ of ideals of G, there is an integer i such that H_k is essential ideal in $H_{k+1}$for every k $\geq$ i.

It is proved (in [5]) that if H_i, K_i($1 \leq i \leq n$) are ideals of G such that the sum of ideals {K_i | 1 $\leq$ i $\leq$ n} is direct and $H_i \subseteq K_i$ for 1 $\leq$ i $\leq$ n, then "$H_i \leq_e K_i$, 1 $\leq$ i $\leq$ n $\Leftrightarrow$ $H_1 \oplus H_2 \oplus \ldots \oplus H_n \leq_e K_1 \oplus K_2 \oplus \ldots \oplus K_n$". In [5], the authors also proved that if an ideal H of G has FGD, then there exists finite number of uniform ideals U_i, 1 $\leq$ i $\leq$ k of G whose sum is direct and essential in H. This number k is independent of choice of U_i' s and k is called the *Goldie Dimension* of H. In this case, we write k = dim H.

For other preliminary definitions and results in near-rings, we refer [4,5, 6, 7, 10, and 11].

Next we collect necessary information related to fuzzyness from the existing literature.

The concept of fuzzy subset was introduced by Zadeh [15]. Later several authors like [12, 13, and 14] were studied the concept: fuzzyness in different algebraic systems, particularly in the theory of rings and near-rings.

We now review some fuzzy logic concepts. Let X be a non empty set. A mapping μ: X$\rightarrow$ [0, 1] is called a fuzzy subset of X. We shall use the notation μ_t, called a *level subset* of μ which is defined as $\mu_t = \{x \in M \mid \mu(x) \geq t\}$ where t $\in$ [0, 1]. Let X and Y are two non empty sets and f a function of X into Y. Let μ and σ be fuzzy subsets of X and Y respectively. Then $f(\mu)$, the *image* of μ under f is a fuzzy subset of Y defined by

$$(f(\mu))(y) = \begin{cases} \sup\limits_{f(x)=y} \mu(x) & \text{if} \quad f^{-1}(y) \neq \phi \\ 0 & \text{if} \quad f^{-1}(y) = \phi \end{cases}$$

and $f^{-1}(\sigma)$, the *pre-image* of σ under f is a fuzzy subset of X defined by $(f^{-1}(\sigma))(x) = \sigma(f(x))$ for all $x \in X$.

2. Fuzzy Ideals of N-Groups

We start this section by defining the concept "fuzzy ideal" of an N-group G.

Definition 2.1. [3] Let μ: G $\rightarrow$ [0, 1] be a mapping. μ is said to be a ***fuzzy ideal of*** G if the following two conditions hold:

1. $\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$,
2. $\mu(g + x - g) = \mu(x)$
3. $\mu(-g) = \mu(g)$
4. $\mu(n(g + x)-ng) \geq \mu(x)$, for all x, g, $g^1 \in G$, n $\in$ N.

If μ satisfies (i), (ii), and (iii), then we say μ, a fuzzy normal subgroup of G.

Proposition 2.2. *Let G be N-group with unity and μ: G $\rightarrow$ [0, 1] is a fuzzy set with $\mu(ng) \geq \mu(g)$ for all $g \in G$, $n \in N$, then the following two conditions are true.*

(i) For all $0 \neq n \in N$, $\mu(ng) = \mu(g)$ if n is left invertible; and
(ii) $\mu(-g) = \mu(g)$.

Proof. (i) Let n^1 be a left inverse of n. Then $n^1n = 1$. Now $\mu(ng) \geq \mu(g)$ $\mu(g) = \mu(1.g) = \mu(n^1ng) \geq \mu(ng)$ (by hypothesis) $\Rightarrow \mu(g) \geq \mu(ng)$. Hence $\mu(ng) = \mu(g)$ for all $g \in G$, and for all left invertible elements $0 \neq n \in N$.

(ii) Follows from (i) by taking n = -1. □

COROLLARY 2.3. *If μ is a fuzzy ideal of G and g, $g^1 \in G$, then $\mu(g - g^1) \geq min\{\mu(g), \mu(g^1)\}$.*

Proof. Given μ is a fuzzy ideal of G. Now $\mu(g - g^1) = \mu(g + (-g^1)) \geq \min\{\mu(g), \mu(-g^1)\}$ (since μ is a fuzzy ideal) $\geq \min\{\mu(g), \mu(g^1)\}$ (by the Proposition 2.2). Therefore $\mu(g - g^1) \geq \min\{\mu(g), \mu(g^1)\}$ for all g, $g^1 \in G$. □

PROPOSITION 2.4. *If μ is a fuzzy ideal of G, and g, $g^1 \in G$ with $\mu(g) > \mu(g^1)$, then $\mu(g + g^1) = \mu(g^1)$. In other words, if $\mu(g) \neq \mu(g^1)$, then $\mu(g + g^1) = min\{\mu(g), \mu(g^1)\}$.*

Proof. By definition, $\mu(g + g^1) \geq \mu(g^1)$.

Take $\mu(g^1) = \mu(g^1 + g - g) \geq \min\{\mu(g^1 + g), \mu(-g)\} = \min\{\mu(g^1 + g), \mu(g)\} = \mu(g + g^1)$ (by hypothesis) and so $\mu(g + g^1) = \mu(g^1)$. □

COROLLARY 2.5. *If μ: $G \to [0, 1]$ is a mapping satisfies the condition $\mu(ng) \geq \mu(g)$ for all $g \in G$ and $n \in N$, then the following two conditions are equivalent:*

1. *$\mu(g - g^1) \geq min\{\mu(g), \mu(g^1)\}$; and*
2. *$\mu(g + g^1) \geq min\{\mu(g), \mu(g^1)\}$.*

Proof. **(i) ⇒ (ii)**: Suppose (i). Now $\mu(g + g^1) = \mu(g - (-g^1)) \geq \min\{\mu(g), \mu(-g^1)\}$ (by supposition) $\geq \min\{\mu(g), \mu(g^1)\}$ (by the given condition with n = -1). Therefore $\mu(g + g^1) \geq \min\{\mu(g), \mu(g^1)\}$.

(ii) ⇒ (i): follows from Corollary 2.3. □

PROPOSITION 2.6. *If μ: $G \to [0, 1]$ is a fuzzy ideal, then (i) $\mu(0) \geq \mu(g)$ for all $g \in G$; and (ii) $\mu(0) = \sup_{g \in G} \mu(g)$.*

Proof. (i) $\mu(0) = \mu$ (x - x) $\geq$ min $\{\mu(x), \mu(-x)\} = \mu(x)$ for all x in G.

(ii) Follows from (i). □

The next part of this section deals with the concept "level ideals". A straightforward proof gives the following theorem.

THEOREM 2.7. [3] *A fuzzy subset μ of G is a fuzzy ideal of G $\Leftrightarrow$ the level set μ_t is an ideal of G for all t $\in$ [0, $\mu(0)$].*

DEFINITION 2.8. Let μ be any fuzzy ideal of G. The ideals μ_t, t $\in$ [0, 1] where $\mu_t = \{x \in G \mid \mu(x) \geq t\}$ are called *level ideals* of μ.

THEOREM 2.9. *Let $I \subseteq G$. Define a fuzzy subset μ by*

$$\mu(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

Then the following conditions are equivalent:

(i) μ is a fuzzy ideal ; and

(ii) I is a ideal of G.

Proof. **(i) $\Rightarrow$ (ii)**: Let x, y $\in$ I. Now $\mu(x) = \mu(y) = 1$.

Now $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ (since μ is fuzzy ideal)

$= \min\{1, 1\} = 1 \Rightarrow \mu(x - y) \geq 1 \Rightarrow x - y \in I$.

Let x $\in$ I. Since μ is fuzzy normal, we have $\mu(y + x - y) = \mu(x) = 1$, for all y $\in$ G. Therefore y + x − y $\in$ I for all y $\in$ G. Hence I is normal. Take x $\in$ I, g $\in$ G and n $\in$ N. Since μ is a fuzzy ideal of G, we have that

$\mu(n(g + x) - ng)) \geq \mu(x) = 1$ and so n(g + x) − ng $\in$ I. Hence I is an ideal of G.

(ii) $\Rightarrow$ (i): Let x, y $\in$ G. If x, y $\in$ I, then x - y $\in$ I and so $\mu(x - y) = 1 \geq \min\{1, 1\} = \min\{\mu(x), \mu(y)\}$. If x $\in$ I and y $\notin$ I, then x - y $\notin$ I and so $\mu(x - y) = 0 \geq \min\{1, 0\} = \min\{\mu(x), \mu(y)\}$. If x $\notin$ I, y $\notin$ I, then $\mu(x - y) \geq 0 = \min\{\mu(x), \mu(y)\}$.

Take x $\in$ I. Since I is an ideal of G, we have that y + x −y $\in$ I and so $\mu(y + x - y) = 1 = \mu(x)$. If $\mu(y + x - y) = 0$, then y + x − y $\notin$ I and so x $\notin$ I.

This shows that $\mu(y + x - y) = 0 = \mu(x)$.

Take x ∈ I, g ∈ G and n ∈ N. Since I is an ideal G, we have n(g + x) − ng ∈ I. Therefore μ(n(g + x) − ng) = 1 = μ(x). If x ∉ I, then μ(n(g + x) − ng) ≥ 0 = μ(x).

Thus (ii) ⇒ (i). □

PROPOSITION 2.10. *Let μ be a fuzzy ideal of G and μ_t, μ_s (with t < s) be two level ideals of μ. Then the following two conditions are equivalent:*

(i) $\mu_t = \mu_s$; and (ii) there is no x ∈ G such that t ≤ μ(x) < s.

Proof. **(i) ⇒ (ii)**: In a contrary way, suppose that there exists an element x ∈ G such that t ≤ μ(x) < s. Then x ∈ μ_t and x ∉ μ_s and so $\mu_t \neq \mu_s$, a contradiction. Hence we get (ii).

(ii) ⇒ (i): Since t < s we have $\mu_t \geq \mu_s$. Let x ∈ μ_t ⇒ μ(x) ≥ t. By given condition (ii), there is no y such that s > μ(y) ≥ t and so μ(x) ≥ s which implies x ∈ μ_s. Thus $\mu_t \leq \mu_s$. □

3. Minimal Elements

We start this section by introducing the new concept "minimal element".

DEFINITION 3.1. An element x ∈ G is said to be a ***minimal element*** if <x> is minimal in the set of all non-zero ideals of G.

THEOREM 3.2. *If G has DCC on ideals, then every nonzero ideal of G contains a minimal element.*

Proof. Let K be a nonzero ideal of G. Since G has DCC on its ideals, it follows that the set of all ideals of G contained in K has a minimal element. So K contains a minimal ideal A (that is, A is a minimal in the set of all non-zero ideals of G contained in K). Let 0 ≠ a ∈ A. Then 0 ≠ <a> ⊆ A and so <a> = A. Since <a> is a minimal ideal, we have that 'a' is a minimal element. □

NOTE 3.3. There are N-Groups, which do not satisfy DCC on its ideals, but contains a minimal element. For this, we observe the following example.

EXAMPLE 3.4. Write N = Z, G = $Z \oplus Z_6$. Now G is an N-group. Clearly G has no DCC on its ideals. Consider g = (0, 2) $\in$ G. Now the ideal generated by g, that is, <g> = Zg = {(0, 0), (0, 2), (0, 4)} is a minimal element in the set of all non-zero ideals of G. Hence g is a minimal element.

THEOREM 3.5. *Every minimal element is an u-element.*

Proof. Let $0 \neq a \in G$ be a minimal element. Consider Na. Let (0) $\neq$ L and I be ideals of G such that L $\subseteq$ <a>, I $\subseteq$ <a> and L $\cap$ I = (0). Since L $\neq$ (0), (0) $\subsetneq$ L $\subseteq$ <a>, and a is minimal, it follows that L = <a>. Now I = I $\cap$ <a> = I $\cap$ L = (0). This shows that L is essential in <a>. Hence <a> is uniform ideal and so a is an u-element. □

NOTE 3.6. The converse of Theorem 3.5 is not true. For this observe the following example given here.

Write G = Z, N = Z. Since Z is a uniform, and 1 is a generator, we have that 1 is an u-element. But 2Z is a proper ideal of 1.Z = Z = G. Hence 1 cannot be a minimal element. Thus 1 is an u-element but not a minimal element.

THEOREM 3.7. *Suppose μ is a fuzzy ideal of G.*

(i) If g $\in$ G, then for any x $\in$ <g> we have μ(x) $\geq$ μ(g); and

(ii) If g is a minimal element, then for any $0 \neq x \in$ <g> we have μ(x) = μ(g).

Proof. (i) By straightforward verification, we conclude that for g $\in$ G, <g>= $\bigcup_{i=0}^{\infty} A_i$ where $A_{k+1} = A_k^* \cup A_k^+ \cup A_k^0$, $A_0 = \{g\}$ and

$A_k^* = \{y + x - y \mid y \in G, x \in A_k\}$,

$A_k^+ = \{n(y + x) - ny \mid n \in N, y \in G, x \in A_k\}$,

$A_k^0 = \{x - y \mid x, y \in A_k\}$,

We prove that $\mu(y) \geq \mu(g)$ for all y $\in A_m$ for m $\geq$ 1. For this, we use induction on m. It is obvious if m = 0. Suppose the induction hypothesis for k. That is, $\mu(y) \geq \mu(g)$ for all y $\in A_k$. Now let v $\in A_k^* \cup A_k^+ \cup A_k^0$. Suppose v $\in A_k^*$. Then v = z + y - z for some y $\in A_k$. Now $\mu(v) = \mu(z + y - z) \geq \mu(y)$ (since μ is a fuzzy ideal of G) $\geq \mu(g)$. Let v $\in A_k^0$. Then v = $y_1 - y_2$ for some $y_1, y_2 \in A_k$. Now

$\mu(v) = \mu(y_1 - y_2) \geq \min\{\mu(y_1), \mu(y_2)\} \geq \mu(g)$, by induction hypothesis.

Suppose $v \in A_k^+$. Then $v = n(y + x) - ny$ for some $n \in N$, $y \in G$, $x \in A_k$. Now $\mu(v) = \mu(n(y + x) - ny) \geq \mu(x)$ (since μ is a fuzzy ideal) $\geq \mu(g)$ (by induction hypothesis). Thus in all cases, we proved that $\mu(v) \geq \mu(g)$ for all $v \in A_{k+1}$. Hence by the principle of mathematical induction, we conclude that $\mu(v) \geq \mu(g)$ for all $v \in A_m$ and for all positive integers m. We proved that $\mu(v) \geq \mu(g)$ for all $v \in A_m$ and for all positive integers m. Hence $\mu(x) \geq \mu(g)$ for all $x \in <g>$.

(ii) Let $g \in G$ be a minimal element. Let $0 \neq x \in <g>$. Now $0 \neq <x> \subseteq <g>$. Since g is a minimal element, we have $<x>=<g>$. Therefore $g \in <x>$ and by (i), we have $\mu(g) \geq \mu(x)$. Thus $\mu(x) = \mu(g)$. □

NOTE 3.8. If G satisfies the descending chain condition on its ideals then we say that "G has DCCI". Let K be an ideal of G. If the set $\{J \mid J$ is an ideal of G, $J \subseteq K\}$ has the descending chain condition, then we say that K has DCC on the ideals of G (we write DCCI G, in short).

LEMMA 3.9. *If x is a u-element in G and G has DCCI, then there exist minimal element* $y \in <x>$ *such that* $<y> \leq_e <x>$.

Proof. Consider the ideal $<x>$. By Theorem 3.2, there exists a minimal element $y \in <x>$. Since $<y>$ is a non-zero ideal of $<x>$, and $<x>$ is uniform ideal, it follows that $<y> \leq_e <x>$. □

DEFINITION 3.10. [11] (i) Let X be a subset of G. X is said to be a *linearly independent* (l. i., in short) set if the sum $\sum_{a \in X} < a >$ is direct. If $\{a_i \mid 1 \leq i \leq n\}$ is a l. i. set, then we say that the elements a_i, $1 \leq i \leq n$ are *linearly independent.* If X is not an l. i. set then we say that X is a *linearly dependent* (l. d., in short) set.

(ii) A subset X of G is said to be u-linearly independent (u-l.i., in short) set if every element of X is an u-element and X is a l.i. set.

(iii) A l. i. set X in G is said to be an *essential basis* for G if $\sum_{a \in X} < a > \leq_e G$. We also say that X forms an essential basis for G.

NOTE 3.11. [11]: (i) G has FGD $\Leftrightarrow$ every l. i. subset X of G is a finite set.

(ii) Suppose that dim G = n and $X \subsetneq G$. If X is a l. i. set, then we have: $|X| = n \Leftrightarrow$ X is a maximal l. i. set $\Leftrightarrow$ X is an essential basis for G.

THEOREM 3.12. *If G has DCCI, then there exist linearly independent minimal elements $x_1, x_2, \ldots, x_n$ in G where n = dim G, and the sum $<x_1> + \ldots + <x_n>$ is direct and essential in G. Also $B = \{x_1, x_2, \ldots, x_n\}$ forms an essential basis for G.*

Proof. Since G has DCCI; by the Proposition 2.2 of [10], G has FGD. Suppose n = dim G. Then by the Theorem 2.7 of [11], there exist u-linearly independent elements $u_1, u_2, \ldots, u_n$ such that the sum $<u_1> + \ldots + <u_n>$ is direct and essential in G. Since G has DCCI, by Lemma 3.9, there exist minimal elements $x_i \in <u_i>$ such that $<x_i> \leq_e <u_i>$ for $1 \leq i \leq n$. Since $u_1, u_2, \ldots, u_n$ are linearly independent, it follows that $x_1, x_2, \ldots, x_n$ are also linearly independent.

Thus we have linearly independent minimal elements $x_1, x_2, \ldots, x_n$ in G where n = dim G. Since $<x_i> \leq_e <u_i>$ by a result mentioned in the introduction, it follows that $<x_1> \oplus \ldots \oplus <x_n> \leq_e <u_1> \oplus \ldots \oplus <u_n> \leq_e G$ and so $<x_1> \oplus \ldots \oplus <x_n> \leq_e G$. Thus $B = \{x_1, x_2, \ldots, x_n\}$ forms an essential basis for G. □

4. Fuzzy Linearly Independent Elements

Now we introduce the concept of fuzzy linearly independent elements with respect to a fuzzy ideal μ of G.

DEFINITION 4.1. Let G be an N-group and μ be a fuzzy ideal of G. $x_1, x_2, \ldots, x_n \in G$ are said to be fuzzy μ-linearly independent (or fuzzy linearly independent with respect to μ) if it satisfies the following two conditions: (i) $x_1, x_2, \ldots, x_n$ are linearly independent; and (ii) $\mu(y_1 + \ldots + y_n) = \min\{\mu(y_1), \ldots, \mu(y_n)\}$ for any $y_i \in <x_i>$, $1 \leq i \leq n$.

THEOREM 4.2. *Let μ be a fuzzy ideal on G. If $x_1, x_2, \ldots, x_n$ are minimal elements in G with distinct μ-values, then $x_1, x_2, \ldots, x_n$ are (i) Linearly independent; and (ii) Fuzzy μ-linearly independent.*

Proof. The proof is by induction on n. If n = 1, then x_1 is linearly independent and also fuzzy linearly independent. Let us assume that the statement is true for (n - 1). Now suppose $x_1, x_2, \ldots, x_n$ are minimal elements with distinct μ values. By induction hypothesis $x_1, x_2, \ldots, x_{n-1}$ are linearly independent and fuzzy linearly independent. If $x_1, \ldots, x_n$ are not linearly independent, then the sum of $<x_1>$, $<x_2>, \ldots, <x_n>$ is not direct. This means $<x_i> \cap (<x_1> \oplus \ldots <x_{i-1}> \oplus <x_{i+1}> \oplus \ldots \oplus <x_n>) \neq \{0\}$. This implies $0 \neq y_i = y_1 + \ldots + y_{i-1} + y_{i+1} + \ldots + y_n$ where $y_j \in <x_j>$ for $1 \leq j \leq n$. Now $\mu(x_i) = \mu(y_i)$ (by Theorem 3.7) $= \mu(y_1 + \ldots + y_{i-1} + y_{i+1} + \ldots + y_n) = \min\{\mu(y_1), \ldots, \mu(y_{i-1}), \mu(y_{i+1}), \ldots, \mu(y_n)\}$ (by induction hypothesis) $= \mu(y_k)$ for some $k \in \{1, 2, \ldots, i\text{-}1, i+1, \ldots, n\} = \mu(x_k)$ (by Theorem 3.7). Thus $\mu(x_i) = \mu(x_k)$ for $i \neq k$, a contradiction. This shows that $x_1, x_2, \ldots, x_n$ are linearly independent.

Now we prove that $x_1, x_2, \ldots, x_n$ are fuzzy linearly independent. Suppose $y_i \in <x_i>$, $1 \leq i \leq n$.

$\mu(y_1 + y_2 + \ldots + y_{n-1}) = \min\{\mu(y_1), \ldots, \mu(y_{n-1})\}$ (by the induction hypothesis) $= \mu(y_j)$ for some j with $1 \leq j \leq n - 1 = \mu(x_j)$ (by the Theorem 3.7). Now $\mu(x_j) \neq \mu(x_n) \Rightarrow \mu(y_1 + y_2 + \ldots + y_{n-1}) = \mu(x_j) \neq \mu(x_n) = \mu(y_n) \Rightarrow \mu(y_1 + y_2 + \ldots + y_{n-1} + y_n) = \min\{\mu(y_1 + \ldots + y_{n-1}), \mu(y_n)\}$ (by Proposition 2.4) $= \min\{\min\{\mu(y_1), \ldots, \mu(y_{n-1}), \mu(y_n)\}\} = \min\{\mu(y_1), \ldots, \mu(y_n)\}$. This shows that $x_1, x_2, \ldots, x_n$ are fuzzy linearly independent with respect to μ. □

5. Fuzzy Dimension

We start this section by defining the concept "fuzzy pseudo basis".

DEFINITION 5.1. (i) Let μ be a fuzzy ideal of G. A subset B of G is said to be a ***fuzzy pseudo basis*** for μ if B is a maximal subset of G such that $x_1, x_2, \ldots, x_k$ are fuzzy linearly independent for any finite subset $\{x_1, x_2, \ldots, x_k\}$ of B.

(ii) Consider the set B = {k | there exist a fuzzy pseudo basis B for μ with |B| = k}. If B has no upper bound, then we say that the ***fuzzy dimension of*** μ is infinite. We denote this fact by S-dim (μ)

$= \infty$. If B has an upper bound, then the ***fuzzy dimension of*** μ is sup B. We denote this fact by S-dim (μ) = sup B. If m = S-dim (μ) = sup B, then a fuzzy pseudo basis B for μ with |B| = m, is called as ***fuzzy basis*** for the fuzzy ideal μ.

PROPOSITION 5.2. *Suppose G has FGD and μ is a fuzzy ideal of G. Then (i) |B| $\leq$ dim G for any fuzzy pseudo basis B for μ; and (ii) S-dim (μ) $\leq$ dim G.*

Proof. Suppose n = dim G.

(i) Suppose B is a fuzzy pseudo basis for μ. If | B | $>$ n, then B contain distinct elements $x_1, x_2, \ldots, x_{n+1}$. Since B is a fuzzy pseudo basis, the elements $x_1, x_2, \ldots, x_{n+1}$ are linearly independent; and by Theorem 2.7 of [11], it follows that n + 1 $\leq$ n, a contradiction. Therefore | B | $\leq$ n = dim G.

(ii) From (i) it is clear that dim M is an upper bound for the set

B = {k | there exist a fuzzy pseudo basis B for μ with |B| = k}.

Therefore S-dim (μ) = sup B $\leq$ dim G. □

DEFINITION 5.3. An N-group G is said to have a ***fuzzy basis*** if there exists an essential ideal A of G and a fuzzy ideal μ of A such that S-dim (μ) = dim G. The fuzzy pseudo basis of μ is called as *fuzzy basis* for G.

REMARK 5.4. If G has FGD, then every fuzzy basis for G is a basis for G.

THEOREM 5.5. *Suppose that G has DCCI. Then G has a fuzzy basis (in other words, there exists an essential ideal A of G and a fuzzy ideal μ of A such that S-dim (μ) = dim G).*

Proof. Since G has DCCI, it has FGD. Suppose dim G = n. By Note 3.11, there exist linearly independent minimal elements x_1, x_2, $\ldots$, x_n such that $\{x_1, x_2, \ldots\ldots x_n\}$forms an essential basis for G. Take $0 \leq t_1 < t_2 < \ldots < t_n \leq 1$. Define $\mu(y_i) = t_i$ for $y_i \in <x_i>$, $1 \leq i \leq n$. Then μ is a fuzzy ideal on $A = <x_1> + <x_2> + \ldots + <x_n> \leq_e G$. By the Theorem 4.2, $x_1, x_2, \ldots, x_n$ are fuzzy μ-linearly independent. So $\{x_1, x_2, \ldots, x_n\}$ is a pseudo basis for μ. Now dim M = n $\leq$ sup B $\leq$ dim G (by the Proposition 5.2) and hence S-dim (μ) = dim G. This shows that G has a fuzzy basis. □

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Department of Mathematics,
Acharya Nagarjuna University,
Nagarjuna Nagar – 522 510,
Andhra Pradesh, India.
E-mail: bhavanari2002@yahoo.co.in

Department of Mathematics,
Manipal Institute of Technology,
Manipal Academy of Higher Education,
Manipal-576 104, India
E-mail: drkuncham@yahoo.com

Department of Mathematics,
Gudlavalleru Engineering College
Gudlavalleru, Andhra Pradesh, India