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SENSITIVITY ANALYSIS OF SOLUTIONS FOR PARAMETRIC NONLINEAR IMPLICIT QUASIVARIATIONAL INCLUSIONS

Weili Wang, Zeqing Liu, and Shin Min Kang

Abstract. In this paper we introduce a new class of parametric nonlinear implicit quasivariational inclusions and obtain some results about the existence and sensitivity analysis of solutions for this kind of quasivariational inclusions.

1. Introduction

Variational inequalities arise in various models for a lot of mathematical, physical, regional, engineering and other problems. It is well known that the theory of variational inequalities provided the most general, natural, simple, unified, and efficient framework for general treatment of a wide class of nonlinear problems. Quasivariational inclusions are a very important generalization of variational inequalities. Many authors (see for example [2, 4–6]) have studied the problem about sensitivity analysis of the solutions for variational inequalities. By making use of the implicit resolvent operator technique, Agarwal, Cho and Huang [1] and Liu, Debnath, Kang and Ume [7] established the sensitivity analysis of solutions for a few classes of quasivariational inclusions.

In this paper we introduce a new class of parametric nonlinear implicit quasivariational inclusions. We establish some results about the existence and sensitivity analysis of solutions for this kind of quasivariational inclusion. Our results extend and improve the corresponding results due to Agarwal, Cho and Huang [1].

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2. Preliminaries

Let H be a real Hilbert space with a norm and inner product denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, and I denote the identity mapping. Let D be a nonempty open subset of H in which the parameter λ takes values and $f, p, g, m : H \times D \to H$, $N : H \times H \times D \to H$. Suppose that $M : H \times H \times D \to 2^H$ is such that for each $(y, \lambda) \in H \times D$, $M(\cdot, y, \lambda) : H \to 2^H$ is a maximal monotone mapping. For a given $\lambda \in D$, we consider the following problem:

Find $u = u(\lambda) \in H$ such that

$$0 \in N(f(u,\lambda), p(u,\lambda), \lambda) + M((g-m)(u,\lambda), u, \lambda). \tag{2.1}$$

The problem (2.1) is called a *parametric nonlinear implicit quasivariational inclusion.*

In case $f(u,\lambda) = p(u,\lambda) = (g-m)(u,\lambda) = u$ for all $u \in H$ and $\lambda \in D$, then the problem (2.1) reduces to the following parametric generalized strongly nonlinear mixed quasi-variational inclusion problem:

Find $u \in H$ such that

$$0 \in N(u,u,\lambda) + M(u,u,\lambda), \tag{2.2}$$

which was introduced and studied by Agarwal, Cho and Huang [1].

If $M(\cdot, y, \lambda) = \partial\varphi(\cdot,\lambda)$ for any $y \in H$ and $\lambda \in D$, where $\varphi : H \times D \to R\bigcup\{+\infty\}$ satisfies that for each $\lambda \in D$, $\varphi(\cdot,\lambda) : M \to R\bigcup\{+\infty\}$ is a proper convex lower semicontinuous function on H and $\partial\varphi(\cdot,\lambda)$ denotes the subdifferential of the function $\varphi(\cdot,\lambda)$, then the problem (2.1) is equivalent to seek $u \in H$ such that

$$\langle N(f(u,\lambda), p(u,\lambda), \lambda), v-u\rangle \geq \varphi(u,\lambda) - \varphi(v,\lambda), \quad \forall v \in H, \tag{2.3}$$

which is called the *parametric nonlinear quasivaritional inequality.*

Let us recall the following concepts.

DEFINITION 2.1. Let $M : H \times H \to 2^H$ be a maximal monotone mapping with respect to the first argument and $\alpha > 0$ be a constant. Then for any given $y \in H$, the *implicit resolvent operator* $J_\alpha^{M(\cdot,y)}$ associated with $M(\cdot,y)$ is defined by

$$J_\alpha^{M(\cdot,y)}(u) = (I + \alpha M(\cdot,y))^{-1}(u) \quad \text{for } u \in H.$$

DEFINITION 2.2. A mapping $f : H \times D \to H$ is said to be

(1) *Lipschitz continuous* with respect to the first argument if there exist a constant $s > 0$ satisfying

$$\|f(u, \lambda) - f(v, \lambda)\| \le s\|u - v\|$$

for $(u, v, \lambda) \in H \times H \times D$;

(2) *t-strongly monotone* with respect to the first argument if there exist a constant $t > 0$ satisfying

$$\langle f(u, \lambda) - f(v, \lambda), u - v\rangle \ge t\|u - v\|^2$$

for $(u, v, \lambda) \in H \times H \times D$.

DEFINITION 2.3. Let m and $g : H \times D \to H$ be mappings. m and g are called *m-g-relaxed monotone* if there exists a constant $q \in [q_0, cd]$ satisfying

$$\langle m(v, \lambda) - m(u, \lambda), g(u, \lambda) - g(v, \lambda)\rangle \le q\|u - v\|^2$$

for $(u, v, \lambda) \in H \times H \times D$, where $q_0 = \inf\{s : \langle m(v, \lambda) - m(u, \lambda), g(u, \lambda) - g(v, \lambda)\rangle \le s\|u - v\|^2 \text{ for } (u, v, \lambda) \in H \times H \times D\}$.

DEFINITION 2.4. Let $f : H \times D \to H$ and $N : H \times H \times D \to H$ be mappings. N is said to be

(1) *l-strongly monotone* with respect to f in the first argument if there exists a constant $l > 0$ such that

$$\langle N(f(u, \lambda), x, \lambda) - N(f(v, \lambda), x, \lambda), u - v\rangle \ge l\|u - v\|^2$$

for $(u, v, x, \lambda) \in H \times H \times H \times D$;

(2) *Lipschitz continuous* if there exist a constant $s > 0$ satisfying

$$\|N(u, x, \lambda) - N(v, x, \lambda)\| \le s\|u - v\|$$

for $(u, v, x, \lambda) \in H \times H \times H \times D$.

In a similar way, we can define the Lipschitz continuity of f with respect to the second argument and the Lipschitz continuity of N with respect to the second and the third arguments, respectively.

LEMMA 2.1. *Let $\alpha > 0$ be a constant and $\lambda \in D$. Then $u = u(\lambda) \in H$ is a solution of the problem (2.1) if and only if the mapping $F : H \times D \to H$ defined by*

$$F(x,\lambda) = x - (g-m)(x,\lambda) + J_\alpha^{M(\cdot,x,\lambda)}((g-m)(x,\lambda) - \alpha N(f(x,\lambda), p(x,\lambda), \lambda)), \quad x \in H \tag{2.4}$$

has a fixed point $u = u(\lambda) \in H$, where $J_\alpha^{M(\cdot,x,\lambda)} = (I + \alpha M(\cdot, x, \lambda))^{-1}$.

PROOF. It is clear that the problem (2.1) has a solution $u = u(\lambda) \in H$ if and only if

$$\begin{aligned}
&- N(f(u,\lambda), p(u,\lambda), \lambda) \in M((g-m)(u,\lambda), u, \lambda) \\
&\Leftrightarrow (g-m)(u,\lambda) - \alpha N(f(u,\lambda), p(u,\lambda), \lambda) \\
&\qquad \in (g-m)(u,\lambda) + \alpha M((g-m)(u,\lambda), u, \lambda) \\
&\Leftrightarrow u = u - (g-m)(u,\lambda) + J_\alpha^{M(\cdot,u,\lambda)}((g-m)(u,\lambda) \\
&\qquad - \alpha N(f(u,\lambda), p(u,\lambda), \lambda)),
\end{aligned}$$

that is, $u = u(\lambda) \in H$ is a fixed point of F. This completes the proof.□

REMARK 2.1. Lemma 2.1 generalizes Lemma 2.1 in [1].

3. Existence of solutions

THEOREM 3.1. *Let $f, p, g, m : H \times D \to H$ be Lipschitz continuous with respect to the first argument with constants a, b, c and d, respectively and f is h-strongly monotone with respect to the first argument. Let g and m be g-m-relaxed monotone with respect to the first argument with constant q, and $g - m$ be r-strongly monotone with respect to the first argument. Assume that $N : H \times H \times D \to H$ is s-strongly monotone with respect to f in the first argument and Lipschitz continuous with respect to the first and the second arguments with constants l and t, respectively. Suppose that $M : H \times H \times D \to 2^H$ satisfies that for each $(y,\lambda) \in H \times D$, $M(\cdot, y, \lambda) : H \to 2^H$ is a maximal monotone mapping and there exists a constant $k > 0$ satisfying*

$$\|J_\alpha^{M(\cdot,x,\lambda)}(z) - J_\alpha^{M(\cdot,y,\lambda)}(z)\| \le k\|x - y\| \tag{3.1}$$

for $(x, y, z, \lambda) \in H \times H \times H \times D$. Let $P = l^2a^2 - t^2b^2$ and $Q = 2\sqrt{1 - 2r + c^2 + d^2 + 2q} + k$. If there exists a constant $\alpha > 0$ satisfying

$$\alpha t b < 1 - Q \tag{3.2}$$

and one of the following conditions

$$(3.3)\quad \begin{aligned} &P > 0, \quad s > (1-Q)tb + \sqrt{PQ(2-Q)}, \\ &\left| \alpha - \frac{s + tb(Q-1)}{P} \right| < \frac{\sqrt{[s+tb(Q-1)]^2 - PQ(2-Q)}}{P}; \end{aligned}$$

$$(3.4)\quad \begin{aligned} &P < 0, \\ &\left| \alpha - \frac{s + tb(Q-1)}{P} \right| > -\frac{\sqrt{[s+tb(Q-1)]^2 - PQ(2-Q)}}{P}, \end{aligned}$$

then for each $\lambda \in D$, *the problem* (2.1) *has a unique solution* $u = u(\lambda) \in H$.

PROOF. Let F be defined by (2.4) and $\lambda \in D$. Since $f, p, g, m : H \times D \to H$ are Lipschitz continuous with respect to the first argument and f is h-strongly monotone with respect to the first arguments, g and m are g-m-relaxed monotone with respect to the first argument and $g - m$ be r-strongly monotone with respect to the first argument, $N : H \times H \times D \to H$ is s-strongly monotone with respect to f in the first argument and Lipschitz continuous with respect to the first and the second arguments, respectively, by (3.1) we have

$$\begin{aligned}
&\|F(u,\lambda) - F(v,\lambda)\| \\
&\le \|u - v - ((g-m)(u,\lambda) - (g-m)(v,\lambda))\| \\
&\quad + \|J_\alpha^{M(\cdot,u,\lambda)}((g-m)(u,\lambda) - \alpha N(f(u,\lambda), p(u,\lambda), \lambda)) \\
&\quad - J_\alpha^{M(\cdot,u,\lambda)}((g-m)(v,\lambda) - \alpha N(f(v,\lambda), p(v,\lambda), \lambda))\| \\
&\quad + \|J_\alpha^{M(\cdot,u,\lambda)}((g-m)(v,\lambda) - \alpha N(f(v,\lambda), p(v,\lambda), \lambda)) \\
&\quad - J_\alpha^{M(\cdot,v,\lambda)}((g-m)(v,\lambda) - \alpha N(f(v,\lambda), p(v,\lambda), \lambda))\| \\
(3.5)\quad &\le 2\|u - v - ((g-m)(u,\lambda) - (g-m)(v,\lambda))\| \\
&\quad + \|u - v - \alpha(N(f(u,\lambda), p(u,\lambda), \lambda) - N(f(v,\lambda), p(v,\lambda), \lambda))\| \\
&\quad + k\|u - v\| \\
&\le 2\|u - v - ((g-m)(u,\lambda) - (g-m)(v,\lambda))\| \\
&\quad + \|u - v - \alpha(N(f(u,\lambda), p(u,\lambda), \lambda) - N(f(v,\lambda), p(u,\lambda), \lambda))\| \\
&\quad + \alpha\|N(f(v,\lambda), p(u,\lambda), \lambda) - N(f(v,\lambda), p(v,\lambda), \lambda)\| \\
&\quad + k\|u - v\| \\
&\le \theta\|u - v\|
\end{aligned}$$

for $(u, v, \lambda) \in H \times H \times D$, where

$$\theta = Q + \sqrt{1 - 2\alpha s + \alpha^2 l^2 a^2} + \alpha t b. \tag{3.6}$$

In light of (3.2) and one of (3.3) and (3.4) we conclude that $\theta < 1$ and (3.5) ensures that F has a unique fixed point $u = u(\lambda) \in H$. It follows from Lemma 2.1 that $u = u(\lambda) \in H$ is the unique solution of the problem (2.1). This completes the proof. □

REMARK 3.1. Theorem 3.1 is an improvement and a refinement of Lemma 3.1 in [1].

As a consequent of Theorem 3.1, we have

COROLLARY 3.1. *Let f, p, N and P be as in Theorem 3.1 and $Q = 0$. Suppose that $\varphi : H \times D \to R \bigcup \{+\infty\}$ satisfies that for each $y \in H$ and $\lambda \in D$, $\varphi(\cdot, \lambda) : M \to R \bigcup \{+\infty\}$ is a proper convex lower semicontinuous function on H. If there exists a constant $\alpha > 0$ satisfying (3.2) and one of (3.3) and (3.4), then for each $\lambda \in D$, the problem (2.3) has a unique solution $u = u(\lambda) \in H$.*

4. Sensitivity analysis

THEOREM 4.1. *Let f, g, m, p, N, M, P, Q and α be as in* Theorem 3.1 *and θ be defined by (3.6). Let f, p, g, m be continuous (resp., uniformly continuous or Lipschitz continuous) with respect to the second arguments and N be continuous (resp., uniformly continuous or Lipschitz continuous) with respect to the third argument. Suppose that there exists a constant $\beta > 0$ such that*

$$\|J_\alpha^{M(\cdot,x,\lambda)}(z) - J_\alpha^{M(\cdot,x,\bar{\lambda})}(z)\| \le \beta\|\lambda - \bar{\lambda}\|, \quad \forall x, z \in H,\ \lambda, \bar{\lambda} \in D. \tag{4.1}$$

If (3.1), (3.2) and one of (3.3) and (3.4) hold, then the solutions of the problem (2.1) are continuous (resp., uniformly continuous or Lipschitz continuous).

PROOF. Let F be defined by (2.4). It follows from Theorem 3.1 that for each $\lambda \in D$, there exists a unique $u \in H$ denoted by $u(\lambda)$ such that it is the unique solution of the problem (2.1). Hence we get that

$$u(\lambda) = F(u(\lambda), \lambda) \quad \text{and} \quad u(\bar{\lambda}) = F(u(\bar{\lambda}), \bar{\lambda}) \quad \text{for each } \lambda, \bar{\lambda} \in D.$$

It is clear that

$$\begin{aligned} &\|u(\lambda) - u(\bar{\lambda})\| \\ (4.2)\qquad &= \|F(u(\lambda),\lambda) - F(u(\bar{\lambda}),\bar{\lambda})\| \\ &\leq \|F(u(\lambda),\lambda) - F(u(\bar{\lambda}),\lambda)\| + \|F(u(\bar{\lambda}),\lambda) - F(u(\bar{\lambda}),\bar{\lambda})\| \end{aligned}$$

for $\lambda, \bar{\lambda} \in D$. In view of (3.5) we obtain that

$$(4.3)\qquad \|F(u(\lambda),\lambda) - F(u(\bar{\lambda}),\lambda)\| \leq \theta\|u(\lambda) - u(\bar{\lambda})\| \quad \text{for } \lambda, \bar{\lambda} \in D.$$

By virtue of (4.1) we deduce that

(4.4)

$$\begin{aligned} &\|F(u(\bar{\lambda}),\lambda) - F(u(\bar{\lambda}),\bar{\lambda})\| \\ &\leq \|(g-m)(u(\bar{\lambda}),\lambda) - (g-m)(u(\bar{\lambda}),\bar{\lambda})\| \\ &\quad + \|J_\alpha^{M(\cdot,u(\bar{\lambda}),\lambda)}((g-m)(u(\bar{\lambda}),\lambda) - \alpha N(f(u(\bar{\lambda}),\lambda), p(u(\bar{\lambda}),\lambda),\lambda)) \\ &\quad - J_\alpha^{M(\cdot,u(\bar{\lambda}),\lambda)}((g-m)(u(\bar{\lambda}),\bar{\lambda}) - \alpha N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\bar{\lambda}))\| \\ &\quad + \|J_\alpha^{M(\cdot,u(\bar{\lambda}),\lambda)}((g-m)(u(\bar{\lambda}),\bar{\lambda}) - \alpha N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\bar{\lambda})) \\ &\quad - J_\alpha^{M(\cdot,u(\bar{\lambda}),\bar{\lambda})}((g-m)(u(\bar{\lambda}),\bar{\lambda}) - \alpha N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\bar{\lambda}))\| \\ &\leq 2\|(g-m)(u(\bar{\lambda}),\lambda) - (g-m)(u(\bar{\lambda}),\bar{\lambda})\| \\ &\quad + \alpha\|(N(f(u(\bar{\lambda}),\lambda), p(u(\bar{\lambda}),\lambda),\lambda) - N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\lambda),\lambda))\| \\ &\quad + \alpha\|(N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\lambda),\lambda) - N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\lambda))\| \\ &\quad + \alpha\|N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\lambda) - N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\bar{\lambda})\| \\ &\quad + \beta\|\lambda - \bar{\lambda}\| \\ &\leq 2\|(g-m)(u(\bar{\lambda}),\lambda) - (g-m)(u(\bar{\lambda}),\bar{\lambda})\| \\ &\quad + \alpha l\|f(u(\bar{\lambda}),\lambda) - f(u(\bar{\lambda}),\bar{\lambda})\| + \alpha t\|p(u(\bar{\lambda}),\lambda) - p(u(\bar{\lambda}),\bar{\lambda})\| \\ &\quad + \alpha\|N(f(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\lambda) \\ &\quad - N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\bar{\lambda}))\| + \beta\|\lambda - \bar{\lambda}\| \end{aligned}$$

for $\lambda, \bar{\lambda} \in D$. Substituting (4.3) and (4.4) into (4.2) we conclude that

$$\begin{aligned} &\|u(\lambda) - u(\bar{\lambda})\| \\ &\leq \frac{1}{1-\theta}[2\|(g-m)(u(\bar{\lambda}),\lambda) - (g-m)(u(\bar{\lambda}),\bar{\lambda})\| \\ (4.5)\qquad &\quad + \alpha l\|f(u(\bar{\lambda}),\lambda) - f(u(\bar{\lambda}),\bar{\lambda})\| + \alpha t\|p(u(\bar{\lambda}),\lambda) - p(u(\bar{\lambda}),\bar{\lambda})\| \\ &\quad + \alpha\|N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\lambda) \\ &\quad - N(f(u(\bar{\lambda}),\bar{\lambda}), p(u(\bar{\lambda}),\bar{\lambda}),\bar{\lambda})\| + \beta\|\lambda - \bar{\lambda}\|] \end{aligned}$$

for $\lambda, \bar{\lambda} \in D$. Observe that f, p, g, m are continuous (resp., uniformly continuous or Lipschitz continuous) with respect to the second argument and N is continuous (resp., uniformly continuous or Lipschitz continuous) with respect to the third argument and $\theta < 1$. Therefore (4.5) guarantees that the solutions of the problem (2.1) are continuous (resp., uniformly continuous or Lipschitz continuous). This completes the proof. □

REMARK 4.1. Theorem 3.1 in [1] is a special case of Theorem 4.1.

COROLLARY 4.1. *Let f, p, N, P, Q be as in* Corollary 3.1 *and θ be defined by* (3.6). *Suppose that there exists a constant $\beta > 0$ satisfying*

$$\|J_\alpha^{\varphi(\cdot,\lambda)}(x) - J_\alpha^{\varphi(\cdot,\bar{\lambda})}(x)\| \leq \beta\|\lambda - \bar{\lambda}\|, \quad \forall \lambda, \bar{\lambda} \in D.$$

If (3.2) *and one of* (3.3) *and* (3.4) *hold, then the solutions of the problem* (2.3) *are continuous (resp., uniformly continuous or Lipschitz continuous).*

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Weili Wang
Basis Courses Teaching Department
Dalian Institute of Light Industry
Dalian, Liaoning 116034, P. R. China

Zeqing Liu
Department of Mathematics
Liaoning Normal University
P.O.Box 200, Dalian, Liaoning 116029, P. R. China
E-mail: zeqingliu@dl.cn

Shin Min Kang
Department of Mathematics and
Research Institute of Natural Science
Gyeongsang National University
Chinju 660-701, Korea
E-mail: smkang@nongae.gsnu.ac.kr