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ON CONGRUENCES OF n-ARY GROUPS

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Abstract. Properties of congruences on n-ary groups are investigated.

1. Introduction

The first properties of congruences on n-ary groups was described by J. D. Monk and F. M. Sioson in [10], where was shown that the congruences of a fixed n-ary group have the following properties: 1) any two congruences commute; 2) the lattice of all congruences is modular; 3) any two congruences having the same class are identical. In [7] K. Głazek and B. Gleichgewicht observed that the class of all n-ary groups is a Mal'cev variety. Moreover, if A is an n-ary group then each subalgebra of the Cartesian square $A \times A$ containing the diagonal $\{(a,a)|a \in A\}$ is a congruence on A. The generalized Zassenhaus Lemma and the generalized Schreier and Hólder-Jordan Theorems (formulated in [2]) holds too. Different connections between congruences of an n-ary group and congruences of its covering group are described in [9]. Many useful facts on congruences of n-ary groups one can find in the author's book [4], where, in particular, is proved that: 4) all classes of the same congruence have the same cardinality; 5) the class containing an n-ary subgroup is a semiinvariant n-ary subgroup. Similarly as in arbitrary groups, 6) any class of a congruence can be expressed by other class of this congruence (see [6]). Note by the way that according to Theorem 32.4 from [13] the

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condition 4) implies 3), which, by [8], implies 2). On the other hand, 2) is a consequence of 1) (cf. Theorem VII.3.4 of [1]).

2. Preliminaries

According to the general convention similar to that introduced in the theory of n-ary systems by G. Čupona the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. In the case $j < i$ this symbol is empty.

If $x_{i+1} = x_{i+2} = \ldots = x_{i+k} = x$, then instead of x_{i+1}^{i+k} we write $\overset{(k)}{x}$. In this convention $[x_1, \ldots, x_n] = [x_1^n]$ and

$$[x_1, \ldots, x_i, \underbrace{x, \ldots, x}_{k}, x_{i+k+1}, \ldots, x_n] = [x_1^i, \overset{(k)}{x}, x_{i+k+1}^n].$$

Similarly $[xB^kC^{n-k-1}]$, where B, C are nonempty subsets of A, means the set

$$\{\, [xb_1^kc_1^{n-k-1}] \mid b_1, \ldots, b_k \in B,\ c_1, \ldots, c_{n-k-1} \in C\}.$$

A sequence $e_1, \ldots, e_{k(n-1)}$ of elements of an n-ary group $(A, [\ ])$ is called *neutral* (cf. [11, 12]) if

$$[e_1^{k(n-1)}a] = [ae_1^{k(n-1)}] = a$$

is valid for every $a \in A$. A sequence β is *inverse* to the sequence α if $\alpha\beta$ and $\beta\alpha$ are neutral sequences of $(A, [\ ])$ (cf. [11]). An element $\bar{a} \in A$ is called *skew* to $a \in A$ if

$$[\overset{(n-1)}{a}\ \bar{a}] = a.$$

A nonempty subset B of A is an *n-ary subgroup* of an n-ary group $(A, [\ ])$ if it is closed with respect to the operation $[\ ]$ and $\bar{a} \in B$ for every $a \in B$.

For a congruence σ of $(A, [\ ])$ by $\sigma(x)$ we denote the class containing $x \in A$. σB denotes the smallest class of σ containing B. If $(B, [\ ])$ of an n-ary subgroup of $(A, [\ ])$, then $(\sigma B, [\ ])$ is an n-ary subgroup too (cf. [4]).

Following Dörnte [3] we say that an n-ary subgroup $(B, [\,])$ of $(A, [\,])$ is *semiinvariant* in $(A, [\,])$ if

$$[xB^{n-1}] = [B^{n-1}x]$$

for every $x \in A$.

In [4] it is proved (Proposition 7.4) that for any semiinvariant n-subsemigroup $(B, [\,])$ of an n-ary group $(A, [\,])$ there exists a congruence ρ_B of $(A, [\,])$ such that $\rho_B(a) = [aB^{n-1}]$ for every $a \in A$. Such congruence is defined by

$$\rho_B = \{\, (a, b) \mid [aB^{n-1}] = [bB^{n-1}] \,\}.$$

The following two technical theorems are proved in [4].

THEOREM 1. *Let $(B, [\,])$ and $(C, [\,])$ be semiinvariant n-ary subgroups of an n-ary group $(A, [\,])$ such that $C \subseteq B$. Then:*

1) *$(B/C, [\,])$ is a semiinvariant n-ary subgroup of n-ary group $(A/C, [\,])$;*

2) *$\rho_C \subseteq \rho_B$;*

3) *$\rho_{B/C} = \rho_B / \rho_C$.*

THEOREM 2. *Let $(B, [\,])$ and $(C, [\,])$ be semiinvariant n-ary subgroups of an n-ary group $(A, [\,])$ such that $B \cap C \neq \emptyset$. Then:*

1) *$\rho_{B\cap C} = \rho_B \cap \rho_C$;*

2) *$\rho_{B\cup C} = \rho_{[CB^{n-1}]} = \rho_B \vee \rho_C$;*

3) *$\rho_{[CB^{n-1}]}(x) = [xB^{n-1}C^{n-1}]$, for all $x \in A$.*

THEOREM 3. *Let ρ be a congruence of an n-ary group $(A, [\,])$. Then for all $a, a_1, \ldots, a_{n-2} \in A$ we have*

1) *$\rho(x) = [xa_1^i \rho(c) a_{i+1}^{n-2}] = [a_1^i \rho(c) a_{i+1}^{n-2} x]$ for every $x \in A$, where*

$$(1) \quad c = [\, \bar{a}_i \overset{(n-3)}{a_i} \ldots \bar{a}_1 \overset{(n-3)}{a_1} \bar{a}_{n-2} \overset{(n-3)}{a_{n-2}} \ldots \bar{a}_{i+1} \overset{(n-3)}{a_{i+1}} \,], \quad 0 \le i \le n-2;$$

2) $\rho(x) = [x \overset{(i)}{a} \rho(\bar{a}) \overset{(n-i-2)}{a}] = [\overset{(i)}{a} \rho(\bar{a}) \overset{(n-i-2)}{a} x]$, $0 \le i \le n-2$;

3) $\rho(x) = [x \overset{(i)}{a} \rho(a) \overset{(n-i-3)}{a} \bar{a}] = [\overset{(i)}{a} \rho(a) \overset{(n-i-3)}{a} \bar{a} x]$, $0 \le i \le n-3$;

4) $\rho(x) = [x \overset{(i-1)}{a} \bar{a} \rho(a) \overset{(n-i-2)}{a}] = [\overset{(i-1)}{a} \bar{a} \rho(a) \overset{(n-i-2)}{a} x]$, $0 \le i \le n-2$.

The last theorem was proved in [6].

3. Properties of the n-ary subgroup $(\rho B, [\,])$

THEOREM 4. *Let ρ be a congruence of an n-ary group $(A, [\,])$. Then for every $a \in A$ and an n-ary subgroup $(B, [\,])$ of $(A, [\,])$ we have:*

$$\rho B = [\rho(a)\overset{(n-3)}{a}\bar{a}B] = [B\overset{(n-3)}{a}\bar{a}\rho(a)] = [\rho(\bar{a})\overset{(n-2)}{a}B] = [B\overset{(n-2)}{a}\rho(\bar{a})].$$

Proof. We prove only $\rho B = [\rho(\bar{a})\overset{(n-2)}{a}B]$. The proof other equalities is similar.

Putting $i = 0$ in 2) from Theorem 3, we obtain

$$\rho(x) = [\rho(\bar{a})\overset{(n-2)}{a}x] \tag{2}$$

for every $x, a \in A$.

Since for $u \in \rho B$ there exists $b \in B$ such that $u \in \rho(b)$, from (2) we get

$$u \in [\rho(\bar{a})\overset{(n-2)}{a}b] \subseteq [\rho(\bar{a})\overset{(n-2)}{a}B],$$

i.e.

$$\rho B \subseteq [\rho(\bar{a})\overset{(n-2)}{a}B].$$

Now let $v \in [\rho(\bar{a})\overset{(n-2)}{aa}B]$. Then $v \in [\rho(\bar{a})\overset{(n-2)}{a}b]$ for some $b \in B$, which, by (2), means that $v \in \rho(b)$. Therefore, $v \in \rho B$ and, in the consequence,

$$[\rho(\bar{a})\overset{(n-2)}{a}B] \subseteq \rho B.$$

Thus $\rho B = [\rho(\bar{a})\overset{(n-2)}{a}B]$. □

As a simple consequence of the above theorem we obtain the following result firstly proved in [5].

COROLLARY 1. *If $(B, [\,])$ is an n-ary subgroup of an n-ary group $(A, [\,])$, then for any $a \in B$ and any congruence ρ of $(A, [\,])$ we have:*

$$\rho B = [\rho(a)B^{n-1}] = [B^{n-1}\rho(a)] = [\rho(\bar{a})B^{n-1}] = [B^{n-1}\rho(\bar{a})].$$

Since for a semiinvariant n-ary subgroup $(C, [\,])$ of an n-ary group $(A, [\,])$ we have $\rho_C(a) = C$ for every $a \in C$, from Theorem 4 we obtain also

Corollary 2. *Let $(B, [\ ])$ and $(C, [\ ])$ be n-ary subgroups of an n-ary group $(A, [\ ])$. If $(C, [\ ])$ is semiinvariant in $(A, [\ ])$, then*

$$\rho_C B = [C^{n-1}B] = [BC^{n-1}].$$

Applying Lemma 5.22 from [4] to the last corollary, we obtain

Corollary 3. *Let $(B, [\ ])$ and $(C, [\ ])$ be n-ary subgroups of an n-ary group $(A, [\ ])$. If $(C, [\ ])$ is semiinvariant in $(A, [\ ])$ and $B \cap C \neq \emptyset$, then*

$$\rho_C B = [CB^{n-1}] = [B^{n-1}C].$$

4. Properties of $(\rho\sigma)B$

Theorem 5. *For any two congruences ρ and σ of an n-ary group $(A, [\ ])$ and any $0 \leq i \leq n-2$, $0 \leq j \leq n-2$ the following identity is satisfied*

$$(\rho\sigma)(x) = [a_1^i \rho(c) a_{i+1}^{n-2} b_1^j \sigma(d) b_{j+1}^{n-2} x] = [x a_1^i \sigma(c) a_{i+1}^{n-2} b_1^j \rho(d) b_{j+1}^{n-2}],$$

where c defined by (1) *and*

$$d = [\bar{b}_j \overset{(n-3)}{b_j} \ldots \bar{b}_1 \overset{(n-3)}{b_1} \bar{b}_{n-2} \overset{(n-3)}{b_{n-2}} \ldots \bar{b}_{j+1} \overset{(n-3)}{b_{j+1}}].$$

Proof. Let $u \in (\rho\sigma)(x)$. Then $(x, u) \in \rho\sigma = \sigma\rho$. So, there exists $z \in A$ such that

$$(x, z) \in \sigma, \quad (z, u) \in \rho. \tag{3}$$

Applying the first condition of Theorem 3, we obtain

$$z \in \sigma(x) = [b_1^j \sigma(d) b_{j+1}^{n-2} x], \quad u \in \rho(z) = [a_1^i \rho(c) a_{i+1}^{n-2} z]. \tag{4}$$

Thus

$$u \in [a_1^i \rho(c) a_{i+1}^{n-2} [b_1^j \sigma(d) b_{j+1}^{n-2} x]], \tag{5}$$

i.e.

$$(\rho\sigma)(x) \subseteq [a_1^i \rho(c) a_{i+1}^{n-2} b_1^j \sigma(d) b_{j+1}^{n-2} x]. \tag{6}$$

Conversely, if (5) holds, then, by Theorem 3, holds also (4), which implies (3). Therefore, $(x, u) \in \rho\sigma = \sigma\rho$ and, in the consequence,

$$[a_1^i \rho(c) a_{i+1}^{n-2} b_1^j \sigma(d) b_{j+1}^{n-2} x] \subseteq (\rho\sigma)(x).$$

This proves the first identity.

To the proof of the second identity we must consider the fact that

$$\sigma(x) = [xa_1^i\sigma(c)a_{i+1}^{n-2}], \quad \rho(z) = [zb_1^j\rho(d)b_{j+1}^{n-2}].$$

The rest is similar. □

In particular, putting in the above theorem $a_1 = b_1, \dots, a_{n-2} = b_{n-2}$ and $i = j$ we obtain

$$(7) \quad (\rho\sigma)(x) = [a_1^i\rho(c)a_{i+1}^{n-2}a_1^i\sigma(c)a_{i+1}^{n-2}x] = [xa_1^i\sigma(c)a_{i+1}^{n-2}a_1^i\rho(c)a_{i+1}^{n-2}]$$

for every $i \in \{0, \dots, n-2\}$. This, for $a_1 = \ldots = a_{n-2} = a$, implies

$$(8) \quad (\rho\sigma)(x) = [\overset{(i)}{a}\rho(c)\overset{(n-2)}{a}\sigma(c)\overset{(n-2-i)}{a}x] = [x\overset{(i)}{a}\sigma(c)\overset{(n-2)}{a}\rho(c)\overset{(n-2-i)}{a}].$$

In the similar way, from (7) we can deduce

$$(9) \quad (\rho\sigma)(x) = [\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)\overset{(n-3)}{a}\bar{a}x] = [x\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)\overset{(n-3)}{a}\bar{a}]$$

and

$$(10) \quad (\rho\sigma)(a) = [\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)]; \quad (\rho\sigma)(\bar{a}) = [\rho(\bar{a})\overset{(n-2)}{a}\sigma(\bar{a})].$$

THEOREM 6. *Let $(B, [\ ])$ be an n-ary subgroup of an n-ary group $(A, [\ ])$. Then for any two congruences ρ and σ of $(A, [\ ])$ and an arbitrary element $a \in A$ we have*

$$\begin{aligned}(\rho\sigma)B &= [\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)\overset{(n-3)}{a}\bar{a}B] = [\rho(a)\overset{(n-3)}{a}\bar{a}B\overset{(n-3)}{a}\bar{a}\sigma(a)]\\ &= [B\overset{(n-3)}{a}\bar{a}\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)] = [\rho(\bar{a})\overset{(n-2)}{a}\sigma(\bar{a})\overset{(n-2)}{a}B]\\ &= [\rho(\bar{a})\overset{(n-2)}{a}B\overset{(n-2)}{a}\sigma(\bar{a})] = [B\overset{(n-2)}{a}\rho(\bar{a})\overset{(n-2)}{a}\sigma(\bar{a})]\end{aligned}$$

for every $a \in A$.

Proof. From Theorem 4 and (10) we obtain

$$\begin{aligned}(\rho\sigma)B = [(\rho\sigma)(a)\overset{(n-3)}{a}\bar{a}B] &= [[\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)]\overset{(n-3)}{a}\bar{a}B]\\ &= [\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)\overset{(n-3)}{a}\bar{a}B],\end{aligned}$$

i.e.

$$(\rho\sigma)B = [\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)\overset{(n-3)}{a}\bar{a}B].$$

But

$$[\sigma(a)\overset{(n-3)}{a}\bar{a}B] = [B\overset{(n-3)}{a}\bar{a}\sigma(a)], \quad [\rho(a)\overset{(n-3)}{a}\bar{a}B] = [B\overset{(n-3)}{a}\bar{a}\rho(a)].$$

Therefore

$$(\rho\sigma)B = [\rho(a)\overset{(n-3)}{a}\bar{a}B\overset{(n-3)}{a}\bar{a}\sigma(a)], \quad (\rho\sigma)B = [B\overset{(n-3)}{a}\bar{a}\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)].$$

Others identities can be proved analogously. □

COROLLARY 4. *Let $(B,[\,])$ be an n-ary subgroup of an n-ary group $(A,[\,])$. Then for any two congruences ρ and σ of $(A,[\,])$ and any $a \in B$ we have*

$$(\rho\sigma)B = [\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)B^{n-1}] = [B^{n-1}\rho(a)\overset{(n-3)}{a}\bar{a}\sigma(a)]$$

$$(\rho\sigma)B = [\rho(\bar{a})\overset{(n-2)}{a}\sigma(\bar{a})B^{n-1}] = [B^{n-1}\rho(\bar{a})\overset{(n-3)}{a}\sigma(\bar{a})].$$

COROLLARY 5. *Let $(B,[\,])$ and $(C,[\,])$ be n-ary subgroups of an n-ary group $(A,[\,])$. If $(C,[\,])$ is semiinvariant in $(A,[\,])$ and σ is a congruence of $(A,[\,])$, then*

1) $(\rho_C\sigma)B = [C^{n-1}\sigma(a)\overset{(n-3)}{a}\bar{a}B] = [C^{n-1}\sigma(\bar{a})\overset{(n-2)}{a}B]$ *for every* $a \in C$;

2) *if* $B \cap C \neq \bigcirc$, *then* $(\rho_C\sigma)B = [C^{n-1}\sigma(a)B^{n-1}] = [C^{n-1}\sigma(\bar{a})B^{n-1}]$ *for every* $a \in B \cap C$.

COROLLARY 6. *Let $(B,[\,])$, $(C,[\,])$ and $(D,[\,])$ be n-ary subgroups of an n-ary group $(A,[\,])$. If $(C,[\,])$ and $(D,[\,])$ are semiinvariant in $(A,[\,])$, then*

1) $(\rho_C\rho_D)B = [C^{n-1}D^{n-1}B]$ *if* $C \cap D \neq \emptyset$;

2) $(\rho_C\rho_D)B = [C^{n-1}DB^{n-1}] = [CD^{n-1}B^{n-1}]$ *if* $B \cap C \cap D \neq \bigcirc$.

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