ON CONGRUENCES OF \( n \)-ARY GROUPS

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ABSTRACT. Properties of congruences on \( n \)-ary groups are investigated.

1. Introduction

The first properties of congruences on \( n \)-ary groups was described by J. D. Monk and F. M. Sioson in [10], where was shown that the congruences of a fixed \( n \)-ary group have the following properties: 1) any two congruences commute; 2) the lattice of all congruences is modular; 3) any two congruences having the same class are identical. In [7] K. Glazek and B. Gleichgewicht observed that the class of all \( n \)-ary groups is a Mal’cev variety. Moreover, if \( A \) is an \( n \)-ary group then each subalgebra of the Cartesian square \( A \times A \) containing the diagonal \( \{(a,a) \mid a \in A\} \) is a congruence on \( A \). The generalized Zassenhaus Lemma and the generalized Schreier and Hólder-Jordan Theorems (formulated in [2]) holds too. Different connections between congruences of an \( n \)-ary group and congruences of its covering group are described in [9]. Many useful facts on congruences of \( n \)-ary groups one can find in the author’s book [4], where, in particular, is proved that: 4) all classes of the same congruence have the same cardinality; 5) the class containing an \( n \)-ary subgroup is a semiinvariant \( n \)-ary subgroup. Similarly as in arbitrary groups, 6) any class of a congruence can be expressed by other class of this congruence (see [6]). Note by the way that according to Theorem 32.4 from [13] the

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condition 4) implies 3), which, by [8], implies 2). On the other hand, 2) is a consequence of 1) (cf. Theorem VII.3.4 of [1]).

2. Preliminaries

According to the general convention similar to that introduced in the theory of $n$-ary systems by G. Čupona the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by $x^j_i$. In the case $j < i$ this symbol is empty.

If $x_{i+1} = x_{i+2} = \ldots = x_{i+k} = x$, then instead of $x^i_{i+k}$ we write $x^k_i$. In this convention $[x_1, \ldots, x_n] = [x^0_1]$ and

$$[x_1, \ldots, x_i, x, \ldots, x_{i+k+1}, \ldots, x_n] = [x^k_i, x, x_{i+k+1}^i].$$

Similarly $[xB^kC^{n-k-1}]$, where $B$, $C$ are nonempty subsets of $A$, means the set

$$\{ [b^k_1c_1^{n-k-1}] | b_1, \ldots, b_k \in B, c_1, \ldots, c_{n-k-1} \in C \}.$$

A sequence $\epsilon_1, \ldots, \epsilon_{k(n-1)}$ of elements of an $n$-ary group $(A, [\,])$ is called neutral (cf. [11, 12]) if

$$[\epsilon_1^{k(n-1)}] = [a\epsilon_1^{k(n-1)}] = a$$

is valid for every $a \in A$. A sequence $\beta$ is inverse to the sequence $\alpha$ if $\alpha \beta$ and $\beta \alpha$ are neutral sequences of $(A, [\,])$ (cf. [11]). An element $a \in A$ is called skew to $a \in A$ if

$$\left[a^{(n-1)} \frac{a}{\bar{a}}\right] = a.$$

A nonempty subset $B$ of $A$ is an $n$-ary subgroup of an $n$-ary group $(A, [\,])$ if it is closed with respect to the operation $[\,]$ and $\bar{a} \in B$ for every $a \in B$.

For a congruence $\sigma$ of $(A, [\,])$ by $\sigma(x)$ we denote the class containing $x \in A$. $\sigma B$ denotes the smallest class of $\sigma$ containing $B$. If $(B, [\,])$ of an $n$-ary subgroup of $(A, [\,])$, then $(\sigma B, [\,])$ is an $n$-ary subgroup too (cf. [4]).
Following Dörrnte [3] we say that an n-ary subgroup \((B, [\cdot])\) of \((A, [\cdot])\) is semiinvariant in \((A, [\cdot])\) if
\[ [xB^n-1] = [B^{n-1}x] \]
for every \(x \in A\).

In [4] it is proved (Proposition 7.4) that for any semiinvariant n-subsemigroup \((B, [\cdot])\) of an n-ary group \((A, [\cdot])\) there exists a congruence \(\rho_B\) of \((A, [\cdot])\) such that \(\rho_B(a) = [aB^{n-1}]\) for every \(a \in A\). Such congruence is defined by
\[ \rho_B = \{ (a, b) \mid [aB^{n-1}] = [bB^{n-1}] \}. \]

The following two technical theorems are proved in [4].

**Theorem 1.** Let \((B, [\cdot])\) and \((C, [\cdot])\) be semiinvariant n-ary subgroups of an n-ary group \((A, [\cdot])\) such that \(C \subseteq B\). Then:
1) \((B/C, [\cdot])\) is a semiinvariant n-ary subgroup of n-ary group \((A/C, [\cdot])\);
2) \(\rho_C \subseteq \rho_B\);
3) \(\rho_{B/C} = \rho_B \cap \rho_C\).

**Theorem 2.** Let \((B, [\cdot])\) and \((C, [\cdot])\) be semiinvariant n-ary subgroups of an n-ary group \((A, [\cdot])\) such that \(B \cap C \neq \emptyset\). Then:
1) \(\rho_{B \cap C} = \rho_B \cap \rho_C\);
2) \(\rho_{B \cup C} = \rho_{B \cap C} = \rho_B \cap \rho_C\);
3) \(\rho_{B \cap C}(x) = [xB^{n-1}C^{n-1}], \) for all \(x \in A\).

**Theorem 3.** Let \(\rho\) be a congruence of an n-ary group \((A, [\cdot])\). Then for all \(a, a_1, \ldots, a_{n-2} \in A\) we have
1) \(\rho(x) = [xa\rho(c)a^{n-i-2}_i] = [a_i\rho(c)a^{n-i-3}_i] \) for every \(x \in A\), where
   \[ c = [a_i^{(n-3)} a_i^{(n-3)} \ldots a_{i-2}^{(n-3)} a_{i-2}^{(n-3)} \ldots a_{i+1}^{(n-3)} a_{i+1}^{(n-3)}], \quad 0 \leq i \leq n-2; \]
2) \(\rho(x) = [x a \rho(c) a^{n-i-2}_i] = [a_i \rho(c) a^{n-i-2}_i] \), \(0 \leq i \leq n-2; \)
3) \(\rho(x) = [x a \rho(c) a^{n-i-2}_i] = [a_i \rho(c) a^{n-i-2}_i] \), \(0 \leq i \leq n-3; \)
4) \(\rho(x) = [x a \rho(c) a^{n-i-2}_i] = [a_i \rho(c) a^{n-i-2}_i] \), \(0 \leq i \leq n-2. \)

The last theorem was proved in [6].
3. Properties of the $n$-ary subgroup $(\rho B, [\ ])$

**Theorem 4.** Let $\rho$ be a congruence of an $n$-ary group $(A, [\ ])$. Then for every $a \in A$ and an $n$-ary subgroup $(B, [\ ])$ of $(A, [\ ])$ we have:

$$\rho B = [\rho(a)^{(n-3)} a \bar{a} B] = [B^{(n-3)} a \rho(\bar{a})] = [\rho(\bar{a})^{(n-2)} a B] = [B (n-2) \rho(\bar{a})].$$

**Proof.** We prove only $\rho B = [\rho(\bar{a})^{(n-2)} a B]$. The proof other equalities is similar.

Putting $i = 0$ in (2) from Theorem 3, we obtain

$$(2) \quad \rho(x) = [\rho(\bar{a})^{(n-2)} a x]$$

for every $x, a \in A$.

Since for $u \in \rho B$ there exists $b \in B$ such that $u \in \rho(b)$, from (2) we get

$$u \in [\rho(\bar{a})^{(n-2)} a b] \subseteq [\rho(\bar{a})^{(n-2)} a B],$$

i.e.

$$\rho B \subseteq [\rho(\bar{a})^{(n-2)} a B].$$

Now let $v \in [\rho(\bar{a})^{(n-2)} a B]$. Then $v \in [\rho(\bar{a})^{(n-2)} a b]$ for some $b \in B$, which, by (2), means that $v \in \rho(b)$. Therefore, $v \in \rho B$ and, in the consequence,

$$[\rho(\bar{a})^{(n-2)} a B] \subseteq \rho B.$$

Thus $\rho B = [\rho(\bar{a})^{(n-2)} a B]$.

As a simple consequence of the above theorem we obtain the following result firstly proved in [5].

**Corollary 1.** If $(B, [\ ])$ is an $n$-ary subgroup of an $n$-ary group $(A, [\ ])$, then for any $a \in B$ and any congruence $\rho$ of $(A, [\ ])$, we have:

$$\rho B = [\rho(a) B^{n-1}] = [B^{n-1} \rho(a)] = [\rho(\bar{a}) B^{n-1}] = [B^{n-1} \rho(\bar{a})].$$

Since for a semiinvariant $n$-ary subgroup $(C, [\ ])$ of an $n$-ary group $(A, [\ ])$, we have $\rho_C(a) = C$ for every $a \in C$, from Theorem 4 we obtain also
Corollary 2. Let \((B, [\cdot])\) and \((C, [\cdot])\) be \(n\)-ary subgroups of an \(n\)-ary group \((A, [\cdot])\). If \((C, [\cdot])\) is semiinvariant in \((A, [\cdot])\), then
\[ \rho_C B = [C^n^{-1}B] = [BC^{n-1}]. \]

Applying Lemma 5.22 from [4] to the last corollary, we obtain

Corollary 3. Let \((B, [\cdot])\) and \((C, [\cdot])\) be \(n\)-ary subgroups of an \(n\)-ary group \((A, [\cdot])\). If \((C, [\cdot])\) is semiinvariant in \((A, [\cdot])\) and \(B \cap C \neq \emptyset\), then
\[ \rho_C B = [CB^{n-1}] = [B^{n-1}C]. \]

4. Properties of \((\rho \sigma)B\)

Theorem 5. For any two congruences \(\rho\) and \(\sigma\) of an \(n\)-ary group \((A, [\cdot])\) and any \(0 \leq i \leq n - 2\), \(0 \leq j \leq n - 2\) the following identity is satisfied
\[ (\rho \sigma)(x) = [a_i^1 \rho(c)a_i^{n-2}b_i^n \sigma(d)b_i^{n-2}x] = [xa_i^1 \sigma(c)a_i^{n-2}b_i^n \rho(d)b_i^{n-2}x], \]
where \(c\) defined by (1) and
\[ d = [b_j b_{j+1} \ldots b_1 b_{n-2} b_{n-2} \ldots b_{j+1}]. \]

Proof. Let \(u \in (\rho \sigma)(x)\). Then \((x, u) \in \rho \sigma \sigma = \sigma \rho\). So, there exists \(z \in A\) such that
\[ (3) \quad (x, z) \in \sigma, \quad (z, u) \in \rho. \]

Applying the first condition of Theorem 3, we obtain
\[ (4) \quad z \in \sigma(x) = [b_i^n \sigma(d)b_i^{n-2}x], \quad u \in \rho(z) = [a_i^1 \rho(c)a_i^{n-2}z]. \]

Thus
\[ (5) \quad u \in [a_i^1 \rho(c)a_i^{n-2}b_i^n \sigma(d)b_i^{n-2}x], \]
i.e.
\[ (6) \quad (\rho \sigma)(x) \subseteq [a_i^1 \rho(c)a_i^{n-2}b_i^n \sigma(d)b_i^{n-2}x]. \]

Conversely, if (5) holds, then, by Theorem 3, holds also (4), which implies (3). Therefore, \((x, u) \in \rho \sigma = \sigma \rho\) and, in the consequence,
\[ [a_i^1 \rho(c)a_i^{n-2}b_i^n \sigma(d)b_i^{n-2}x] \subseteq (\rho \sigma)(x). \]
This proves the first identity.

To the proof of the second identity we must consider the fact that
\[ \sigma(x) = [x a_1^i \sigma(c) a_{i+1}^{n-2}] , \quad \rho(z) = [z b_j^i \rho(d) b_{j+1}^{n-2}] . \]
The rest is similar. \( \square \)

In particular, putting in the above theorem \( a_1 = b_1, \ldots, a_{n-2} = b_{n-2} \) and \( i = j \) we obtain
\[ (7) \quad (\rho \sigma)(x) = [a_1^i \rho(c) a_{i+1}^{n-2} a_1^i \sigma(c) a_{i+1}^{n-2} x] = [x a_1^i \sigma(c) a_{i+1}^{n-2} a_1^i \rho(c) a_{i+1}^{n-2}] \]
for every \( i \in \{0, \ldots, n-2\} \). This, for \( a_1 = \ldots = a_{n-2} = a \), implies
\[ (8) \quad (\rho \sigma)(x) = [a a_1^{(n-2)} a \sigma(c) a x] = [x a a_1^{(n-2)} a \rho(c) a] . \]

In the similar way, from (7) we can deduce
\[ (9) \quad (\rho \sigma)(x) = [\rho(a)^{(n-3)} a a_1^{(n-2)} a \sigma(a) a x] = [x \rho(a)^{(n-3)} a a_1^{(n-2)} a \sigma(a)] \]
and
\[ (10) \quad (\rho \sigma)(a) = [\rho(a)^{(n-3)} a \sigma(a)]; \quad (\rho \sigma)(\bar{a}) = [\rho(\bar{a})^{(n-2)} a \sigma(\bar{a})] . \]

**Theorem 6.** Let \((B, [ ] )\) be an \( n\)-ary subgroup of an \( n\)-ary group \((A, [ ] )\). Then for any two congruences \( \rho \) and \( \sigma \) of \((A, [ ] )\) and an arbitrary element \( a \in A \) we have
\[ (\rho \sigma)B = [\rho(a)^{(n-3)} a a_1^{(n-2)} a \sigma(a) a \bar{a} B] = [\rho(a)^{(n-3)} a a \rho(a)^{(n-3)} a \sigma(a) a \bar{a} B] \]
\[ = [\rho(a)^{(n-3)} a a \sigma(a) a \bar{a} B] = [\rho(a)^{(n-3)} a a \sigma(a) a \bar{a} B] , \]
for every \( a \in A \).

**Proof.** From Theorem 4 and (10) we obtain
\[ (\rho \sigma)B = [(\rho \sigma)(a)^{(n-3)} a \bar{a} B] = [\rho(a)^{(n-3)} a \sigma(a) a \bar{a} B] \]
\[ = [\rho(a)^{(n-3)} a \sigma(a) a \bar{a} B] , \]
i.e.
\[ (\rho \sigma)B = [\rho(a)^{(n-3)} a \sigma(a) a \bar{a} B] . \]
But
\[ \sigma(a)^{(n-3)} a \bar{a} B = [B^{(n-3)} a \bar{a} \sigma(a)], \quad \rho(a)^{(n-3)} a \bar{a} B = [B^{(n-3)} a \bar{a} \rho(a)]. \]

Therefore
\[(\rho \sigma) B = [\rho(a)^{(n-3)} a \bar{a} B^{(n-3)} \bar{a} \sigma(a)], \quad (\rho \sigma) B = [B^{(n-3)} a \bar{a} \rho(a)^{(n-3)} a \bar{a} \sigma(a)]. \]

Others identities can be proved analogously. 

**Corollary 4.** Let \((B, [\cdot])\) be an \(n\)-ary subgroup of an \(n\)-ary group \((A, [\cdot])\). Then for any two congruences \(\rho\) and \(\sigma\) of \((A, [\cdot])\) and any \(a \in B\) we have
\[(\rho \sigma) B = [\rho(a)^{(n-3)} a \bar{a} \sigma(a) B^{n-1}] = [B^{n-1} \rho(a)^{(n-3)} a \bar{a} \sigma(a)] \]
\[(\rho \sigma) B = [\rho(\bar{a})^{(n-2)} a \sigma(\bar{a}) B^{n-1}] = [B^{n-1} \rho(\bar{a})^{(n-3)} a \sigma(\bar{a})]. \]

**Corollary 5.** Let \((B, [\cdot])\) and \((C, [\cdot])\) be \(n\)-ary subgroups of an \(n\)-ary group \((A, [\cdot])\). If \((C, [\cdot])\) is semiinvariant in \((A, [\cdot])\) and \(\sigma\) is a congruence of \((A, [\cdot])\), then
1) \((\rho \sigma) C^{n-1} \sigma(a)^{(n-3)} a \bar{a} B = [C^{n-1} \sigma(\bar{a})^{(n-2)} a \sigma(\bar{a}) B^{n-1}] \) for every \(a \in C\);
2) if \(B \cap C \neq \emptyset\), then \((\rho \sigma) C^{n-1} \sigma(a)^{(n-3)} a \bar{a} B = [C^{n-1} \sigma(\bar{a}) B^{n-1}] = [C^{n-1} \sigma(\bar{a}) B^{n-1}] \) for every \(a \in B \cap C\).

**Corollary 6.** Let \((B, [\cdot]), (C, [\cdot])\) and \((D, [\cdot])\) be \(n\)-ary subgroups of an \(n\)-ary group \((A, [\cdot])\). If \((C, [\cdot])\) and \((D, [\cdot])\) are semiinvariant in \((A, [\cdot])\), then
1) \((\rho \sigma) C = [C^{n-1} D \bar{a} B^{n-1}] \) if \(C \cap D \neq \emptyset\);
2) \((\rho \sigma) D = [C^{n-1} D B^{n-1}] = [C D^{n-1} B^{n-1}] \) if \(B \cap C \cap D \neq \emptyset\).

**References**


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