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THREE-TERM CONTIGUOUS FUNCTIONAL RELATIONS FOR BASIC HYPERGEOMETRIC SERIES ${}_2\phi_1$

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Abstract. The authors aim mainly at giving fifteen three-term contiguous relations for the basic hypergeometric series ${}_2\phi_1$ corresponding to Gauss's contiguous relations for the hypergeometric series ${}_2F_1$ given in Rainville([6], p.71). They also apply them to obtain two summation formulas closely related to a known q-analogue of Kummer's theorem.

1. Introduction and preliminaries

It is customary to define the *basic hypergeometric series* (or *q-hyperge ometric series*) by

$$
\begin{aligned}
\phi(a,\, b;\, c;\, q,\, z) \equiv {}_2\phi_1(a,\, b,\, c;\, q,\, z) &\equiv {}_2\phi_1\left[\begin{matrix} a,\, b \\ c \end{matrix};\, q,\, z\right] \\
&= \sum_{n=0}^{\infty} \frac{(a;q)_n\,(b;q)_n}{(q;q)_n\,(c;q)_n}\, z^n \quad (|z|<1;\ |q|<1),
\end{aligned} \tag{1.1}
$$

where

$$
(a;q)_n := \begin{cases} 1, & n=0, \\ \prod_{m=0}^{n-1}(1-aq^m), & n \in \mathbb{N} := \{1,\, 2,\, 3,\, \ldots\}, \end{cases} \tag{1.2}
$$

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is the *q-shifted factorial* and it is assumed that $c \neq q^{-m}$ $(m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$. When $n = \infty$ in (1.2),

$$(a;q)_\infty = \prod_{m=0}^{\infty} (1 - aq^m). \tag{1.3}$$

If we simply denote

$$\phi = {}_2\phi_1 \left[\begin{matrix} a,\, b \\ c \end{matrix} ;\, q,\, z \right] \quad \text{and} \quad \phi_q = {}_2\phi_1 \left[\begin{matrix} aq,\, bq \\ cq \end{matrix} ;\, q,\, z \right],$$

then we express the transforms of ϕ as follows:

$$\phi(aq) = {}_2\phi_1 \left[\begin{matrix} aq,\, b \\ c \end{matrix} ;\, q,\, z \right] \quad \text{and} \quad \phi\left(\frac{a}{q}\right) = {}_2\phi_1 \left[\begin{matrix} \frac{a}{q},\, b \\ c \end{matrix} ;\, q,\, z \right].$$

Similarly, $\phi(bq)$, $\phi(\frac{b}{q})$, $\phi(cq)$, and $\phi(\frac{c}{q})$ are expressed.

Heine [4] derived the following three-term relations:

$$\phi(aq) - \phi = \frac{a\,(1-b)}{1-c} z\phi_q. \tag{1.4}$$

$$\phi\left(\frac{c}{q}\right) - \phi = \frac{c(1-a)(1-b)}{(q-c)(1-c)} z\phi_q. \tag{1.5}$$

$$\phi(aq) - \phi\left(\frac{c}{q}\right) = \frac{(1-b)(aq-c)}{(q-c)(1-c)} z\phi_q. \tag{1.6}$$

$$\phi(aq) - \phi(bq) = \frac{a-b}{1-c} z\phi_q, \tag{1.7}$$

which can, upon replacing a and b by $\frac{a}{q}$ and $\frac{b}{q}$ respectively, be written as follows

$$\phi\left(\frac{b}{q}\right) - \phi\left(\frac{a}{q}\right) = \frac{a-b}{q(1-c)} z\phi(cq). \tag{1.8}$$

$$\phi\left(aq, \frac{b}{q}, c\right) - \phi = \frac{az(1-b/aq)}{1-c}\phi(aq, b, cq). \tag{1.9}$$

$$b(1-a)\phi(aq) - a(1-b)\phi(bq) = (b-a)\phi. \tag{1.10}$$

$$a\left(1-\frac{b}{c}\right)\phi\left(\frac{b}{q}\right) - b\left(1-\frac{a}{c}\right)\phi\left(\frac{a}{q}\right) = (a-b)\left(1-\frac{abz}{cq}\right)\phi. \tag{1.11}$$

$$\begin{aligned}&\left[1+q-a-\frac{aq}{c}+\frac{a^2}{c}\left(1-\frac{b}{a}\right)z\right]\phi \\ &= q\left(1-\frac{a}{c}\right)\phi\left(\frac{a}{q}\right) + (1-a)\left(1-\frac{ab}{c}z\right)\phi(aq).\end{aligned} \tag{1.12}$$

$$\begin{aligned}&(c-1)[c(q-c)+(ca+cb-ab-abq)z]\phi \\ &= (1-c)(q-c)(abz-c)\phi\left(\frac{c}{q}\right)) + (c-a)(c-b)z\phi(cq).\end{aligned} \tag{1.13}$$

Bailey [2] and Daum [3] independently discovered the summation formula

$$_2\phi_1\left[\begin{matrix} a, b \\ \frac{aq}{b} \end{matrix}; q, -\frac{q}{b}\right] = \frac{(-q\,;q)_\infty\,(aq;q^2)_\infty\left(\frac{aq^2}{b^2}\,;q^2\right)_\infty}{\left(\frac{aq}{b}\,;q\right)_\infty\left(-\frac{q}{b}\,;q\right)_\infty}, \tag{1.14}$$

which is a q-analogue of Kummer's theorem

$$_2F_1(a, b;\, 1+a-b;\, -1) = \frac{\Gamma(1+a-b)\,\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma(1+a)\,\Gamma\left(1+\frac{1}{2}a-b\right)}. \tag{1.15}$$

In 1973, Andrews [1] gave a very simple proof of the Bailey-Daum summation formula (1.14).

Gauss defined as *contiguous* to the hypergeometric function $F(a, b; c; z)$ each of the six function obtained by increasing or decreasing one of the parameters by unity. For simplicity in printing, we use the notations

$$\begin{aligned} F &= {}_2F_1(a, b; c; z), \\ F(a+) &= F(a+1, b; c; z), \\ F(a-) &= F(a-1, b; c; z), \end{aligned}$$

together with similar notations $F(b+)$, $F(b-)$, $F(c+)$, and $F(c-)$ for the other four of the six functions contiguous to F.

The main objective of this paper is to give fifteen three-term contiguous relations for the basic hypergeometric series ${}_2\phi_1$ corresponding to Gauss's contiguous relations for the hypergeometric series ${}_2F_1$ given in Rainville [6, p.71]:

$$(a-b)F = aF(a+) - bF(b+), \tag{1.16}$$

$$(a-c+1)F = aF(a+) - (c-1)F(c-), \tag{1.17}$$

$$[a+(b-c)z]F = a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+), \tag{1.18}$$

$$(1-z)F = F(a-) - c^{-1}(c-b)zF(c+), \tag{1.19}$$

$$(1-z)F = F(b-) - c^{-1}(c-a)zF(c+), \tag{1.20}$$

$$[2a-c+(b-a)z]F = a(1-z)F(a+) - (c-a)F(a+), \tag{1.21}$$

$$(a+b-c)F = a(1-z)F(a+) - (c-b)F(b-), \tag{1.22}$$

$$(c-a-b)F = (c-a)F(a-) - b(1-z)F(b+), \tag{1.23}$$

$$(b-a)(1-z)F = (c-a)F(a-) - (c-b)F(b-), \tag{1.24}$$

$$[1-a+(c-b-1)z]F = (c-a)F(a-) - (c-1)(1-z)F(c-), \tag{1.25}$$

$$[2b-c+(a-b)z]F = b(1-z)F(b+) - (c-b)F(b+), \tag{1.26}$$

$$[b+(a-c)z]F = b(1-z)F(b+) - c^{-1}(c-a)(c-b)zF(c+), \tag{1.27}$$

$$(b-c+1)F = bF(b+) - (c-1)F(c-), \tag{1.28}$$

$$[1-b+(c-a-1)z]F = (c-b)F(b-) - (c-1)(1-z)F(c-), \tag{1.29}$$

$$[c-1+(a+b+1-2c)z]F = (c-1)(1-z)F(c-) - c^{-1}(c-a)(c-b)zF(c+). \tag{1.30}$$

We also apply them to obtain two summation formulas closely related to q-analogue of Kummer's theorem (1.15).

2. The three - term ${}_2\phi_1$ relations

We are ready to obtain three-term relations for ${}_2\phi_1$ by employing the identities given in Section 1 or the Heine's transformation formula for ${}_2\phi_1$:

$$(2.1) \qquad {}_2\phi_1\left[\begin{matrix} a,\, b \\ c \end{matrix}\, ;\, q,\, z\right] = \frac{(\frac{ab}{c}z\, ;\, q)_\infty}{(z\, ;\, q)_\infty}\, {}_2\phi_1\left[\begin{matrix} \frac{c}{a},\, \frac{c}{b} \\ c \end{matrix}\, ;\, q,\, \frac{ab}{c}z\right].$$

The fifteen three-term relations for ${}_2\phi_1$ are here listed in the same order as the Gauss's contiguous relations for ${}_2F_1$ (1.16)–(1.30):

$$(2.2) \qquad (a-b)\,\phi = a(1-b)\phi(bq) - b(1-a)\phi(aq),$$

$$(2.3) \qquad (aq-c)\phi = a(q-c)\phi\left(\frac{c}{q}\right) - c(1-a)\phi(aq),$$

$$(2.4) \qquad \begin{aligned} [c^2(1-a) - a^2(b-c)z]\phi &= c(1-a)(c-abz)\phi(aq) \\ &\quad - a(1-c)^{-1}(a-c)(b-c)z\phi(cq), \end{aligned}$$

$$(2.5) \qquad (abz-cq)(1-c)\phi = az(b-c)\phi(cq) - qc(1-c)\phi\left(\frac{a}{q}\right),$$

$$(2.6) \qquad (abz-cq)(1-c)\phi = bz(a-c)\phi(cq) - qc(1-c)\phi\left(\frac{b}{q}\right),$$

$$(2.7) \qquad \begin{aligned} &\left[1+q-a-\frac{aq}{c}+\frac{a^2}{c}(1-\frac{b}{a})z\right]\phi \\ &= q\left(1-\frac{a}{c}\right)\phi\left(\frac{a}{q}\right) + (1-a)\left(1-\frac{ab}{c}z\right)\phi(aq), \end{aligned}$$

$$(2.8) \qquad \begin{aligned} &\left[b(1-a)+aq\left(1-\frac{b}{c}\right)\right]\phi \\ &= aq(1-\frac{b}{c})\phi\left(\frac{b}{q}\right) + b(1-a)\left(1-\frac{ab}{c}z\right)\phi(aq), \end{aligned}$$

$$\begin{aligned}&\left[a(1-b)+bq\left(1-\frac{a}{c}\right)\right]\phi\\&=bq(1-\frac{a}{c})\phi\left(\frac{a}{q}\right)+a(1-b)\left(1-\frac{ab}{c}z\right)\phi(bq),\end{aligned}\tag{2.9}$$

$$(a-b)\left(1-\frac{ab}{cq}z\right)\phi=a\left(1-\frac{b}{c}\right)\phi\left(\frac{b}{q}\right)-b\left(1-\frac{a}{c}\right)\phi\left(\frac{a}{q}\right),\tag{2.10}$$

$$\begin{aligned}&\left[c(q-a)+a^2\left(1-\frac{bq}{c}z\right)\right]\phi\\&=a(q-c)\left(1-\frac{ab}{c}z\right)\phi\left(\frac{c}{q}\right)+q(c-a)\,\phi\left(\frac{a}{q}\right),\end{aligned}\tag{2.11}$$

$$\begin{aligned}&\left[1+q-b-\frac{bq}{c}+\frac{b^2}{c}\left(1-\frac{a}{b}\right)z\right]\phi\\&=q\left(1-\frac{b}{c}\right)\phi\left(\frac{b}{q}\right)+(1-b)\left(1-\frac{ab}{c}z\right)\phi(bq),\end{aligned}\tag{2.12}$$

$$\begin{aligned}&[c^2(1-b)(1-c)-b^2(1-c)(a-c)z]\phi\\&=c(1-b)(1-c)(c-abz)\phi(bq)-b(b-c)(a-c)z\phi(cq),\end{aligned}\tag{2.13}$$

$$(bq-c)\phi=b(q-c)\phi\left(\frac{c}{q}\right)-c(1-b)\phi(bq),\tag{2.14}$$

$$\begin{aligned}&\left[c(q-b)+b^2\left(1-\frac{aq}{c}z\right)\right]\phi\\&=b(q-c)\left(1-\frac{ab}{c}z\right)\phi\left(\frac{c}{q}\right)+q(c-b)\,\phi\left(\frac{b}{q}\right),\end{aligned}\tag{2.15}$$

$$\begin{aligned}&(c-1)[c(q-c)+(ca+cb-ab-abq)z]\,\phi\\&=(1-c)(q-c)(abz-c)\phi\left(\frac{c}{q}\right)+(c-a)(c-b)z\,\phi(cq).\end{aligned}\tag{2.16}$$

PROOF OF (2.2). Multiplying both sides of (1.4) and (1.7) by $a-b$ and $a(1-b)$, respectively, and subtracting one resulting equation from the other one leads immediately to (2.2).

Similarly, the other remaining fourteen formulas (2.4)–(2.16) may be proved.

REMARK. Taking the limits of both sides of (2.2)–(2.16) as $q \to 1$ yields Gauss's contiguous relations (1.16)–(1.30), respectively.

3. Applications

In this section we will show how three-term contiguous relations for ${}_2\phi_1$ in Section 2 can be applied to obtain certain summation formulas for ${}_2\phi_1$. For example, if we replace c by cq in (2.3), we obtain

$$c(1-a)\phi(aq,cq)+(a-c)\phi(cq)=a(1-c)\phi, \tag{3.1}$$

which, upon dividing each side by $a-c$, gives

$$\phi(cq)=\frac{a(1-c)}{a-c}\phi-\frac{c(1-a)}{a-c}\phi(aq,cq). \tag{3.2}$$

Substituting $\frac{aq}{b}$ for c in (3.2), after a little simplification, we get

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix}a,b\\ \frac{aq^2}{b}\end{matrix};q,-\frac{q}{b}\right]&=\frac{1-\frac{aq}{b}}{1-\frac{q}{b}}\,{}_2\phi_1\left[\begin{matrix}a,b\\ \frac{aq}{b}\end{matrix};q,-\frac{q}{b}\right]\\ &\quad-\frac{\frac{q}{b}(1-a)}{1-\frac{q}{b}}\,{}_2\phi_1\left[\begin{matrix}aq,b\\ \frac{aq^2}{b}\end{matrix};q,-\frac{q}{b}\right].\end{aligned} \tag{3.3}$$

Now, if we apply the Bailey-Daum summation formula (1.14) to the two ${}_2\phi_1$ on the right-hand side of (3.3), we get the following interesting identity:

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix}a,b\\ \frac{aq^2}{b}\end{matrix};q,-\frac{q}{b}\right]&=\frac{(\frac{q^2}{b};q)_\infty(-q;q)_\infty}{(\frac{q}{b};q)_\infty(\frac{aq^2}{b};q)_\infty(-\frac{q}{b};q)_\infty}\\ &\quad\times\left[(aq;q^2)_\infty\left(\frac{aq^2}{b^2};q^2\right)_\infty-\frac{q}{b}(a;q^2)_\infty\left(\frac{aq^3}{b^2};q^2\right)_\infty\right].\end{aligned} \tag{3.4}$$

For another example, if we replace z by $-\frac{q}{b}$ in (2.5) and substitute $\frac{a}{b}$ for c in the resulting identity, we get

$$\begin{aligned}{}_2\phi_1\left[\begin{matrix}a,b\\ \frac{a}{b}\end{matrix};q,-\frac{q}{b}\right]&=\frac{1}{1+b}\,{}_2\phi_1\left[\begin{matrix}\frac{a}{q},b\\ \frac{a}{b}\end{matrix};q,-\frac{q}{b}\right]\\ &\quad+\frac{b^2-a}{(1+b)(b-a)}\,{}_2\phi_1\left[\begin{matrix}a,b\\ \frac{aq}{b}\end{matrix};q,-\frac{q}{b}\right].\end{aligned} \tag{3.5}$$

The application of the Bailey-Daum summation formula (1.14) to the two ${}_2\phi_1$ on the right-hand side of (3.5), after a little simplification, yields another interesting identity:

(3.6)

$$
{}_2\phi_1\left[\begin{matrix} a, b \\ \frac{a}{b} \end{matrix} ; q, -\frac{q}{b}\right] = \frac{(-q;q)_\infty}{(\frac{a}{b};q)_\infty(-\frac{1}{b};q)_\infty} \times \left[\frac{(a;q^2)_\infty\,(\frac{aq}{b^2};q^2)_\infty}{b} + \left(aq;q^2\right)_\infty \left(\frac{a}{b^2};q^2\right)_\infty\right].
$$

It is seen that the identities (3.4) and (3.6) are closely related to the Bailey-Daum summation formula (1.14) which is a q-analogue of Kummer's theorem (1.15). Similarly other identities may also be derived.

We conclude this paper by remarking that the identities (3.5) and (3.6) have been obtained by Kim and Rathie [7] in a different way.

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