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# ON THE STABILITY OF FUNCTIONAL EQUATIONS IN $n$-VARIABLES AND ITS APPLICATIONS

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Abstract. In this paper we investigate a generalization of the Hyers-Ulam-Rassias stability for a functional equation of the form $f(\varphi(X)) = \phi(X)f(X)$, where $X$ lie in n-variables. As a consequence, we obtain a stability result in the sense of Hyers, Ulam, Rassias, and Găvruta for many other equations such as the gamma, beta, Schröder, iterative, and $G$-function type's equations.

## 1. Introduction

In 1940, the stability problem raised by S. M. Ulam [27] was solved in the case of the additive mapping by D. H. Hyers [6]. The result of Hyers has been generalized to the unbounded case by Bourgin [3] and Th. M. Rassias [18]. The result of latter also has been generalized by P. Găvruta [4], R. Ger [5], and others(see Refs.).

The stability type in which we are investigated is the sense of Găvruta for the case of the suitable function as follows: If for an approximate solution $f$ of the equation $E_1(h) = E_2(h)$, i.e., for a function $f$ such that $|E_1(f) - E_2(f)| \leq \phi$ holds with a given function $\phi$ there exists a function $g$ such that $E_1(g) = E_2(g)$ and $|g(x) - f(x)| \leq \Phi(x)$ for some fixed function $\Phi$.

The functional equation in which we are interested in this article is derived from the gamma functional equation $f(x+1) = xf(x)$, which was considered by S.-M.Jung([11, 12, 13]). This equation generalizes the gamma type functional equation $f(x+p) = \varphi(x)f(x)$ and the beta type functional equation $f(x+p, y+p) = \varphi(x,y)f(x,y)$ by author ([10, 14, 15, 16, 17]).

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In this paper, we will investigate a generalization of the Hyers-Ulam-Rassias stability in the sense of Găvruta for the functional equation

$$f(\varphi(X)) = \phi(X)f(X), \tag{1.1}$$

where $\varphi, \phi$ are given functions, while $f$ is the unknown function and $X$ depends upon n-variables. Namely, the aim of this paper is the extension to the domain of n-variables and applications to many other functional equations as gamma, G, Schrölder, iterative, and beta types functional equations.

In section 2, we study the stability in the sense of Găvruta for the functional equations (1.1).

In section 2′, we consider the special case of section 2 with $\varphi(X) = X + P$.

In section 3, our results shown in sections 2, 2′ are applied to the gamma, $G$, Schrölder, iterative and beta types functional equations and some examples suitably restricted to a domain in one or two variables.

Throughout this paper, let $B$ be a Banach space over the field $K$, where $K$ will be either the field $R$ of real numbers or the field $C$ of complex numbers. Each positive real number $\delta$ is fixed, and the constant $c > 0$. $R_+$ denotes the set of all nonnegative real numbers. Given the nonempty set $S$ and the function $\varphi : S^n \to S^n$, we put $\varphi_0(X) := X$ and $\varphi_n(X) := \varphi(\varphi_{n-1}(X))$ for all positive integers $n$ and all points $X \in S^n$. The functions $\phi : S^n \to K\backslash\{0\}$ and $\varepsilon : S^n \to R_+$ are defined.

## 2. Generalization of Hyers-Ulam-Rassias stability of Eq.(1.1)

Let $\varphi, \phi$ and $\varepsilon$ be given functions such that

$$\omega(X) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|} < \infty, \qquad \forall X \in S^n. \tag{2.1}$$

THEOREM 1. *Let the functions $\varphi, \phi, \varepsilon$ satisfy the condition* (2.1). *If a function $f : S^n \longrightarrow B$ satisfies the inequality*

$$||f(\varphi(X)) - \phi(X)f(X)|| \le \varepsilon(X) \qquad \forall X \in S^n, \tag{2.2}$$

*then there exists an unique solution $g : S^n \longrightarrow B$ of the equation* (1.1) *with*

$$||g(X) - f(X)|| \le \omega(X). \tag{2.3}$$

PROOF. For any $X \in S^n$ and for every positive integer $n$, let $\omega_n : S^n \longrightarrow R_+$ and $g_n : S^n \longrightarrow B$ be the functions defined by

$$\omega_n(X) := \sum_{k=0}^{n-1} \frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|} \quad \text{and} \quad g_n(X) := \frac{f(\varphi_n(X))}{\prod_{j=0}^{n-1} \phi(\varphi_j(X))}$$

for all $X \in S^n$, respectively.

By (2.2), it follows that

$$||\frac{f(\varphi(X))}{\phi(X)} - f(X)|| \leq \frac{\varepsilon(X)}{|\phi(X)|} \quad \text{for all} \quad X \in S^n.$$

Substituting $X$ by $\varphi_n(X)$ in this inequality, and then dividing both sides of the obtained inequality by $\prod_{j=0}^{n-1} |\phi(\varphi_j(X))|$, we get

$$||g_{n+1}(X) - g_n(X)|| = \frac{\varepsilon(\varphi_n(X))}{\prod_{j=0}^{n} |\phi(\varphi_j(X))|}. \tag{2.4}$$

By induction on $n$ we prove that

$$||g_n(X) - f(X)|| \leq \omega_n(X) \tag{2.5}$$

for all $X \in S^n$, and for all positive integers $n$. For the case $n = 1$, the inequality (2.5) is an immediate consequence of (2.2).

Assume that the inequality (2.5) holds true for some $n$. Then we obtain the inequality for $n + 1$. This is an immediate consequence of

$$\begin{aligned} ||g_{n+1}(X) - f(X)|| &\leq ||g_{n+1}(X) - g_n(X)|| + ||g_n(X) - f(X)|| \\ &\leq \frac{\varepsilon(\varphi_n(X))}{\prod_{j=0}^{n} |\phi(\varphi_j(X))|} + \omega_n(X) \\ &= \omega_{n+1}(X). \end{aligned}$$

We claim that $\{g_n(X)\}$ is a Cauchy sequence. Indeed, by (2.4) and (2.1), we have for $n > m$ that

$$\begin{aligned} ||g_n(X) - g_m(X)|| &\leq \sum_{k=m}^{n-1} ||g_{k+1}(X) - g_k(X)|| \\ &\leq \sum_{k=m}^{n-1} \frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|} \longrightarrow 0 \end{aligned}$$

as $m \longrightarrow \infty$.

Hence, we can define a function $g : S^n \longrightarrow B$ by

$$g(X) := \lim_{n\to\infty} g_n(X). \tag{2.6}$$

From the definition of $g_n$, we have $g_n(\varphi(X)) = \phi(X)g_{n+1}(X)$ and therefore the function $g$ satisfies (1.1).

We show from (2.5) that $g$ satisfies the inequality (2.3) as follows:

$$\begin{aligned}||g(X) - f(X)|| &= \lim_{n\to\infty} ||g_n(X) - f(X)|| \\ &\le \lim_{n\to\infty} \omega_n(X) \\ &= \omega(X) \qquad \forall X \in S^n.\end{aligned}$$

If $h : S^n \longrightarrow B$ is another such function, which satisfies (1.1) and (2.3), then we have

$$\begin{aligned}&||g(X) - h(X)|| \\ &= ||g(\varphi_n(X)) - h(\varphi_n(X))|| \cdot \prod_{j=0}^{n-1} \frac{1}{|\phi(\varphi_j(X))|} \\ &\le 2\omega_n(\varphi_n(X)) \cdot \prod_{j=0}^{n-1} \frac{1}{|\phi(\varphi_j(X))|} \\ &= 2\Big(\sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_{n+k}(X))}{\prod_{j=0}^{k} |\phi(\varphi_{n+j}(X))|}\Big) \cdot \prod_{j=0}^{n-1} \frac{1}{|\phi(\varphi_j(X))|} \\ &= 2\sum_{k=n}^{\infty} \frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|}\end{aligned}$$

for all $X \in S^n$ and all positive integers $n$, which tends to zero as $n \to \infty$, since $\omega(X)$ is bounded. This implies the uniqueness of $g$. $\square$

Setting $\varepsilon(X) = \delta$ in Theorem 1, we have the Hyers-Ulam stability of equation (1.1).

Let the functions $\varphi, \phi$ satisfy

$$\mu(X) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{|\phi(\varphi_j(X))|} < \infty \qquad \forall X \in S^n. \tag{2.7}$$

COROLLARY 1. *Let $\varphi, \phi$ satisfy condition* (2.7). *If a function $f : S^n \longrightarrow B$ satisfies the inequality*

$$\|f(\varphi(X)) - \phi(X)f(X)\| \le \delta \tag{2.8}$$

*for all $X \in S^n$, then there exists an unique solution $g : S^n \longrightarrow B$ of the equation* (1.1) *with*

$$\|g(X) - f(X)\| \le \delta\mu(X). \tag{2.9}$$

## 2′. Stability in the case $\varphi(X) = X + P$ of the Eq.(1.1)

As a special case of section 2, we consider that $\varphi(X) = X + P$, $S = (0, \infty)$, and $B = R$. Then, we can obtain the same results for the functional equation:

$$f(X + P) = \phi(X)f(X), \tag{1.1′}$$

where $X = (x_1, x_2, \cdots, x_n)$, $P = (p_1, p_2, \cdots, p_n) \in (0, \infty)^n$, and each positive real number $x_i$ is a variable, each positive real number $p_i$ is fixed, and $n$ is a natural number. The statement $X > 0$ means that each component $x_i$ of $X$ lies in the interval $(0, \infty)$, and the statement $X > N_0$ means that $x_i > n_0$ for each component $x_i$ of $X$ and for a fixed natural number $n_0$.

All results shown in section 2 are replaced with the analogous results in the form $\varphi(X) = X + P$.

Let the functions $\phi$ and $\varepsilon$ satisfy the inequalities

$$\omega'(X) := \sum_{k=0}^{\infty} \frac{\varepsilon(X + kP)}{\prod_{j=0}^{k} |\phi(X + jP)|} < \infty \tag{2.1′}$$

and

$$\mu'(X) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{|\phi(X + jP)|} < \infty \qquad \forall\, X > 0. \tag{2.7′}$$

THEOREM 1′. *Let $\phi, \varepsilon$ satisfy condition* (2.1′). *If a function $f : (0, \infty)^n \longrightarrow R$ satisfies the inequality*

$$|f(X + P) - \phi(X)f(X)| \le \varepsilon(X) \qquad \forall X > N_0,$$

*then there exists an unique solution* $g : (0,\infty)^n \longrightarrow R$ *of the equation* (1.1) *with*

$$|g(X) - f(X)| \le \omega'(X) \qquad \forall X > N_0.$$

PROOF. Setting $S = (0,\infty), B = R, \varphi(X) = X + P$ in Theorem 1, then the claimed result of this theorem is satisfied except for the condition that replaces $X \in S^n$ by $X > N_0$. For this, we define the new function $g_0 : (n_0,\infty)^n \longrightarrow R$ by

$$g_0(X) := \lim_{n\to\infty} g_n(X)$$

in substituting $g$ defined in (2.6) for $g_0$.

Now, we extend the function $g_0$ to the domain $(0,\infty)^n$. We define for each $0 < X \le N_0$,

$$g(X) := \frac{g_0(X+kP)}{\prod_{n=0}^{k-1} \phi(X+nP)},$$

where $k$ is the smallest natural number satisfying the inequalities $x_i + kp_i > n_0$ for each $i$.

Then, $g(X+P) = \phi(X)g(X)$ for all $X > 0$ and $g(X) = g_0(X)$ for all $X > N_0$. Also the inequality

$$|g(X) - f(X)| < \omega'(X)$$

holds for all $X > 0$. □

COROLLARY 1′. *Let* $\varphi$ *satisfies condition* (2.7′). *If a function* $f : (0,\infty)^n \longrightarrow R$ *satisfies the inequality*

$$|f(X+P) - \phi(X)f(X)| \le \delta, \qquad \forall X > N_0,$$

*then there exists an unique solution* $g : (0,\infty)^n \longrightarrow R$ *of the equation* (1.1) *with*

$$|g(X) - f(X)| \le \delta\mu'(X), \qquad \forall X > N_0.$$

## 3. Applications to the gamma type, $G$-function, Schröder, iterative and the beta type functions

The results shown in the sections 2, 2′ can be applied to the well known stability results for the gamma, $G$, beta, Schröder, iterative functional equations, and also to certain other forms. It suffices to show how to bring the equation (1.1) into the concrete forms of those functional equations.

(1) BETA TYPE FUNCTIONAL EQUATIONS

We restrict the functional equation (1.1) in the case of a double variable. Then, we can obtain the same results for the beta type functional equation, as follows :

$$f(x+p, y+q) = \phi(x,y)f(x,y) \tag{3.1}$$

$$f(x+1, y+1)^{-1} = \frac{(x+y)(x+y+1)}{xy} f(x,y)^{-1} \tag{3.2}$$

which provide some special cases of the equation (1.1).

The beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ is a solution of the beta functional equation $f(x+1, y+1) = \frac{xy}{(x+y)(x+y+1)} f(x)$, which is closely related to the gamma function $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$. The relationship between them is given by $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(y,x)$.

In the case of 2-variables, the following Corollary 2 follow from Theorems 1 or Theorems 1′ with $S = (0,\infty), B = R$.

In Theorem 1, the condition (2.1) with $\varphi(x,y) = (x+p, y+q)$ applies

$$\omega_\beta(x,y) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+kp, y+kq)}{\prod_{j=0}^{k} |\phi(x+jp, y+jq)|} < \infty \qquad \forall x, y > 0. \tag{3.3}$$

COROLLARY 2. [10] *Let $\phi, \varepsilon$ satisfy condition* (3.3). *If a function $f : (0,\infty) \times (0,\infty) \longrightarrow R$ satisfies the inequality*

$$|f(x+p, y+q) - \phi(x,y)f(x,y)| \le \varepsilon(x,y) \qquad \forall x, y > n_0,$$

*then there exists an unique solution $g : (0,\infty) \times (0,\infty) \longrightarrow R$ of the equation* (3.1) with

$$|g(x,y) - f(x,y)| \le \omega_\beta(x,y) \qquad \forall x, y > n_0.$$

The following Corollary 3 follow from the Corollary 2 with $\phi(x,y) = \frac{(x+y)(x+y+1)}{xy}$ and $p = q = 1$. The condition (3.3) is replaced by

(3.4)
$$\omega_{\beta_1}(x,y) := \sum_{k=0}^{\infty} \varepsilon(x+k, y+k) \times \prod_{j=0}^{k} \frac{(x+j)\ (y+j)}{((x+j)+(y+j))((x+j)+(y+j)+1)} < \infty$$

for all $x, y > 0$.

COROLLARY 3. [15] *Let a function* $\varepsilon$ *satisfies condition* (3.4). *If the function* $f : (0,\infty) \times (0,\infty) \longrightarrow R$ *satisfies the inequality*

$$|f(x+1,y+1)^{-1} - \frac{(x+y)(x+y+1)}{xy} f(x,y)^{-1}| \le \varepsilon(x,y) \qquad \forall x, y > n_0,$$

*then there exists an unique reciprocal of the beta functional equation* $g : (0,\infty) \times (0,\infty) \longrightarrow R$ *of the equation* (3.2) *with*

$$|g(x,y)^{-1} - f(x,y)^{-1}| \le \omega_{\beta_1}(x,y) \qquad \forall x, y > n_0.$$

(2) GAMMA AND OTHER TYPE FUNCTIONAL EQUATIONS

Let restrict the functional equation (1.1) for the case of a single variable. Then, we can consider similar results for the gamma, G. Schröder, iterative, other types functional equations as follows :

$$f(\varphi(x)) = \phi(x) f(x) \tag{3.5}$$

$$f(\varphi(x)) = x f(x) \tag{3.6}$$

$$f(\varphi(x)) = c f(x), \quad c : \text{constant} \tag{3.7}$$

$$f(\varphi(x)) = (x+c) f(x) \tag{3.8}$$

$$f(\varphi(x)) = \varphi(x)f(x) \tag{3.9}$$

$$f(\varphi(x)) = f(x)^2 \tag{3.10}$$

$$f(f(x)) = f(x)^2 \tag{3.11}$$

$$f(x+p) = \phi(x)f(x) \tag{3.12}$$

$$f(x+1) = (x+1)f(x) \tag{3.13}$$

$$f(x+1) = xf(x), \tag{3.14}$$

which are special cases of the equation (1.1) by various applying of the functions $\varphi, \phi$.

The gamma function given by $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ is a solution of the gamma functional equation (3.14), also the functional equation (3.7) is called the Schröder functional equation.

REMARK 2. The Hyers-Ulam stability and the generalized Hyers-Ulam-Rassias stability for all of the above functional equations follow immediately from Theorems 1, 2′ with $S = (0, \infty), B = R$ or $K, p = 1, \phi(x) = x, \varphi_j(x) = x + j, \varepsilon(x) = \delta$ by restricting to a single variable. Since the Hyers-Ulam stability and the generalized Hyers-Ulam-Rassias stability of the equations (3.12), (3.13), (3.14) are studied in the papers ([1, 10, 11, 12, 13, 14, 17, 26]).

The equation (3.6) can be considered as the generalized form of Schröder functional equation (3.7). In the case $c > 1$, Trif proved the Hyers-Ulam stability of the equation (3.7).

Restricting the condition (2.1) to a single variable, we get for all x ∈ S

$$\omega'(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\phi(\varphi_j(x))|} < \infty \tag{2.1′}$$

$$\omega_g(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\varphi_j(x)|} < \infty \tag{3.15}$$

$$\omega_c(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{c^{k+1}} < \infty \qquad c: \text{positive constant.} \tag{3.16}$$

$$\omega_\Gamma(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+kp)}{\prod_{j=0}^{k} |\phi(x+jp)|} < \infty. \tag{3.17}$$

COROLLARY 6. [26] *Let $\varphi, \phi, \varepsilon$ satisfy condition (2.1′). If a function $f : S \longrightarrow B$ satisfies the inequality*

$$||f(\varphi(x)) - \phi(x)f(x)|| \le \varepsilon(x),$$

*then there exists an unique solution $g : S \longrightarrow B$ of the functional equation (3.5) with*

$$||g(x) - f(x)|| \le \omega'(x).$$

COROLLARY 7. *Let a function $f : S \longrightarrow B$ satisfies the inequality*

$$||f(\varphi(x)) - \phi(x)f(x)|| \le \delta,$$

*where the functions $\varphi, \phi$ satisfy $\mu'_\delta(x) := \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k} |\phi(\varphi_j(x))|} < \infty$ for all x in S.*

*Then, there exists an unique solution $g : S \longrightarrow B$ of the functional equation (3.5) with*

$$||g(x) - f(x)|| \le \delta\mu'_\delta(x).$$

COROLLARY 8. *Let $\varphi, \varepsilon$ satisfy condition (3.15). If a function $f : S \longrightarrow B$ satisfies the inequality*

$$||f(\varphi(x)) - xf(x)|| \le \varepsilon(x),$$

*then there exists an unique solution $g : S \longrightarrow B$ of the equation (3.6) with*

$$||g(x) - f(x)|| \le \omega_g(x).$$

COROLLARY 9. *Let $\varphi$ satisfies condition $||\varphi_j(x)|| > ||\varphi(x)|| > 1$ for all $j$. If a function $f : S \longrightarrow B$ satisfies the inequality*

$$||f(\varphi(x)) - xf(x)|| \leq \delta,$$

*then there exists an unique solution $g : S \longrightarrow B$ of the equation* (3.6) *with*

$$||g(x) - f(x)|| \leq \frac{\delta||\varphi(x)||}{x(||\varphi(x)|| - 1)}.$$

*In particular, if $\varphi$ satisfies the inequality $||\varphi_j(x)|| > ||x|| > 1$, then there exists an unique solution $g : S \longrightarrow B$ of the generalized Schröder functional equation* (3.6) *satisfying*

$$||g(x) - f(x)|| \leq \frac{\delta}{||x|| - 1}.$$

COROLLARY 10. *Let $\varphi, \varepsilon$ satisfy condition* (3.16). *If a function $f : S \longrightarrow B$ satisfies the inequality*

$$||f(\varphi(x)) - cf(x)|| \leq \varepsilon(x),$$

*then there exists the unique Schröder function $g : S \longrightarrow B$ satisfying* (3.7) *with*

$$||g(x) - f(x)|| \leq \omega_c(x).$$

COROLLARY 11. [26] *Let $c > 1$. If a function $f : S \longrightarrow B$ satisfies the inequality*

$$||f(\varphi(x)) - cf(x)|| \leq \delta,$$

*then there exists the unique Schröder function $g : S \longrightarrow B$ satisfying* (3.7) *with*

$$||g(x) - f(x)|| \leq \frac{\delta}{c - 1}.$$

COROLLARY 12. *If a function $f : S \longrightarrow B$ satisfies the inequality*

$$||f(\varphi(x)) - (c + x)f(x)|| \leq \varepsilon(x),$$

*where the functions $\varphi$ satisfy $\mu_c(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\varphi_j(x)+c|} < \infty$ for all x in S.*

*then there exists the unique function $g : S \longrightarrow B$ of the equation* (3.8) *with*

$$||g(x) - f(x)|| \leq \mu_c(x).$$

The following corollary will be using in (3) $G$-function and (4) Examples.

COROLLARY 13. [14] *Let $\varphi, \phi, \varepsilon$ satisfy condition* (3.17). *If a function $f : S \longrightarrow B$ satisfies the inequality*

$$||f(x+p) - \phi(x)f(x)|| \leq \varepsilon(x),$$

*then there exists an unique solution $g : S \longrightarrow B$ of the functional equation* (3.12) *with*

$$||g(x) - f(x)|| \leq \omega_\Gamma(x).$$

The condition (3.15) with $\varphi_j(x) = x + j$ can be represented by

$$\omega_{\gamma_1}(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+k)}{\prod_{j=0}^{k} |x + (j+1)|} < \infty, \qquad \forall x > 0. \tag{3.18}$$

and

$$\omega_{\gamma}(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+k)}{\prod_{j=0}^{k} |x+j|} < \infty, \qquad \forall x > 0. \tag{3.19}$$

COROLLARY 14. *Let $\varepsilon$ satisfy condition* (3.18). *If a function $f : (0, \infty) \longrightarrow K$ satisfies the inequality*

$$|f(x+1) - (x+1)f(x)| \leq \varepsilon(x),$$

*then there exists an unique solution $g : (0, \infty) \longrightarrow K$ of the equation* (3.13) *with*

$$|g(x) - f(x)| \leq \omega_{\gamma_1}(x).$$

The following Corollary 15 is the Hyers-Ulam-Rassias stability of the gamma functional equation (3.14), and if we takes $\varepsilon(x) = \delta$, then the approximate difference of the Hyers-Ulam stability is $\frac{e\delta}{x}$.

COROLLARY 15. [10, 12, 13, 14] *Let* $\varepsilon$ *satisfy condition* (3.19). *If a function* $f : (0, \infty) \longrightarrow K$ *satisfies the inequality*

$$|f(x+1) - xf(x)| \leq \varepsilon(x),$$

*then there exists an unique solution* $g : (0, \infty) \longrightarrow K$ *of the gamma functional equation* (3.14) *with*

$$|g(x) - f(x)| \leq \omega_\gamma(x).$$

COROLLARY 16. *If a function* $f : S \longrightarrow B$ *satisfies the inequality*

$$||f(\varphi(x)) - \varphi(x)f(x)|| \leq \varepsilon(x),$$

*where the functions* $\varphi$ *satisfy* $\omega_\varphi(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\varphi_{j+1}(x)|} < \infty$ *for all x in* $S$, *then there exists the unique function* $g : S \longrightarrow B$ *of the equation* (3.9) *with*

$$||g(x) - f(x)|| \leq \omega_\varphi(x).$$

COROLLARY 17. *If a function* $f : S \longrightarrow B$ *satisfies the inequality*

$$||f(\varphi(x)) - f(x)^2|| \leq \varepsilon(x),$$

*where the functions* $\varphi$ *satisfy* $\omega_f(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |f(\varphi_j(x))|} < \infty$ *for all x in* $S$, *then there exists an unique function* $g : S \longrightarrow B$ *of the equation* (3.10) *with*

$$||g(x) - f(x)|| \leq \omega_f(x).$$

Let consider the same case $f = \varphi = \phi$. Namely $f$ has the iterative properties such as $f^n(x) := f(f^{n-1}(x))$ and $f^0(x) := x$.

COROLLARY 18. *If a function* $f : B \longrightarrow B$ *satisfies the inequality*

$$||f^2(x) - f(x)^2|| \leq \varepsilon(x),$$

*where the functions* $\varphi$ *satisfy* $\omega_{it}(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(f^k(x))}{\prod_{j=0}^{k} |f^{j+1}(x))|} < \infty$ *for all x in S, then there exists an unique function* $g : S \longrightarrow B$ *of the equation* (3.11) *with*

$$||g(x) - f(x)|| \leq \omega_{it}(x).$$

(3) $G$-FUNCTIONAL EQUATION

The $G$-function introduced by E.W. Barnes [2]

$$G(z) = (2\pi)^{\frac{z-1}{2}} e^{-\frac{z(z-1)}{2}} e^{-\gamma \frac{(z-1)^2}{2}} \prod_{k=1}^{\infty} \left[\left(1 + \frac{z-1}{k}\right)^k e^{1-z+\frac{(z-1)^2}{2k}}\right]$$

does satisfy the equation $G(x+1) = \Gamma(x)G(x)$ and $\Gamma(1) = G(1) = 1$, where $\gamma$ is the Euler-Mascheroni's constant defined by $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n\right) \cong 0.577215664\cdots$.

The properties and values of $G$-function depend on those of the gamma function. Since the double gamma function $\Gamma_2$ is defined by the reciprocal of the $G$-function(see [2]), $\Gamma_2(x) = 1/G(x)$, and its functional equation is $\Gamma_2(x+1) = \Gamma_2(x)/\Gamma(x)$. Therefore the stability problem for the $G$-function is equivalent to the stability for the reciprocal of the double gamma function.

Putting $\phi(x) = \Gamma(x)$ and $p = 1$ in equation (3.12), we obtain

$$f(x+p) = \Gamma(x)f(x), \tag{3.20}$$

$$f(x+1) = \Gamma(x)f(x). \tag{3.21}$$

Since G-function satisfies the equation (3.21), which is called the $G$-functional equation.

The condition (2.1′) in a single variable is represented by

$$\omega_{G_p}(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+kp)}{\prod_{j=0}^{k} |\Gamma(x+jp)|} < \infty \tag{3.22}$$

$$\omega_G(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+k)}{\prod_{j=0}^{k} |\Gamma(x+j)|} < \infty. \tag{3.23}$$

COROLLARY 19. *Let the functions $\Gamma, \varepsilon$ satisfy condition* (3.22). *If a function $f : (0,\infty) \longrightarrow R$ satisfies the inequality*

$$|f(x+p) - \Gamma(x)f(x)| \le \varepsilon(x) \qquad \forall x > n_0,$$

*then there exists an unique solution $g : (0,\infty) \longrightarrow R$ of the equation* (3.20) *with*

$$|g(x) - f(x)| \le \omega_{G_p}(x) \qquad \forall x > n_0.$$

COROLLARY 20. [16] *Let the functions $\Gamma, \varepsilon$ satisfy condition* (3.23). *If a function $f : (0,\infty) \longrightarrow R$ satisfies the inequality*

$$|f(x+1) - \Gamma(x)f(x)| \le \varepsilon(x) \qquad \forall x > n_0,$$

*then there exists an unique G-function $g : (0,\infty) \longrightarrow R$ satisfying the equation* (3.21) *with*

$$|g(x) - f(x)| \le \omega_G(x) \qquad \forall x > n_0.$$

REMARK. The Hyers-Ulam stability of all equations (3.1), (3.2), (3.5)~(3.14), (3.20), (3.21) in Section 3 follows immediately from each Corollaries with $\varepsilon(x) = \delta$.

(4) EXAMPLES

The results of Sections 2, 3 may be applied to the following examples.

EXAMPLE 1. Set $\phi(X) = x_1 * x_2 * \cdots * x_n$ in Corollary 1′, where $*$ is an operation on the set $S$. If a function $f : (0,\infty)^n \longrightarrow \mathbb{R}$ satisfies the inequality

$$|f(X+P) - (x_1 * x_2 * \cdots * x_n)f(X)| \le \delta \qquad \forall X > N_0,$$

then there exists an unique solution $g : (0,\infty)^n \longrightarrow \mathbb{R}$ of the equation $f(X+P) = (x_1 * x_2 * \cdots * x_n)f(X)$ with

$$|g(X) - f(X)| \le \begin{cases} \omega_*(x) \text{ if } \omega_*(x) = \sum_{k=0}^{\infty}\prod_{j=0}^{k} \frac{\delta}{|(x_1+jp_1)*\cdots*(x_n+jp_n)|} < \infty \\ \frac{\delta}{x_1 * x_2 * \cdots * x_n - 1} \text{ if } x_1 * \cdots * x_n > 1 \\ \frac{\delta e}{x_1 * x_2 * \cdots * x_n} \text{ if } p_1 * p_2 * \cdots * p_n > 1 \\ \frac{\delta}{x_1 \cdots x_n}\sum_{k=0}^{\infty}\frac{1}{(k!)^n} \text{ if } P = (1,\cdots,1),\ x_1 * \cdots * x_n = x_1 \cdots x_n \\ \frac{\delta}{x_1 + \cdots + x_n}\sum_{k=0}^{\infty}\frac{1}{(k! n^k)} \text{ if } P = (1,\cdots,1), \\ \qquad x_1 * \cdots * x_n = x_1 + \cdots + x_n. \end{cases}$$

From Corollary 13, we have the following Ex 2 as a corollary, the other examples is derived from it.

EXAMPLE 2.(COROLLARY). If a function $f : (0, \infty) \longrightarrow \mathbb{R}$ satisfies the inequality

$$|f(x+1) - \phi(x)f(x)| \leq \delta \qquad \forall x \geq n_0,$$

where $\phi$ is a function such that

$$\mu_g(x) := \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k} |\phi(x+j)|} < \infty, \tag{3.24}$$

then there exists an unique solution $g : (0, \infty) \longrightarrow R$ of the equation (3.12) with

$$|g(x) - f(x)| \leq \delta\mu_g(x).$$

The definition of the function $\phi$ in the following examples satisfies the condition (3.24). Thus the functional equation $f(x+1) = \phi(x)f(x)$ in following each case has the Hyers-Ulam stability.

EXAMPLE 3. $\phi(x) = c > 1$, where $c$ is constant.

EXAMPLE 4. $\phi(x) = (1 + \frac{1}{x})^x$. Note that $\lim_{x\to\infty}(1 + \frac{1}{x})^x = e > 1$.

EXAMPLE 5. $\phi(x) = x^n$, for $x > 1$, $n \in \mathbb{N}$.

EXAMPLE 6. $\phi(x) = \arctan(x)$, since $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$.

EXAMPLE 7. $\phi(x) = \arcsin(x)$ for $x > 1$, since $\lim_{x\to\infty} \arcsin(x) = \frac{\pi}{2}$.

Similarly, we can also consider $\sinh(x), \cosh(x), \log(x)$ with a suitable domain for each function.

EXAMPLE 8. In Hyers-Ulam stability of Corollary 12, putting $\varphi(x) = x+1$, and $c, x > 0$, we obtain the approximate difference $|f(x) - g(x)| \leq \frac{e\delta}{|x+c|}$ for the functional equation $f(x+1) = (c+x)f(x)$.

## References

[1] H. Alzer, *Remark on the stability of the Gamma functional equation*, Results Math. **35** (1999), 199–200.

[2] E. W. Barnes, *The theory of the G-function*, Quart. J. Math. **31** (1899), 264–314.

[3] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[4] P. Găvruta, *A Generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[5] R. Ger, *Superstability is not natural*, Rocznik Nauk.-Dydakt. Prace Mat. **159** (1993), 109–123.

[6] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[7] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of the Functional Equations in Several Variables*, Birkhäuser Verlag, 1998.

[8] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math. **44** (1992), 125–153.

[9] G. Isac and Th. M. Rassias, *On the Hyers-Ulam stability of $\psi$-additive mappings*, J. Approx. Theory **72** (1993), 131–137.

[10] K. W. Jun, G. H. Kim, Y. W. Lee, *Stability of generalized Gamma and Beta functional equations*, Aequationes Math. **60** (2000), 15–24.

[11] S.-M. Jung, *On the general Hyers-Ulam stability of gamma functional equation*, Bull. Korean Math. Soc. **34** (1997), 437–446.

[12] ———, *On the modified Hyers-Ulam-Rassias stability of the functional equation for gamma function*, Mathematica **39** (1997), 233–237.

[13] ———, *On the stability of gamma functional equation*, Results Math. **33** (1998), 306–309.

[14] G. H. Kim, *On the stability of generalized Gamma functional equation*, Internat. J. Math. & Math. Sci. **23** (2000), 513–520.

[15] ———, *A generalization of the Hyers-Ulam-Rassias stability of the Beta functional equation*, Publ. Math. Debrecen **59** (2001), 111–119.

[16] ———, *A generalization of Hyers-Ulam-Rassias stability of a G-functional equation*, Math. Inequal. Appl., to appear.

[17] G. H. Kim, Y.W. Lee, *The stability of the beta functional equation*, Studia Univ. "Babes-Bolyai", Mathematica **40** (2000), 89–96.

[18] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[19] ———, *On the modified Hyers-Ulam sequence*, J. Math. Anal. Appl. **158** (1991), 106–113.

[20] ———, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.

[21] ———, *On the stability of functional equations in Banach space*, J. Math. Anal. Appl. **251** (2000), 264–284.

[22] Th. M. Rassias (ed.), *Functional Equations and Inequalities*, Kluwer Academic Publishers, Dordrecht, 2000.

[23] ———, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 2003.

[24] Th. M. Rassias and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.

[25] Th. M. Rassias and J. Tabor, *Stability of Mappings of Hyers-Ulam Type*, Hadronic Press Inc., Florida, 1994.

[26] T. Trip, *On the stability of a general gamma-type functional equation*, Publ. Math. Debrecen **60** (2002), 47–62.

[27] S. M. Ulam, *Problems in Modern Mathematics, Chap. VI*, Science editions, Wiley, New York, 1964.

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