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# Parametric Sensitivity Analysis Using Fourier Transformation
# 푸리에 변환을 이용한 파라미터 민감도 해석

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**Key Words** : Fourier transformation(푸리에 변환), Frequency domain(주파수 영역), Sensitivity analysis(민감도 해석), Sensitivity derivative(민감도 도함수), Logarithmic sensitivity function(지수 민감도 함수), Sensitivity matrix(민감도 행렬)

**Abstract** : 주파수 영역 민감도 해석법은 동적 시스템의 전달함수에 대한 설계 파라미터의 변화에 의한 효과를 파악하기 위해 사용되어 왔으며, 이때의 민감도 함수는 시스템 설계 파라미터에 대한 시스템 전달 함수의 편미분 값이다. 일반적으로 종래의 주파수 영역 민감도 해석은 직접 미분법이나 라플라스 변환이 사용되어 왔다. 라플라스 변환을 사용하는 경우에 시스템의 차수가 증가할수록 역행렬 조작은 매우 많은 시간을 필요로 하며 또한 어려운 작업이다. 본논문에서는 이러한 다점을 보완하기 위하여 푸리에 변환을 이용한 민감도 기법을 제시하였다. 파라미터의 변화에 대한 진폭-주파수 특성의 민감도 해석을 간단한 2자유도 모델과 로터 다이나믹 시스템에 적용하였다.

## 1. 서 론

Recently, increasing concern has shown to the sensitivity analysis for the redesign and modification of a mechanical system. A theoretical study on the sensitivity analysis was first executed by mathematicians. In its early times, the sensitivity analysis was used to investigate the effect of parameter variations on the solution of differential equations. Adelman and Haftka[1] reviews methods for calculating the derivatives of static displacements and stresses, eigenvalues and eigenvectors, transient structural response of structural systems, and derivatives of optimum structural designs with respect to problem parameters. Much less is known about the application of sensitivity methods to the analysis and synthesis of mechanical systems. The sensitivity derivatives of vehicle dynamic system with respect to design variables and system parameters were examined by Nalecz[2),3)] and Nalecz and Wicher[4)]. More recently, sensitivity methods have been applied to the yaw rate analysis for a front wheel steering vehicle by Jang and Han[5), 6)].

The performance of a mechanical system design is strongly affected by the accuracy of the mathematical model and the influence of the design parameters. The sensitivity analysis in frequency domain has been used to find the influence of parameter changes on the transfer function of a system. The sensitivity analysis provides information on which parameter variations in the model have the largest potential effect on the system's dynamic characteristics.

Especially, sensitivity analysis appears to be very useful in analysis and synthesis of complex mechanical systems. The past work on sensitivity analysis in the frequency domain was used the direct method (analytical method) or the Laplace transformation. In general, however, for the complex structural systems such as those modeled by finite elements, the explicit analytical relationships between the matrix elements and design parameters may not be available and then the sensitivity matrix should be computed by

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using numerical methods. In the case of the Laplace transformation, matrix inverse operation is very time-consuming and difficult when the order of system is increased. To overcome these disadvantages, the sensitivity method by using Fourier transformation is proposed in this work.

The logarithmic sensitivity analysis in frequency domain is particularly useful to the designer in case of the sensitivity function has the normalizing coefficients of the transfer function in amplitude-frequency characteristics.

This paper presents a procedure that the designer can use to determine whether or not the design parameters are adequate or what additional modifications of the design parameters are required. To confirm the validity of the procedure, the simulation results of a two degree of freedom dynamic system and a rotor system are shown.

The C++ language has been used for the implementation of computer code for the integration of equations of motion and estimation of the sensitivity derivatives.

## 2. Sensitivity Analysis in Frequency Domain

In the case of a degree of freedom for linear dynamic system, the state equation takes the form with initial condition

$$\dot{x} = Ax + Bu,\ x(t_0) = x^0 \tag{1}$$

where x is the state vector $(2n \times 1)$, u is the input vector $(2m \times 1)$

$$x = x(p, t) \tag{2}$$

$$u = u(t) \tag{3}$$

and p is the $r$ dimensional parameter vector

In general, matrix A is a $(2n \times 2n)$ state matrix and B is a $(2n \times 2m)$ input matrix as follows:

$$A = A(p) \tag{4}$$

$$B = B(p) \tag{5}$$

To determine the influence of parameter vectors on system responses, the parameter vector p shown above represents the actual parameter vector

$$p = p_0 + \Delta p \tag{6}$$

where $p_0$ is the nominal parameter vector, and $\Delta p$ represents small parameter deviations.

Taking the Fourier transformations of $k$-th element of x, and $l$-th element of u

$$X^k(f, T) = \int_0^T x^k(t) e^{-j2\pi ft} dt \tag{7}$$

$$U^l(f, T) = \int_0^T u^l(t) e^{-j2\pi ft} dt \tag{8}$$

From Eqs. (7) and (8), the auto-power spectrums of $x^k$ and are $u^l$ are

$$S_{xx}^k(f) = \lim_{T\to\infty} E[|X^k(f, T)|^2], \tag{9}$$

$$S_{uu}^l(f) = \lim_{T\to\infty} E[|U^l(f, T)|^2] \tag{10}$$

and the cross power spectrum $x^k$ and $u^l$ are is

$$S_{ux}^{lk}(f) = \lim_{T\to\infty} E[U^{l*}(f, T) X^k(f, T)] \tag{11}$$

where the asterisk (*) denotes complex conjugate, so that $X^{k*}$ and $U^{l*}$is the complex conjugate of $X^k$ and $U^l$, respectively. These $S(f)$ quantities are two-sided spectral density functions where $-\infty < f < \infty$.

The Fourier transformations of $x^k$ and $u^l$ over a length $T$ are related by

$$X^k(f, T) = H^{lk}(f) U^l(f, T) \tag{12}$$

where $H^{lk}(f)$ is the transfer function and it follows that

$$U^{l*}(f, T) X^k(f, T) = H^{lk}(f) U^{l*}(f, T) U^l(f, T) \tag{13}$$

$$S_{ux}^{lk}(f) = H^{lk}(f) S_{uu}^l(f) \tag{14}$$

Hence, the transfer function becomes

$$H^{lk}(f) = \frac{S^{lk}_{ux}(f)}{S^{l}_{uu}(f)} \tag{15}$$

The real and imaginary part of complex transfer function $H^{lk}(f)$ is defined

$$H^{lk}(f) = C^{lk}(f) + jD^{lk}(f) \tag{16}$$

where, $j = (-1)^{1/2}$.

Taking the partial derivative of Eq. (1) with respect to a parameter vector p, we obtain the sensitivity equations with initial conditions

$$\dot{\mathrm{w}} = \frac{\partial \dot{\mathrm{x}}}{\partial \mathrm{p}} = \mathrm{A}_0 \frac{\partial \mathrm{x}}{\partial \mathrm{p}} + \frac{\partial \mathrm{A}}{\partial \mathrm{p}} \mathrm{x}_0 = \frac{\partial \mathrm{B}}{\partial \mathrm{p}} \mathrm{u} \ ,$$

$$\mathrm{w}^0 = \frac{\partial \mathrm{x}^0}{\partial \mathrm{p}} = 0 \tag{17}$$

where $\mathrm{A}_0 = \mathrm{A}(\mathrm{p}_0)$, $\mathrm{x}_0 = \mathrm{x}(t, \mathrm{p}_0)$ and the partial derivatives $\partial \mathrm{A}/\partial \mathrm{p}$ and $\partial \mathrm{B}/\partial \mathrm{p}$ are taken at the nominal parameter value $\mathrm{p}_0$. And $\mathrm{w} = \partial \mathrm{x}/\partial \mathrm{p}$ is known as the sensitivity equation in time domain. Eq. (17) is called first order sensitivity function in time domain[3].

The component of sensitivity equation for parameter $p_j$ and state variable $x_k$ takes the form

$$w^k_j = \frac{\partial x_k}{\partial p_j} \tag{18}$$

From Eq. (18), taking the Fourier transformation of $w^k_j$ we have

$$W^k_j(f, T) = \int_0^T w^k_j(t) e^{-j2\pi f t} dt \tag{19}$$

and the auto power spectrum of $w^k_j$ is

$$S^k_{ww,j}(f) = \lim_{T \to \infty} \frac{1}{T} E[|W^k_j(f, T)|^2] \tag{20}$$

and the cross power spectrum of $w^k_j$ and $u^l$ is

$$S^{lk}_{uw,j}(f) = \lim_{T \to \infty} \frac{1}{T} E[|U^{l*}(f, T) W^k_j(f, T)] \tag{21}$$

The transfer function of $H^{lk}_j(f)$ is defined as

$$W^k_j(f, T) = H^{lk}_j(f) U^l(f, T) \tag{22}$$

It follows that

$$U^{l*}(f, T) W^k_j(f, T) = H^{lk}_j(f) U^{l*}(f, T) U^l(f, T) \tag{23}$$

$$S^{lk}_{uw,j}(f) = H^{lk}_j(f) S^l_{uu}(f) \tag{24}$$

It is now possible to obtain expressions for the derivatives of $k, l$-th elements of the transfer function $H^{lk}(f)$:

$$H^{lk}_j(f) = \frac{S^{lk}_{uw,j}(f)}{S^l_{uu}(f)} \tag{25}$$

The real and imaginary part of the derivative of the complex transfer function $H^{lk}_j(f)$ is represented as

$$H^{lk}_j(f) = c^{lk}_j(f) + jd^{lk}_j(f) \tag{26}$$

Logarithmic sensitivity functions, which possess normalizing coefficients, are useful when sensitivity analysis is carried out in the frequency domain. From Eq. (26), the logarithmic sensitivity function is defined as

$$S^{lk}_j = \frac{\partial \ln H^{lk}}{\partial \ln p_j} = \frac{\partial H^{lk}}{\partial p_j} \frac{p_j}{H^{lk}} = H^{lk}_j \left( \frac{p_j}{H^{lk}} \right) \tag{27}$$

For some dynamic systems it is beneficial to provide sensitivity analysis of amplitude-frequency characteristics instead of transfer functions. In such case the amplitude-frequency characteristic for $k, l$-th transfer function[2] is represented as

$$S^{lk}_j|_{|H^{lk}|} = \mathrm{Re}\ \{S^{lk}_j\} \tag{28}$$

where

$$|H^{lk}| = \sqrt{[\mathrm{Re}\ \{H^{lk}\}]^2 + [\mathrm{Im}\ \{H^{lk}\}]^2} \tag{29}$$

In the special case that we have only experimental measurement of input-output data without exact mathematical model, the procedure from Eq. (7) to Eq. (27) can be applied for sensitivity analysis.

## 3. Numerical Analysis

The developed procedure has been applied to determine the sensitivity of the amplitude-frequency characteristic to changes of selected parameters of a two degree of freedom for viscously damped mechanical system and a rotor dynamic system on linear bearings.

From Eq. (18), the sensitivity equation was calculated by utilizing the approximate method which employs the 5 point central difference method. This expression is shown below:

$$w_j^k = \frac{-x_k(p_j+2\Delta p_j)+8x_k(p_j+\Delta p_j)}{12\Delta p_j} \frac{-8x_k(p_j-\Delta p_j)+x_k(p_j-2\Delta p_j)}{12\Delta p_j} \quad (30)$$

where $\Delta p_j$ is a small variation of the $j$-th parameter. The size of $\Delta p_j$ must be small enough to ensure a linear variation in response, and large enough to avoid round-off errors[7]. In this work, a variation of $\Delta p_j = p_j \times 10^{-2}$ was used, which implies one percent variation of nominal value of the design parameters. Each term on the right-hand side of Eq. (30) is evaluated by the numerical integrations, and 5 integrations of the equations of motion during each time step are required in 5 point central difference method.

In the process of Fourier transformation, proper choice of windowing method is very important[8], since the accuracy of sensitivity functions is influenced by the windowing. There were occasional significant differences in sensitivity values when different windowing methods were used. If one is interested in accurate quantitative values of sensitivity functions, the window methods should be carefully selected. Here, a conventional Hanning tapering procedure is used to compute the spectral estimates for sidelobe suppression.

### 3.1 Viscously damped mechanical system

The first example presented here is intended to illustrate the procedure for sensitivity analysis proposed in this paper. The mechanical model assumed for sensitivity analysis is a two degree-of-freedom vehicle model developed by Nalecz and Wicher[4]. They utilized this model to investigate the effects of a variety of design parameters on the simple vehicle model. The viscously damped mechanical system with excitation is shown in Fig. 1. The state space form of equations of motion is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{c_1}{m_1} & \frac{k_1}{m_1} & \frac{c_1}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_1}{m_2} & -\frac{k_1+k_2}{m_2} & -\frac{c_1+c_2}{m_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{c_2}{m_2} & \frac{k_2}{m_2}\end{bmatrix}\begin{bmatrix}\dot{u}\\ u\end{bmatrix} \quad (31)$$

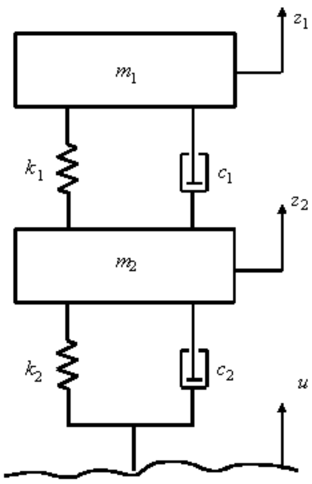


**Fig. 1 Model of a two degree-of-freedom for viscously damped mechanical system**

The sensitivity functions are determined for a chosen parameter vector

$$\mathbf{p} = (m_1,\ m_2, c_1, c_2, k_1, k_2) \quad (32)$$

The model and the values of parameters are the same as assumed in the sample shown in

Nalecz and Wicher[4)], where the model of a simplified vehicle suspension has been examined. The first resonance frequency and the second resonance frequency of this model are about 7 rad/s and 68 rad/s, respectively. The computation has been carried out for the frequency range 0-80 rad/s, which include the first and second resonances of the system considered.

The first order logarithmic sensitivity functions computed as the first order logarithmic derivatives of the amplitude-frequency characteristics of mass $m_1$ with respect to parameters $m_1$, $m_2$, $c_1$, $c_2$, $k_1$ and $k_2$ have been plotted in Fig. 2. The numerical sensitivity results and exact values of sensitivity results for parameter  are given in Table 1. From the results in Table 1, it is concluded that all three methods of design sensitivity analysis work pretty well. Both sensitivity methods with Laplace transformation and with Fourier transformation are reliable, but sensitivity methods imposing Fourier transformation is more accurate.

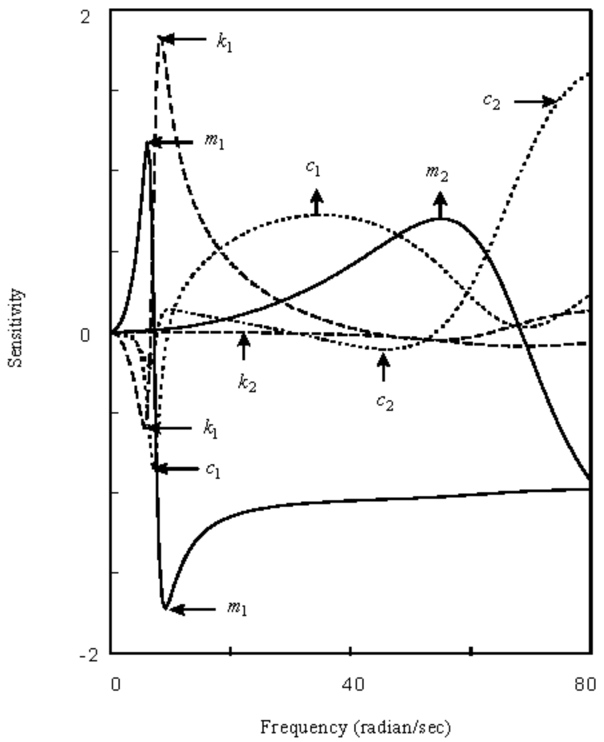


Fig. 2 Sensitivity functions with respect to mplitude-frequency characteristics for mass $m_1$

Table 1 Sensitivity results for parameter $m_1$

| Frequency (rad/s) | Sensitivity Value | | |
|---|---|---|---|
| | Direct Differentiation | Finite Difference Method | |
| | | with Laplace Transformation | with Fourier Transformation |
| 1 | 0.0197 | 0.0195 | 0.0196 |
| 5 | 0.7513 | 0.7773 | 0.7516 |
| 10 | -1.6449 | -1.7423 | -1.6447 |
| 20 | -1.1467 | -1.1493 | -1.1467 |
| 30 | -1.0730 | -1.0713 | -1.0730 |
| 40 | -1.0484 | -1.0456 | -1.0484 |
| 50 | -1.0332 | -1.0301 | -1.0332 |
| 60 | -1.0123 | -1.0089 | -1.0122 |
| 70 | -0.9879 | -0.9860 | -0.9879 |
| 80 | -0.9808 | -0.9827 | -0.9809 |

Analysis of the first order sensitivity functions indicates that the largest effect of changes in parameters $m_1$ and $k_1$ on the vibration amplitude of mass $m_1$ occurs in the neighborhood of the first resonance frequency. Mass $m_2$ and stiffness $c_2$ has the largest influence on the vibration amplitude of $m_1$ near the second resonance frequency. In the frequency range between the first and second resonances, the sensitivity functions for $m_1$ and $c_1$ possess the highest values. For the frequencies above the second resonance, the parameters $m_1$, $m_2$ and $c_2$ have the large influence on the sensitivity curve. The parameter change of $k_2$ does not exhibit any influences on the vibration amplitude of mass $m_1$ in the whole frequency range.

### 3.2 Rotor dynamic system on linear bearings

A turbine rotor can be modeled by way of a three-mass $m_1$, $m_2$ and $m_3$ connected with a massless shaft of diameter $d$ as shown in Fig. 3[9)]. The two end masses represent the journals, which are supported by linear bearings having stiffness $k_1$ and $k_2$, linear viscous damping constants $c_1$ and $c_2$, The parameter $m$ is an unbalance mass on the mass $m_2$ at a radial distance $r$ from the center of the shaft and $\omega_s$ is the speed of rotation. The horizontal and vertical vibration are not coupled. The state space form of equations of motion is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{m_1}(\frac{k}{4}+k_1) & -\frac{c_1}{m_1} & \frac{k}{2m_1} & 0 & -\frac{k}{4m_1} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k}{2m_2} & 0 & \frac{-k}{m_2} & 0 & \frac{k}{2m_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ mr\omega_s^2 \\ 0 \\ 0 \end{bmatrix} e^{j\omega_s t} \tag{33}$$

where the vertical spring constant of the shaft itself with respect to a rigid bearing is

$$k = \frac{48EI}{L^3} \tag{34}$$

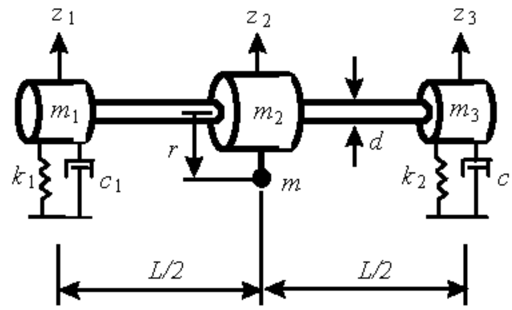

**Fig. 3** Model of a rotor dynamic system on linear bearings

The sensitivity functions are determined for a chosen parameter vector

$$\mathbf{p} = (m_1, m_2, m_3, m, c_1, c_2, k_1, k_2, k) \tag{35}$$

The model and the values of parameters are the same as assumed in the sample shown in Dimarogonas and Haddad[9]. The first resonance frequency, the second resonance frequency and the third resonance frequency of this model are about 10.5 rad/s, 18.2 rad/s, and 28.6 rad/s, respectively. The computation has been carried out for the frequency range 0-80 rad/s, which include the first, second and third resonances of the system considered.

The sensitivity results for parameter $k$ are given in Table 2. The sensitivity results obtained here has a good agreement with the results of direct differentiation method. But, in this case, sensitivity methods imposing Laplace transformation is more accurate.

**Table 2** Sensitivity results for parameter $k$

| Frequency (rad/s) | Sensitivity Value | | |
|---|---|---|---|
| | Direct Differentiation | Finite Difference Method | |
| | | with Laplace Transformation | with Fourier Transformation |
| 1 | -0.0001 | -0.0001 | 0 |
| 5 | -0.0023 | -0.0023 | -0.0022 |
| 10 | -0.0092 | -0.0092 | -0.0092 |
| 20 | -0.0391 | -0.0391 | -0.0391 |
| 30 | -0.0640 | -0.0640 | -0.0640 |
| 40 | -0.2103 | -0.2103 | -0.2104 |
| 50 | -0.4613 | -0.4613 | -0.4612 |
| 60 | -1.4825 | -1.4825 | -1.4830 |
| 70 | 1.8907 | 1.8907 | 1.8835 |
| 80 | 0.6405 | 0.6405 | 0.6387 |

The amplitude-frequency characteristics, together with the logarithmic sensitivity functions of the components of the parameter vector Eq. (35) with respect to the amplitude of the displacement of mass $m_1$ have been computed as shown in Fig. 4.

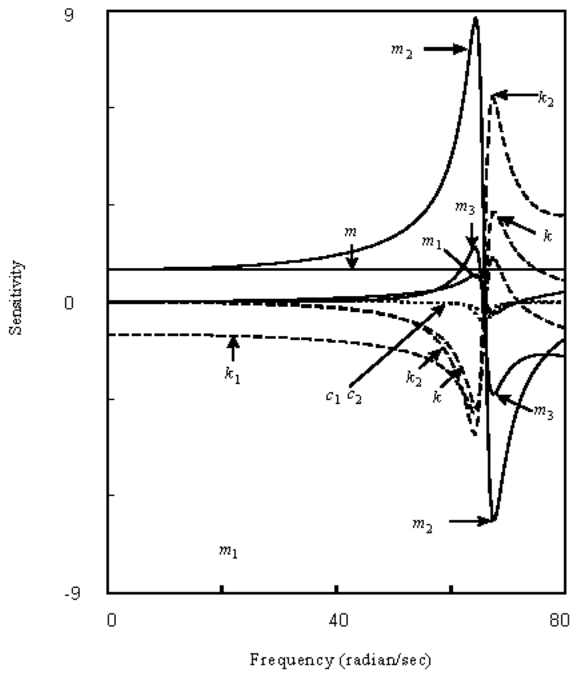

**Fig. 4** Sensitivity functions with respect to amplitude-frequency characteristics for mass $m_1$

The parameter changes of $m_1$, $m_2$, $m_3$, $c_1$, $c_2$, $k_2$ and $k$ do not exhibit any influence on the vibration amplitude of mass $m_1$ in the whole resonant range(rad/s). In the frequency range between the first and third resonances the sensitivity function of stiffness $k_1$ has a negative value which indicates that an increase of stiffness $k_1$ decreases the vibration amplitude. The stiffness $k_2$ and the mass $m_2$ have significant influences for $\omega > 60$ rad/s. In the whole frequency range, the result of the sensitivity functions for $m_1$, $c_1$ and $c_2$ are nearly zero which indicates that these parameters have little or no influences on the vibration amplitude of mass $m_1$.

## 5. Conclusions

Sensitivity analysis is a very useful method for the study of the system characteristics by the parameter variation. In view of the factors determining the human response to vibrations, the sensitivity analysis in frequency domain is more important. In this work, the sensitivity analysis in frequency domain is proposed by using Fourier transformation. Also, we have used and evaluated the sensitivity algorithms of both Laplace transformation and Fourier transformation. The examples of a two degree-of-freedom viscously damped mechanical system and a rotor dynamic system on linear bearings have illustrated the usefullness of the proposed sensitivity method. It is concluded that the sensitivity method with Fourier transformation is accurate and reliable.

Some numerical experience is worthy of discussion for the accuracy and the efficiency of sensitivity analysis with Fourier transformation. If the number of frequency interval is large, it will affect the accuracy of the sensitivity values. However, too small the number of frequency grids results in more computational time. Selection of the frequency interval is an art. First of all, the frequency interval for the sensitivity analysis should be selected as large as possible to obtain the accurate solution.

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