ON THE FEKETE-SZEGÖ PROBLEM FOR STRONGLY α-LOGARITHMIC CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $C^{*}(\alpha, \beta)$ denote the class of normalized strongly $\alpha$-logarithmic close-to-convex functions of order $\beta$, defined in the open unit disk $U$ by

$$\arg \left\{ \left( \frac{f(z)}{g(z)} \right)^{1-\alpha} \left( \frac{zf'(z)}{g(z)} \right)^{\alpha} \right\} \leq \frac{\pi}{2}, \quad \{ \alpha, \beta \geq 0 \}$$

where $g \in S^{*}$ the class of normalized starlike functions. In this paper, we prove sharp Fekete-Szegö inequalities for functions $f \in C^{*}(\alpha, \beta)$.

1. Introduction

Let $S$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

which are univalent in the open unit disk $U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. Let $S^{*}$ be the subclass of $S$ consisting of all starlike functions in $U$.

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A classical result of Fekete and Szegö [4] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter $\mu$, for $f \in S$. There are also several results of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions (see, e.g., [1,6-10]).

Denote by $K(\beta)$ the class of strongly close-to-convex functions of order $\beta$. Thus $f \in K(\beta)$ if and only if there exists $g \in S^*$ such that

$$\left| \arg \frac{zf'(z)}{g(z)} \right| \leq \frac{\pi}{2} \beta, \quad (\beta \geq 0; \ z \in \mathbb{U})$$

A great deal of attention has been given in recent years to the class $K(\beta)$ introduced by Pommerenke [13]. For $0 \leq \beta \leq 1$, the class $K(\beta)$ is a subclass of close-to-convex functions introduced by Kaplan [6] and hence contains only univalent functions. However, Goodman [5] showed that $K(\beta)$ can contain functions with infinite valence for $\beta > 1$. The Fekete-Szegö problems for $K(1)$ and $K(\beta)$ has been also solved by Keogh and Merkes [7] and London[10], respectively. We now introduce a new class which covers the class $K(\beta)$ in terms of powers as follows:

**Definition.** A function $f \in S$, given by (1.1) is said to be strongly $\alpha$-logarithmic close-to-convex of order $\beta$ if there exists a function $g \in S^*$ such that

$$\left| \arg \left\{ \left( \frac{f(z)}{g(z)} \right)^{1-\alpha} \left( \frac{zf'(z)}{g(z)} \right)^{\alpha} \right\} \right| \leq \frac{\pi}{2} \beta, \quad (\alpha, \beta \geq 0; \ z \in \mathbb{U}). \quad (1.2)$$

We denote by $CS^\alpha(\beta)$ the class of strongly $\alpha$-logarithmic close-to-convex functions of order $\beta$. We note that $CS^1(\beta) = K(\beta)$. In particular, $CS^0(1)$ is the class of close-to-star functions introduced by Reade [15].

The purpose of the present paper is to prove sharp Fekete-Szegö inequalities of the functions belonging to the class $CS^\alpha(\beta)$, which extend the results by Abdel-Gawad and Thomas [1], Keogh and Merkes [7] and London [10].
2. Results

In proving our main result, we need the following lemma.

**Lemma.** Let $p$ be analytic in $U$ and satisfying $\text{Re}\{p(z)\} > 0$ for $z \in U$, with $p(z) = 1 + p_1z + p_2z^2 + \cdots$. Then

\[
|p_n| < 2 \quad \text{(2.1)}
\]

and

\[
\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \quad \text{(2.2)}
\]

The inequality (2.1) can be first proved by Carathéodory [2] (also see Duren [3], p.41) and the inequality (2.2) can be found in [14, p.166].

With the help of Lemma, we now derive

**Theorem.** Let $f \in CS^\alpha(\beta)$ and be given by (1.1). Then for $\alpha \geq 0$ and $\beta \geq 0$, we have

\[
(1 + 2\alpha)|\alpha_3 - \mu a_2^2| \leq \begin{cases} 
1 + \frac{2(1+\beta)^3((1+3\alpha)-(1+2\alpha)\mu)}{(1+\alpha)^2} & \text{if } \mu \leq \frac{(1+\beta)(1+2\alpha) - (1+\alpha)^2}{2(1+\beta)(1+2\alpha)}, \\
1 + 2\beta + \frac{2(1+3\alpha) - (1+2\alpha)\mu}{(1+\alpha)^2 - \mu(1+3\alpha) - 2(1+2\alpha)\mu} & \text{if } \frac{(1+\beta)(1+3\alpha) - (1+\alpha)^2}{2(1+\beta)(1+2\alpha)} \leq \mu \leq \frac{1+3\alpha}{2(1+2\alpha)}, \\
1 + 2\beta & \text{if } \frac{1+3\alpha}{2(1+2\alpha)} \leq \mu \leq \frac{(1+\beta)(1+3\alpha) + (1+\alpha)^2}{2(1+\beta)(1+2\alpha)}, \\
-1 + \frac{2(1+\beta)^2(2(1+2\alpha)\mu - (1+3\alpha))}{(1+\alpha)^2} & \text{if } \mu \geq \frac{(1+\beta)(1+3\alpha) + (1+\alpha)^2}{2(1+\beta)(1+2\alpha)}.
\end{cases}
\]

For each $\mu$, there are functions in $CS^\alpha(\beta)$ such that equality holds in all cases.

**Proof.** Let $f \in CS^\alpha(\beta)$. Then it follows from (1.2) that we may write
where \( g \) is starlike and \( p \) has positive real part. Let \( g(z) = z + b_2z^2 + b_3z^3 + \cdots \), and let be given as in Lemma. Then by equating coefficients, we obtain

\[
(\alpha + 1)a_2 = b_2 + \beta p_1
\]

and

\[
(1 + 2\alpha)a_3 = b_3 + \frac{\alpha(1 - \alpha)}{2(1 + \alpha)^2} b_2^2 + \frac{\beta(1 + 3\alpha)}{(1 + \alpha)^2} p_1 b_2
\]

\[
+ \frac{\beta(\beta(1 + 3\alpha) - (1 + \alpha)^2)}{2(1 + \alpha)^2} p_1^2 + \beta p_2.
\]

So, with

\[
x = \frac{(1 + 3\alpha) - 2(1 + 2\alpha)\mu}{(1 + \alpha)^2},
\]

we have

\[
(1 + 2\alpha)(a_3 - \mu a_2^2) = b_3 + \frac{1}{2}(x - 1)b_2^2
\]

\[
+ \beta(p_2 + \frac{1}{2}(\beta x - 1)p_1^2) + \beta xp_1 b_2.
\]

Since rotations of \( f \) also belong to \( \mathbb{CS}^\alpha(\beta) \), we may assume, without loss of generality, that \( a_3 - \mu a_2^2 \) is positive. Thus we now estimate \( \text{Re}(a_3 - \mu a_2^2) \).

For some functions \( h(z) = 1 + k_1z + k_2z^2 + \cdots (z \in U) \) with positive real part, we have \( zg'(z) = g(z)h(z) \). Hence, by equating coefficients, \( b_2 = k_1 \) and \( b_3 = (k_2 + k_1^2)/2 \). So, by using Lemma and letting \( k_1 = 2re^{i\theta}(0 \leq \rho \leq 1, \ 0 \leq \phi \leq 2\pi) \) and \( p_1 = 2re^{i\theta}(0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi) \) in (2.3), we obtain
\[
\text{Re}(1 + 2\alpha)(a_3 - \mu a_2^2) \leq 1 - \rho^2 + (1 + 2x)\rho^2 \cos 2\phi \\
+ 2\beta(1 - \rho^2) + 2\beta^2 x^2 \rho^2 \cos 2\theta \\
+ 4\beta x\rho \cos(\theta + \phi)
\]

and we now proceed to maximize the right-hand side of (2.4). This function will be denote \( \psi \) whenever all parameters except \( x \) are held constant.

Assume that

\[
\frac{(1 + \beta)(1 + 3\alpha) - (1 + \alpha)^2}{2(1 + \beta)(1 + 2\alpha)} \leq \mu \leq \frac{1 + 3\alpha}{2(1 + 2\alpha)},
\]

so that \( 0 \leq x \leq 1/(1 + \beta) \). The expression \(-t^2 + t^2\beta x \cos 2\theta + 2xt\) is the largest when \( t + x/(\beta x \cos 2\theta) \), we have

\[
\psi(x) \leq 1 + 2x + 2\beta \left(1 + \frac{x^2}{1 - \beta x}\right) \\
= 1 + 2\beta + \frac{2((1 + 3\alpha) - 2(1 + 2\alpha)\mu)}{(1 + \alpha)^2 - \beta((1 + 3\alpha) - 2(1 + 2\alpha)\mu)}
\]

and with (2.4) this establishes the second inequality in the theorem. Equality occurs if and only if

\[
p_1 = \frac{2((1 + 3\alpha) - 2(1 + 2\alpha)\mu)}{(1 + \alpha)^2 - \beta((1 + 3\alpha) - 2(1 + 2\alpha)\mu)}, \quad p_2 = b_2 = 2, \quad b_3 = 3
\]

and the corresponding function \( f \) is defined by

\[
f(z)^{1-\alpha}(zf'(z))^\alpha = \frac{z}{(1 - z)^2} \left(\lambda \frac{1 + z}{1 - z} + (1 - \lambda) \frac{1 - z}{1 + z}\right)^\beta,
\]

where
\[ \lambda = \frac{(1 + \alpha)^2 + (1 - 2\beta)((1 + 3\alpha) - 2(1 + 2\alpha)\mu)}{2((1 + \alpha)^2 - \beta((1 + 3\alpha) - 2(1 + 2\alpha)\mu))}. \]

We now to prove the first inequality. Let

\[ \mu \leq \frac{(1 + \beta)(1 + 3\alpha) - (1 + \alpha)^2}{2(1 + \beta)(1 + 2\alpha)}, \]

so that \( x \geq 1/(1 + \beta) \). With \( x_0 = 1/(1 + \beta) \), we have

\[
\begin{align*}
\psi(x) &\leq \psi(x_0) + 2(x - x_0)(1 + \beta)^2 \\
&\leq 1 + \frac{2(1 + \beta)^2((1 + 3\alpha) - 2(1 + 2\alpha)\mu)}{(1 + \alpha)^2},
\end{align*}
\]

as required. Equality occurs only if \( p_1 = p_2 = 2, \ b_2 = 2, \ b_3 = 3 \) and the corresponding function \( f \) is defined by

\[ (f(z))^{1-\alpha}(zf'(z))^{\alpha} = \frac{z}{(1-z)^2} \left( \frac{1+z}{1-z} \right)^{\beta}. \]

Let \( x_1 = -1/(1 + \beta) \). We note that \( \psi(x_1) \leq 1 + 2\beta \). Then \( \psi(x) \) satisfies

\[
\begin{align*}
\psi(x) &\leq \psi(x_1) + 2|x - x_1|(1 + \beta)^2 \\
&\leq -1 + \frac{2(1 + \beta)^2(2(1 + 2\alpha)\mu - (1 + 3\alpha))}{(1 + \alpha)^2},
\end{align*}
\]

if \( x \leq x_1 \), that is,

\[ \mu \geq \frac{(1 + \beta)(1 + 3\alpha) + (1 + \alpha)^2}{2(1 + \beta)(1 + 2\alpha)}. \]

Equality occurs only if \( p_1 = 2i, \ p_2 = -2, \ b_2 = 2i, \ b_3 = -3 \) and the corresponding function \( f \) is defined by

\[ (f(z))^{1-\alpha}(zf'(z))^{\alpha} = \frac{z}{(1-iz)^2} \left( \frac{1+iz}{1-iz} \right)^{\beta}. \]
Finally, since
\[
\psi(\lambda x_1) = \lambda \psi(x_1) + (1 - \lambda) \psi(0) \leq 1 + 2\beta
\]
for \(0 \leq \lambda \leq 1\), we obtain \(\psi(x) \leq 1 + 2\beta\) for \(x_1 \leq x \leq 0\), i.e.,
\[
\frac{1 + 3\alpha}{2(1 + 2\alpha)} \leq \mu \leq \frac{(1 + \beta)(1 + 3\alpha) + (1 + \alpha)^2}{2(1 + \beta)(1 + 2\alpha)}.
\]
Equality occurs only if \(p_1 = b_2 = 0\), \(p_2 = 2\), \(b_3 = 1\) and the corresponding function \(f\) is defined by
\[
(f(z))^{1-\alpha}(zf'(z))^\alpha = \frac{z(1 + z^2)^\beta}{(1 - z^2)^{1+\beta}}.
\]
Therefore we complete the proof of Theorem.

From Theorem, we have immediately the following

**Corollary.** Let \(f \in CS^0(\beta)\) and be given by (1.1). Then for \(\beta \geq 0\), we have
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
1 + 2(1 + \beta)^2(1 - 2\mu) & \text{if } \mu \leq \frac{\beta}{2(1+\beta)}, \\
1 + 2\beta + \frac{2(1 - 2\mu)}{1 - \beta(1 - 2\mu)} & \text{if } \frac{\beta}{2(1+\beta)} \leq \mu \leq \frac{1}{2}, \\
1 + 2\beta & \text{if } \frac{1}{2} \leq \mu \leq \frac{2 + \beta}{2(1+\beta)}, \\
-1 + 2(1 + \beta)^2(2\mu - 1) & \text{if } \mu \geq \frac{2 + \beta}{2(1+\beta)}.
\end{cases}
\]
For each \(\mu\), there are functions in \(CS^0(\beta)\) such that equality holds in all cases.

**Remark.** If we take \(\alpha = 1\) in Theorem, then we have the result by London [10], which covers the results of Keogh and Merkes [7] and Abdel-Gawad and Thomas [1].
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