ALMOST EVERYWHERE CONVERGENCE THEOREM FOR SEMINORMED FUZZY CO-INTEGRALS

-준노름 퍼지보적분의 거의 모든 점에서의 수렴정리-

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요약

본 논문에서는 퍼지측도가 공(空) 가법성을 갖는 경우 준노름퍼지 보적분의 거의모든점에서의 수렴정리가 성립함을 보였다.

■ 중심어 : | 퍼지측도 | t-준(코)노름 | 준노름퍼지적분 | 준코노름퍼지보적분 |

Abstract

In this paper, an almost everywhere convergence theorem for seminormed fuzzy co-integrals is showed provided that the fuzzy measure is null-additive.

■ keyword : | Fuzzy Measure | t-semi(co)norm | Seminormed Fuzzy Integral | Seminormed Fuzzy Co-integral |

I. Introduction

Almost everywhere convergence theorem holds for a sequence of bounded Lebesgue integrable functions. This theorem also holds for seminormed fuzzy integrals provided that the fuzzy measure is null-additive[5]. In [1], the seminormed fuzzy co-integral was introduced as a complementary concept of seminormed fuzzy integral, and an application for decision making problems was proposed.

The purpose of this paper is to show that almost everywhere convergence theorem for seminormed fuzzy co-integrals holds provided that the fuzzy measure is null-additive.

II. The Seminormed Fuzzy Co-integrals

Let X be a nonempty set, $\mathcal{A}$ be a σ-algebra of subsets of X. A set function $g : \mathcal{A} \to [0, 1]$ is called a *fuzzy measure* on A if

(1) $g(\emptyset) = 0$, $g(X) = 1$;

(2) (monotonicity) $A, B \in \mathcal{A}$ with $A \subset B$ implies $g(A) \le g(B)$;

(3) (continuity from below) $\{A_n\} \in \mathcal{A}$ $A_1 \subset A_2 \subset \cdots$ imply

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$$\lim_{n\to} g(A_n) = g(\cup_{n=1}^{\infty} A_n);$$

(4) (continuity from above) $\{A_n\} \in \mathcal{A}$ $A_1 \supset A_2 \supset \cdots$, and $g(A_1) < \infty$ imply

$$\lim_{n\to\infty} g(A_n) = g(\cap_{n=1}^{\infty} A_n).$$

We note that if a fuzzy measure g is both continuity from below and continuity from above, then g is called a continuous fuzzy measure.

We call $(X, \mathcal{A}, g)$ a fuzzy measure space if g is a fuzzy measure on $\mathcal{A}$.

A fuzzy measure g is called *null-additive* if

$$g(A \cup B) = g(A)$$

whenever

$$A, B \in \mathcal{A},\ A \cap B = \emptyset, \text{and } g(B) = 0.$$

Note that classical measures satisfy null-additivity for the countable additivity.

Let $\mathcal{B}$ is the σ-algebra of Borel subsets of [0, 1], and $\mathcal{A}$ a measure for X. A real-valued function $h: X \to [0, 1]$ is $(\mathcal{A}, \mathcal{B})$ *-measurable* (i.e., measurable with respect to $\mathcal{A}$ and $\mathcal{B}$) if

$$h^{-1}(B) = \{x \mid h(x) \in B\} \in \mathcal{A}$$

for any $B \in \mathcal{B}$. We shall say measurable for $(\mathcal{A}, \mathcal{B})$-measurable if there is no confusion likely. From now on, we consider only the set

$$L^0(X) = \{h: X \to [0,1] \mid h \text{ is measurable with respect to } \mathcal{A} \text{ and } \mathcal{B}\}.$$

For any given $h \in L^0(X)$ and $\alpha \in [0, 1]$, we write

$$H_\alpha = \{x \in X \mid h(x) \geq \alpha\},$$

$$H^\alpha = \{x \in X \mid h(x) < \alpha\}.$$

Definition 2.1.([3]) A function $\top : [0,1] \times [0,1] \to [0,1]$ is called a *t-seminorm* if it satisfies:

(1) $\top(x, 1) = \top(1, x) = x$, for each $x \in [0,1]$,

(2) if $x_1 \leq x_3$, $x_2 \leq x_4$ for each $x_1, x_2, x_3, x_4 \in [0, 1]$, then $\top(x_1, x_2) \leq \top(x_3, x_4)$.

There exists a concept of *t-semiconorm* $\perp$ defined by

$$\perp(x, y) = 1 - \top(1-x, 1-y)$$

for seminorm $\top$.

A *t-semiconorm* is a function $\perp : [0,1] \times [0,1] \to [0,1]$ which satisfies :

(1) $\perp(x, 0) = \perp(0, x) = x$ for each $x \in [0,1]$,

(2) if $x_1 \leq x_3$, $x_2 \leq x_4$ for each $x_1, x_2, x_3, x_4 \in [0, 1]$, then $\perp(x_1, x_2) \leq \perp(x_3, x_4)$.

Cleary, $\top(x, 0) = \top(0, x) = 0$ and $\perp(x, 1) = \perp(1, x) = 1$.

Definition 2.2.([3]) Let $A \in \mathcal{A}$, $h \in L^0(X)$. The *seminormed fuzzy integral* of h on A with respect to g is defined by

$$\int_A h \top g = \sup_{\alpha \in [0,1]} \top [\alpha, g(A \cap H_\alpha)].$$

When A = X, the seminormed fuzzy integral is denoted by $\int h \top g$. Cleary, the seminormed fuzzy integral is the fuzzy integral for the case

$$\top(x, y) = (x \wedge y)([5]).$$

Definition 2.3.([1]) Let $A \in \mathcal{A}$, $h \in L^0(X)$. The *seminormed fuzzy co-integral* of h on A, with respect to g is defined by

$$(co)\int_A h \top g = \inf_{\alpha \in [0,1]} \perp [(1-\alpha), g(A \cap H^\alpha)]$$

When A=X, the seminormed fuzzy co-integral is denoted by $(co)\int h \top g$.

Definition 2.4. Let (X, A, g) be a fuzzy measure space, and $A \in \mathcal{A}$. A property P holds *almost everywhere* on A if the set of points of A where P fails to hold has fuzzy measure zero.

For the Lebesgue integral, if $h_1 = h_2$ almost everywhere on A, then $\int_A h_1 d\mu = \int_A h_2 d\mu$. Also, we have the following theorem for the seminormed fuzzy co-integral.

Theorem 2.5. Let $(X, \mathcal{A}, g)$ be a fuzzy measure space, $A \in \mathcal{A}$, and $h_1, h_2 \in L^0(X)$. If $h_1 = h_2$ almost everywhere on A and g is null-additive then,

$$(co)\int_A h_1 \top g = (co)\int_A h_2 \top g.$$

Proof. Let

$$A_1 = A \cap \{x \in A : h_1(x) = h_2(x)\}$$

and $H_1^\alpha = \{x \in X \mid h_1(x) < \alpha\}$.

Then

$$g(A - A_1) = 0,$$

and hence

$$g((A - A_1) \cap H_1^\alpha) = 0$$

by the monotonicity of g.

Since g is null-additive and

$A \cap H_1^\alpha = (A_1 \cap H_1^\alpha) \cup ((A - A_1) \cap H_1^\alpha)$,

$g(A \cap H_1^\alpha) = g(A_1 \cap H_1^\alpha)$.

Hence

$$\begin{aligned}(co)\int_A h_1 \top g &= \inf_{\alpha \in [0,1]} \perp [(1-\alpha), g(A \cap H_1^\alpha)] \\ &= \inf_{\alpha \in [0,1]} \perp [(1-\alpha), g(A_1 \cap H_1^\alpha)] \\ &= (co)\int_{A_1} h_1 \top g\end{aligned}$$

Similarly, we can show that

$$(co)\int_A h_2 \top g = (co)\int_{A_1} h_2 \top g$$

Therefore

$$(co)\int_A h_1 \top g = (co)\int_A h_2 \top g$$

since $h_1 = h_2$ on A_1. □

Corollary 2.6. Let $(X, \mathcal{A}, g)$ be a fuzzy measure space and $h \in L^0(X)$. If $A_1 \subset A$ and $g(A - A_1) = 0$. Then

$$(co)\int_A h \top g = (co)\int_{A_1} h \top g.$$

Theorem 2.7.([1]) Let $(X, \mathscr{A}, g)$ be a fuzzy measure space. If $h_1 \leq h_2$, then

$$(co)\int_A h_1 \top g \geqq (co)\int_A h_2 \top g.$$

III. Almost Convergence Theorem for Seminormed Fuzzy Co-integrals

In Classical measure theory, there are the monotone convergence theorem, the uniform convergence theorem and so on, all of which are well known. For the fuzzy integral sequence, there are a lot of convergence theorems as well. In this section, we will two convergence theorems of fuzzy integal sequence under some conditions. In these theorems, we will use a symbol "$\searrow$" (or "$\nearrow$") to denote "decreasing converge to" (or "increasing converge to") for both function sequences and number sequences.

Theorem 3.1.([1]) (Monotone Convergence Theorem) Let $(X, \mathscr{A}, g)$ be a fuzzy measure space, $\top$ be a continuous t-seminorm. If $h_n \searrow h$ on A, (or if $h_n \nearrow h$,)

Then

$$(co)\int_A h_n \top g \nearrow (co)\int_A h \top g$$

$$(\text{or } (co)\int_A h_n \top g \searrow (co)\int_A h \top g).$$

Now we give an almost everywhere convergence theorem for seminormed fuzzy co-integral.

Theorem 3.2 Let $(X, \mathscr{A}, g)$ be a fuzzy measure space, $\top$ a continuous t-seminorm, and g null-additive. If h_n converges to h almost everywhere on A, then

$$\lim_{n\to\infty} (co)\int_A h_n \top g = (co)\int_A h \top g.$$

Proof. Let

$$A_1 = A \cap \{x \in A: h_n(x) \to h(x)\}.$$

Then $g(A - A_1) = 0$.

And for any $x \in A_1$, $inf_{k \geq n} h_k(x) \nearrow h(x)$ and $sup_{k \geq n} h_k(x) \searrow h(x)$.

We have

$$(co)\int_{A_1} \inf_{k \geq n} h_k \top g \searrow (co)\int_{A_1} h \top g$$

and $$(co)\int_{A_1} sup_{k \geq n} h_k \top g \nearrow (co)\int_{A_1} h \top g$$

by Theorem 3.1.

Since

$inf_{k \geq n} h_k(x) \leqq h_n(x) \leqq sup_{k \geq n} h_k(x)$

for any $x \in A_1$ and $n \in N$, it follows that

$$(co)\int_{A_1} \inf_{k \geq n} h_k \top g \geq (co)\int_{A_1} h_n \top g \geq (co)\int_{A_1} \sup_{k \geq n} h_k \top g$$

by Theorem 2.7.

Hence,

$$\lim_{n\to\infty} (co)\int_{A_1} h_n \top g = (co)\int_{A_1} h \top g.$$

Note that

$$(co)\int_A h_n \top g = (co)\int_{A_1} h_n \top g$$

and

$$(co)\int_A h \top g = (co)\int_{A_1} h \top g$$

by Corollary 2.6.

Thus, we have

$$\lim_{n\to\infty} (co)\int_A h_n \top g = (co)\int_A h \top g. \quad \square$$

Corollary 3.3. *Let* $(X, \mathcal{A}, g)$ *be a fuzzy measure space,* $\top$ *a continuous t-seminorm and* g *null-additive. If* h_n *converges to* h *on* A*, then*

$$\lim_{n\to\infty} (co)\int_A h_n \top g = (co)\int_A h \top g.$$

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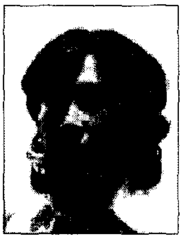

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