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SKEW POLYNOMIAL RINGS OVER σ-QUASI-BAER AND σ-PRINCIPALLY QUASI-BAER RINGS

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ABSTRACT. Let R be a ring R and σ be an endomorphism of R. R is called *σ-rigid* (resp. *reduced*) if $a\sigma(a) = 0$ (resp. $a^2 = 0$) for any $a \in R$ implies $a = 0$. An ideal I of R is called a *σ-ideal* if $\sigma(I) \subseteq I$. R is called *σ-quasi-Baer* (resp. right (or left) *σ-p.q.-Baer*) if the right annihilator of every σ-ideal (resp. right (or left) principal σ-ideal) of R is generated by an idempotent of R. In this paper, a skew polynomial ring $A = R[x;\sigma]$ of a ring R is investigated as follows: For a σ-rigid ring R, (1) R is σ-quasi-Baer if and only if A is quasi-Baer if and only if A is $\bar{\sigma}$-quasi-Baer for every extended endomorphism $\bar{\sigma}$ on A of σ; (2) R is right σ-p.q.-Baer if and only if R is σ-p.q.-Baer if and only if A is right p.q.-Baer if and only if A is p.q.-Baer if and only if A is $\bar{\sigma}$-p.q.-Baer if and only if A is right $\bar{\sigma}$-p.q.-Baer for every extended endomorphism $\bar{\sigma}$ on A of σ.

1. Introduction and some definitions

Throughout this paper, R will denote an associative ring with identity, σ will be an endomorphism of R, and A will be the skew polynomial ring $R[x;\sigma]$, i.e., A is a ring of polynomials over R in an indeterminate x with multiplication subject to the relation $xr = \sigma(r)x$ for all $r \in R$. When σ is identity 1, we write $R[x]$ for $R[x;1]$. In [11] Kaplansky introduced the Baer rings (i.e., rings in which the right annihilator of every nonempty subset is generated (as a right ideal) by an idempotent) to abstract various properties of rings of operators on Hilbert spaces. In [8], Clark introduced the quasi-Baer rings (i.e., rings in which the right annihilator of every right ideal is generated (as a right ideal) by an

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idempotent) which are generalizations of Baer rings and used them to characterize a finite dimensional twisted matrix units semigroup algebra over an algebraically closed field. Further works on quasi-Baer rings appear in [12], [3], [4] and [5]. The study of Baer and quasi-Baer rings has its roots in functional analysis. Recently, in [6] Birkenmeier, Kim and Park defined a *right* (or *left*) *principally quasi-Baer* (simply, called *right* (or *left*) *p.q.-Baer*) ring as a generalization of quasi-Baer ring by the rings in which the right (or left) annihilator of every right (or left) principal ideal of R is generated by an idempotent of R. R is called a *p.q.-Baer* ring if it is both right p.q.-Baer and left p.q.-Baer. Another generalization of Baer ring is a p.p.-ring. A ring R is called a *right* (resp. *left*) *p.p.*-ring if the right (resp. left) annihilator of any element of R is generated by an idempotent of R. R is called a *p.p.*-ring if it is both right and left p.p.-ring.

A subset S of a ring R is called a *σ-set* if S is a σ-stable set, i.e., $\sigma(S) \subseteq S$. In particular, if a singleton set $S = \{a\}$ of R is σ-set, i.e., $\sigma(a) = a$, then a is called a σ-element of R. A *left* (*right*, *two-sided*) ideal I of R is called a left (right, two-sided) *σ-ideal* if I is a σ-set. By analog, we can define a *σ-Baer* ring (resp. *σ-quasi-Baer*-ring) by the ring in which the right annihilator of every σ-set (resp. σ-ideal) is generated by an idempotent. We also define a *right* (or *left*) *σ-p.q.-Baer* ring (resp. *right* (or *left*) *σ-p.p.*-ring) by the ring in which the right (or left) annihilator of every right (or left) principal σ-ideal (resp. σ-element) is generated by an idempotent. R is called a *σ-p.q.-Baer* ring (resp. *σ-p.p.*-ring) if it is both right σ-p.q.-Baer (resp. right σ-p.p.) and left σ-p.q.-Baer (resp. left σ-p.p.). In this paper, we denote the right (resp. left) annihilator of a subset S of a ring R by $r_R(S) = \{a \in R \mid Sa = 0\}$ (resp. $l_R(S) = \{a \in R \mid aS = 0\}$). We recall that R is a *σ-rigid* (resp. *reduced*) ring if for some endomorphism σ of R, $a\sigma(a) = 0$ (resp. $a^2 = 0$) implies that $a = 0$ for each $a \in R$. We can note that any σ-rigid ring is reduced and this endomorphism σ is a monomorphism. Now we can observe the following implications: Baer (resp. quasi-Baer) $\Rightarrow$ σ-Baer (resp. σ-quasi-Baer); right (or left) p.q.-Baer (resp. right (or left) p.p.) $\Rightarrow$ right (or left) σ-p.q.-Baer (resp. right (or left) σ-p.p.); σ-Baer $\Rightarrow$ σ-quasi-Baer $\Rightarrow$ σ-p.q.-Baer. All the implications are strict by the following examples;

EXAMPLE 1. [9, Example 9] Let Z be the ring of integers and consider the ring $Z \oplus Z$ with the usual addition and multiplication. Then the subring $R = \{(a, b) \in Z \oplus Z \mid a \equiv b (\text{mod } 2)\}$ of $Z \oplus Z$ is a commutative reduced ring which has only two idempotents (0, 0) and (1, 1). Observe that R is not p.p. (and then R is not Baer). Indeed, for $a = (2,0) \in R$, $r_R(a) = (0) \oplus 2Z$ which is not generated by an idempotent of R. Since R is reduced, R is not p.q.-Baer and hence it is not quasi-Baer. Let $\sigma : R \to R$ be a map defined by $\sigma((a,b)) = (b,a)$ for all $(a,b) \in R$. Then σ is an endomorphism of R. Note that all the σ-sets of R are $S \oplus S$ for some subset S of Z. Let $T = S \oplus S$. If $T = (0)$, then $r_R(T) = R = (1,1)R$. If $T \neq (0)$, then $r_R(T) = (0) = (0,0)R$. Hence R is σ-Baer, and so R is σ-quasi-Baer, σ-p.q.-Baer and σ-p.p.

EXAMPLE 2. Let Z be the ring of integers. Let $R = \begin{pmatrix} Z & Z \\ 0 & Z \end{pmatrix}$ be the upper 2×2 triangular matrix ring over Z. Since Z is quasi-Baer, R is quasi-Baer by [12, Proposition 9]. But it is neither left p.p. nor right p.p. by [7, Example 8.1] and hence it is not p.p.. Consider an endomorphism $\sigma : R \to R$ given by

$$\sigma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & c \end{pmatrix} \text{ for all } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R.$$

We claim that R is σ-p.p. but it is not σ-Baer. First, note that every σ-element of R is of the form

$$\alpha = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.$$

Let $\beta = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in r_R(\alpha)$ be arbitrary. Then $\alpha\beta = \begin{pmatrix} ax & ay \\ 0 & cz \end{pmatrix} = 0$.

Consider the following four cases;

(i) If a and $c \neq 0$, then $x = y = z = 0$. Thus $r_R(\alpha) = (0)$, which is generated by idempotent 0 of R.

(ii) If $a \neq 0$ and $c = 0$, then $x = y = 0$ and z is arbitrary. Thus

$$r_R(\alpha) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} \in R \right\} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} R,$$

i.e., it is generated by an idempotent $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ of R.

(iii) If $a = 0$ and $c \neq 0$, then x, y are arbitrary and $z = 0$. Thus

$$r_R(\alpha) = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \in R \right\} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} R,$$

i.e., it is generated by an idempotent $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ of R.

(iv) If a and $c = 0$, then x, y and z are arbitrary. Thus $r_R(\alpha) = R$, which is generated by idempotent 1 of R. Hence R is a right σ-p.p. ring. Similarly, we can show that R is a left σ-p.p. ring.

Consequently, R is a σ-p.p. ring.

EXAMPLE 3. [6, Example 1.3] Let Z_2 be the field of two elements and consider $R = \{(x_n) \in \prod_{i=1}^{\infty} Z_2 \mid x_n \text{ is eventually constant}\}$. Then R is a Boolean ring which is not self-injective. By [12, p.79, p.249 and p.250], R is not Baer and hence it is not quasi-Baer since R is reduced. But R is p.q.-Baer and hence it is p.p. since R is reduced.

(1) Let $\sigma_1 : R \to R$ be defined by $\sigma_1((x_1, x_2, \dots)) = (x_2, x_3, \dots)$. Then σ_1 is an endomorphism of R. Note that the σ_1-ideals of R are only R and (0). Hence R is σ_1-quasi-Baer.

(2) Let $\sigma_2 : R \to R$ be defined by $\sigma_1((x_1, x_2, x_3, \dots)) = (0, x_2, x_3, \dots)$. Then σ_1 is an endomorphism of R. Note that every ideal of R is a σ_2-ideal of R. Hence R is not σ_2-quasi-Baer. But R is σ_2-p.q.-Baer.

(3) Let $\sigma_3 : R \to R$ be defined by $\sigma_3((x_1, x_2, x_3, \dots)) = (x_2, x_1, x_3, \dots)$ and consider a projection $\pi : R \to R$ given by $\pi((x_1, x_2, \dots)) = (x_3, x_4, \dots)$. Then σ_3 is an endomorphism of R. Note that every ideal of R is not always σ_3-ideal of R, for example, $(0) \times Z_2 \times \pi(I)$ is an ideal of R for some ideal I of R but it is not σ_3-ideal of R. On the other hand, for any ideal I of R, $J = Z_2 \times Z_2 \times \pi(I)$ and $K = (0) \times (0) \times \pi(I)$ are σ_3-ideals of R. Then $r_R(J) = (0) \times (0) \times r_R(\pi(I))$ and $r_R(K) = Z_2 \times Z_2 \times r_R(\pi(I))$. Since R is not quasi-Baer, $\pi(R)$ is not quasi-Baer and so R is not σ_3-quasi-Baer. But R is σ_3-p.q.-Baer.

We begin with the following lemmas;

LEMMA 1.1. *Let R be a ring with an endomorphism σ. Then*

(1) *If I is a right σ-ideal of R, then RI is a right σ-ideal of R;*
(2) *If I is a left σ-ideal of R, then IR is a left σ-ideal of R.*

Proof. (1) Let I be a right σ-ideal of R. Clearly, RI is a right ideal of R. Let $t \in RI$ be arbitrary. Then $t = \sum_{i=1}^{n} a_i b_i$ for some $a_i \in R$, $b_i \in I$ and some integer $n \in Z^+$. Since I is a right σ-ideal of R, $\sigma(I) \subseteq I$. For each i, $\sigma(a_i b_i) = \sigma(a_i)\sigma(b_i) \in RI$, and so $\sigma(RI) \subseteq RI$. Hence RI is a right σ-ideal of R.

(2) It follows from the similar argument given as in (1). □

LEMMA 1.2. *Let R be a ring with an endomorphism σ. Then R is σ-quasi-Baer if and only if the right annihilator of every right σ-ideal of R is generated by an idempotent.*

Proof. For any right σ-ideal I of R, RI is a σ-ideal of R and $r_R(I) = r_R(RI)$ since R has an identity. □

LEMMA 1.3. *Let R be a σ-rigid ring. Then R is σ-Baer if and only if R is σ-quasi-Baer.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose that R is σ-quasi-Baer. Let S be any σ-set of R. Consider the right ideal $< S >$ of R generated by S. Since S is a σ-set of R, $< S >$ is a right σ-ideal of R. Since R is σ-quasi-Baer, $r_R(< S >) = eR$ for some idempotent $e \in R$ by Lemma 1.2. We will show that $r_R(S) = r_R(< S >)$. Clearly, $r_R(< S >) \subseteq r_R(S)$. Let $b = \sum_{i=1}^{n} s_i x_i \in < S >$ be arbitrary. If $a \in r_R(S)$, then $s_i a = 0$ for all $s_i \in S$. Since R is reduced, $s_i a = 0$ if and only if $a s_i = 0$ if and only if $s_i R a = 0$. Then $0 = \sum_{i=1}^{n} (a s_i) x_i = \sum_{i=1}^{n} (s_i x_i) a = ba$, and so $a \in r_R(< S >)$. Thus $r_R(S) = r_R(< S >) = eR$. Hence R is σ-Baer. □

COROLLARY 1.4. *Let R be a reduced ring. Then R is Baer if and only if R is quasi-Baer.*

Proof. It follows from Lemma 1.3 by letting $\sigma = 1$. □

LEMMA 1.5. *Let R be a σ-rigid ring. Then the following statements are equivalent:*

(1) *R is a right σ-p.p.-ring;*
(2) *R is a σ-p.p.-ring;*
(3) *R is a right σ-p.q.-Baer ring;*
(4) *R is a σ-p.q.-Baer ring;*
(5) *For any σ-element $a \in R$ and any positive integer n, $r_R(a^n R) = eR$ for some idempotent $e \in R$.*

Proof. Since R is σ-rigid, $r_R(a) = l_R(a) = r_R(aR) = l_R(Ra) = r_R(a^n R)$ for any σ-element $a \in R$ and any positive integer n. Hence we have the result. □

In [1], Armendariz has shown that if R is reduced, then R is a Baer ring if and only if the polynomial ring $R[x]$ is a Baer ring. In this paper, we will generalize the result by showing that if R is σ-rigid, then R is a σ-quasi-Baer ring if and only if the skew polynomial ring $R[x;\sigma]$ is a quasi-Baer ring; R is a right (or left) σ-p.q.-Baer ring if and only if the skew polynomial ring $R[x;\sigma]$ is a right (or left) p.q.-Baer ring.

LEMMA 1.6. *Let R be a σ-rigid ring. Then for all a, b, c, and $d \in R$,*

(1) *$a\sigma(b) = 0$ if and only if $\sigma(b)a = 0$;*
(2) *If $ab = 0$ and $bc + da = 0$, then $bc = da = 0$;*
(3) *If $ab = 0$ and $ad + cb = 0$, then $ad = cb = 0$;*
(4) *If $ab = 0$, then $a\sigma(b) = \sigma(a)b = 0$;*
(5) *If $a\sigma^k(b) = 0$ for some positive integer k, then $ab = 0$.*

Proof. (1) is clear.

(2) If $ab = 0$ and $bc+da = 0$, then $0 = (bc+da)b = (bc)b+(da)b = bcb$, and so $bc = 0$. Hence $da = 0$.

(3) It is similar to the proof of (2).

(4) Suppose that $ab = 0$. Since R is reduced, $ba = 0$. Thus

$$a\sigma(b)\sigma(a\sigma(b)) = a\sigma(ba)\sigma^2(b) = 0.$$

Since R is σ-rigid, $a\sigma(b) = 0$. Similarly, if $ab = 0$, then $\sigma(a)b = 0$.

(5) If $a\sigma^k(b) = 0$ for some positive integer k, then by using (4) repeatedly we have $\sigma^k(ab) = \sigma^k(a)\sigma^k(b) = 0$, and so $ab = 0$ because σ is a monomorphism. □

For a ring R with an endomorphism σ, there exists an endomorphism of $A = R[x;\sigma]$ which extends σ. For example, consider a map $\bar{\sigma}$ on A defined by $\bar{\sigma}(f(x)) = \sigma(a_0) + \sigma(a_1)x + \cdots + \sigma(a_n)x^n$ for all $f(x) = a_0 + a_1x + \cdots + a_nx^n \in A$. Then $\bar{\sigma}$ is an endomorphism of A and $\bar{\sigma}(a) = \sigma(a)$ for all $a \in R$, which means that $\bar{\sigma}$ is an extension of σ. We call the endomorphism of $A = R[x;\sigma]$ which extends σ an *extended endomorphism* of σ. Let Σ_σ be the set of all extended endomorphisms on A of σ. Note that $\Sigma_\sigma \neq \emptyset$ since $\bar{\sigma} \in \Sigma_\sigma$.

LEMMA 1.7. *Let R be a ring with an endomorphism σ and let Σ_σ be the set of all extended endomorphisms on $A = R[x;\sigma]$ of σ. Then*

(1) *If I is a σ-ideal of R, then IA is a θ-ideal of A for all $\theta \in \Sigma_\sigma$;*
(2) *If I is a right principal σ-ideal of R, then IA is a right principal θ-ideal of A for all $\theta \in \Sigma_\sigma$;*
(3) *If I is a left principal σ-ideal of R, then AI is a left principal θ-ideal of A for all $\theta \in \Sigma_\sigma$.*

Proof. It is straitforward. □

LEMMA 1.8. *Let R be a ring with an endomorphism σ and let Σ_σ be the set of all extended endomorphisms on $A = R[x;\sigma]$ of σ. Then R is σ-rigid if and only if A is θ-rigid for all $\theta \in \Sigma_\sigma$. In this case, $\sigma(e) = e$ for every idempotent $e \in R$.*

Proof. Assume that R is σ-rigid and A is not θ-rigid for some $\theta \in \Sigma_\sigma$. Then there exists a nonzero $f \in A$ such that $f\theta(f) = 0$. Since R is σ-rigid, $f \notin R$. Let $f = \sum_{i=0}^m a_i x^i$ where $a_i \in R, a_m \neq 0$ for some $m \geq 1$. Since $f\theta(f) = 0$, $a_m\sigma^m(a_m) = 0$. Since R is σ-rigid, $a_m^2 = 0$ by Lemma 1.6, and then $a_m = 0$ since R is reduced, a contradiction. Hence A is θ-rigid for all $\theta \in \Sigma_\sigma$. The converse is true by the definition of extended endomorphism of σ. Let e be any idempotent of R. In case that A is θ-rigid for each $\theta \in \Sigma_\sigma$ (and then A is reduced). Hence e is central idempotent in A, and thus $ex = xe = \sigma(e)x$, which implies that $\sigma(e) = e$. □

Note that for a reduced ring R, $A = R[x;\sigma]$ is not necessarily reduced. Indeed, consider the reduced ring R and σ introduced in Example 1. Let $f = (0,2)x \in A$. Then $f^2 = (0,2)x(0,2)x = (0,2)\sigma(0,2)x^2 = (0,2)(2,0)x^2 = (0,0)x^2 = 0$. But $f \neq 0$. Hence A is not reduced.

We need the following corollary as a special case of [9, Proposition 6].

COROLLARY 1.9. *Let R be a σ-rigid ring. Then for any*

$$f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j \in R[x;\sigma],$$

$fg = 0$ if and only if $a_i b_j = 0$ for each i, j.

2. Skew polynomial rings over σ-quasi-Baer and σ-p.q.-Baer rings

We recall from [2] an idempotent $e \in R$ is *left* (resp. *right*) *semicentral* in R if $eae = ae$ (resp. $eae = ea$), for all $a \in R$. Equivalently, an idempotent $e \in R$ is left (resp. right) semicentral if eR (resp. Re) is an ideal of R. Since the right annihilator of a right σ-ideal is an ideal, we can note that the right annihilator of a right σ-ideal is generated by a left semicentral idempotent in a σ-quasi-Baer ring. Observe that if $e_1, e_2, \ldots, e_m$ are left (or right) semicentral idempotents of R, then $e = e_1 e_2 \cdots e_m$ is an idempotent of R. Thus we can obtain the following lemma;

LEMMA 2.1. *Let R be a ring with an endomorphism σ. Then R is a right (resp. left) σ-p.q.-Baer if and only if the right (resp. left) annihilator of every finitely generated right (resp. left) σ-ideal of R is generated by an idempotent of R.*

Proof. It is enough to show the left-handed version because the right-handed version is similarly proved. Suppose that R is right σ-p.q.-Baer and let $I = \sum_{i=1}^{m} a_i R$ be any finitely generated right σ-ideal of R. Then $r_R(I) = \bigcap_i^m e_i R$ where $r_R(a_i R) = e_i R$. By the above observation, $r_R(I)$ is an ideal of R and e_i is a left semicentral idempotent of R . Since each e_i is left semicentral idempotents of R, $e = e_1 e_2 \cdots e_m$ is idempotent of R, and so $r_R(I) = eR$. The converse is clear. □

LEMMA 2.2. *Let R be a σ-rigid ring. If $e \in R$ is a left semicentral idempotent, then e is also a left semicentral idempotent in $R[x;\sigma]$.*

Proof. Let $f = \sum_{i=0}^{m} a_i x^i \in R[x;\sigma]$ be arbitrary. Since R is σ-rigid, $\sigma(e) = e$ for any idempotent $e \in R$ by Lemma 1.8. Since e is a left semicentral idempotent, $ea_ie = a_ie$ for each i. Then $fe = \sum_{i=0}^{m} a_i \sigma^i(e) x^i = \sum_{i=0}^{m} a_i e x^i = \sum_{i=0}^{m} e a_i e x^i = efe$. Hence e is a left semicentral idempotent in $R[x;\sigma]$. □

THEOREM 2.3. *Let R be a ring with an endomorphism σ and let Σ_σ be the set of all extended endomorphisms on $A = R[x;\sigma]$ of σ. If R is σ-rigid, then the following are equivalent:*

(1) *R is σ-quasi-Baer;*
(2) *A is quasi-Baer;*
(3) *A is θ-quasi-Baer for all $\theta \in \Sigma_\sigma$.*

Proof. (1) $\Rightarrow$ (2). Suppose that R is σ-quasi-Baer. Let I be an arbitrary ideal of A. If $g \in r_A(I)$, then $fg = 0$ for all $f \in I$. Let $f = \sum_{i=0}^{m} a_i x^i, g = \sum_{j=0}^{n} b_j x^j$. Then by Corollary 1.9, $a_i b_j = 0$ for all i, j. Consider the set I_c of all coefficients of polynomials in I. Then I_c is an ideal of R and $b_0, b_1, \ldots, b_n \in r_R(I_c)$. We can observe that I_c is an σ-ideal of R. Indeed, for any $f = \sum_{i=0}^{m} a_i x^i \in I$, $xf = \sum_{i=0}^{m+1} \sigma(a_i) x^i$, and so $\sigma(a_i) \in I_c$ for each i. Thus I_c is a σ-ideal of R. Since R is σ-quasi-Baer and I_c is a σ-ideal of R, $r_R(I_c) = eR$ for some idempotent $e \in R$. Thus $g = ge$ and hence $r_A(I) \subseteq eA$. Now $I_c e = 0$. Since $\sigma(e)=$

e, by Lemma 1.8, we have $Ie = 0$ so $eA \subseteq r_A(I)$. Therefore $r_A(I) = eA$. Hence A is quasi-Baer.

(2) $\Rightarrow$ (3). It is clear.

(3) $\Rightarrow$ (1). Suppose that A is θ-quasi-Baer for all $\theta \in \Sigma_\sigma$. Let I be any σ-ideal of R. Then by Lemma 1.7, IA is a θ-ideal of A. Since A is θ-quasi-Baer, $r_A(IA) = eA$ for some semicentral idempotent $e \in A$. Since A is θ-rigid (and so A is reduced) by Lemma 1.8, e is a central idempotent in A, and hence e is an idempotent in R by [10, Theorem 3.15]. Since $r_R(I) = r_A(IA) \cap R = eR$, R is σ-quasi-Baer. □

REMARK. (1) If σ is an automorphism, we can check the condition "R is σ-rigid" does not need by using a similar method in the proof of Theorem 1.2 in [6]. (2) there is an example of a σ-quasi-Baer ring R and an endomorphism σ of R such that $R[x;\sigma]$ is not quasi-Baer (refer Example 1.4 in [6]).

COROLLARY 2.4. *Let R be a ring with an endomorphism σ and let Σ_σ be the set of all extended endomorphisms on $A = R[x;\sigma]$ of σ. If R is σ-rigid, then the following are equivalent:*

(1) *R is σ-Baer;*
(2) *A is Baer;*
(3) *A is θ-quasi-Baer for all $\theta \in \Sigma_\sigma$.*

Proof. It follows from Lemma 1.3 and Theorem 2.3. □

COROLLARY 2.5. *[1, Theorem A] Let R be a reduced ring and let $A = R[x]$. Then R is Baer if and only if $R[x]$ is Baer.*

Proof. It follows from Corollary 1.4 and Corollary 2.4. □

THEOREM 2.6. *Let R be a ring with an endomorphism σ and let Σ_σ be the set of all extended endomorphisms on $A = R[x;\sigma]$ of σ. If R is σ-rigid, then the following are equivalent:*

(1) *R is right σ-p.q.-Baer;*
(2) *R is σ-p.q.-Baer;*
(3) *A is right p.q.-Baer;*
(4) *A is p.q.-Baer;*
(5) *A is θ-p.q.-Baer for all $\theta \in \Sigma_\sigma$;*
(6) *A is right θ-p.q.-Baer for all $\theta \in \Sigma_\sigma$.*

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 1.5. (3) $\Leftrightarrow$ (4) also follows from Lemma 1.5 by letting $\sigma = 1$. (4) $\Rightarrow$ (5) $\Rightarrow$ (6) is clear. It remains to show that (1) $\Rightarrow$ (3) and (6) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). Suppose that R is right σ-p.q.-Baer. Let I be any right principal ideal of A generated by $h = \sum_{k=0}^{n} a_k x^k$. If $g \in r_A(I)$, then $fg = 0$ for all $f \in I$. Let $f = \sum_{i=0}^{l} c_i x^i, g = \sum_{j=0}^{m} b_j x^j$. Then by Lemma 1.6, $c_i b_j = 0$ for all i, j. Let I_c be the set of all coefficients of all $f \in I$. Note that I_c is a right σ-ideal of R and $b_0, b_1, \ldots, b_n \in r_R(I_c)$ as given in the proof of Theorem 2.3. Since I is a right principal ideal of A, I_c is a right finitely generated ideal of R with a generating set $\{a_0, \ldots, a_n\}$. Since R is right σ-p.q.-Baer and I_c is a right finitely generated σ-ideal of R, $r_R(I_c) = eR$ for some idempotent e of R by Lemma 2.1. Hence $r_A(I) = eA$, and so A is right p.q.-Baer.

(6) $\Rightarrow$ (1). Suppose that A is right θ-p.q.-Baer for all $\theta \in \Sigma_\sigma$. Let I be any right principal σ-ideal of R. Then by Lemma 1.1, IA is a right principal θ-ideal of A. Since A is θ-p.q.-Baer, $r_A(IA) = eA$ for some semicentral idempotent $e \in A$. Since A is θ-rigid (and so reduced) by Lemma 1.8, e is a central idempotent in A, and hence e is an idempotent in R by [10, Theorem 3.15]. Since $r_R(I) = r_A(IA) \cap R = eR$, R is right σ-p.q.-Baer. □

COROLLARY 2.7. *Let R be a ring with an endomorphism σ and let Σ_σ be the set of all extended endomorphisms on $A = R[x;\sigma]$ of σ. If R is σ-rigid, then the following are equivalent:*

(1) *R is right σ-p.p.;*
(2) *R is σ-p.p.;*
(3) *A is right p.p.;*
(4) *A is p.p.;*
(5) *A is θ-p.p. for all $\theta \in \Sigma_\sigma$;*
(6) *A is right θ-p.p. for all $\theta \in \Sigma_\sigma$.*

Proof. It follows from the Lemma 1.5 and Theorem 2.6. □

COROLLARY 2.8. *[1, Theorem B] Let R be a reduced. Then R is p.p.-Baer if and only if $R[x]$ is p.p.-Baer;*

Proof. It follows from the Lemma 1.5 (by letting $\sigma = 1$) and Corollary 2.7. □

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References

[1] E. P. Armendariz, *A note on extensions of Baer and p.p. rings*, Austral. Math. Soc. **18** (1974), 470–473.
[2] S. K. Berberian, *Baer *-rings*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.
[3] G. F. Birkenmeier, *Baer rings and quasi-continuous rings have a MSDN*, Pacific J. Math. **97** (1981), 283–292.
[4] ———, *Idempotents and completely semiprime ideals*, Comm. Algebra **11** (1983), 567–580.
[5] ———, *Decompositions of Baer-like rings*, Acta Math. Hungar. **59** (1992), 319–326.
[6] G. F. Birkenmeier, J. K. Kim and J. K. Park, *On extensions of quasi-Baer and principally quasi-Baer rings*, J. Pure Appl. Algebra **159** (2001), 25–42.
[7] A. W. Chatters and C. R. Hajarnavis, *Rings with Chain Conditions*, Pitman, Boston, 1980.
[8] W. E. Clark, *Twisted matrix units semigroup algebras*, Duke Math. J. **34** (1967), 417–424.
[9] C. Hong, N. Kim and T. Kwak, *Ore extensions of Baer and p.p-rings*, J. Pure Appl. Algebra **151** (2000), 215–226.
[10] A. A. M. Kamal, *Idempotents in polynomial rings*, Acta Math. Hungar. **59** (1992), no. 3-4, 355–363.
[11] I. Kaplansky, *Rings of Operators*, Lecture Notes in Math., Benjamin, New York, 1965.
[12] P. Pollingher and A. Zaks, *On Baer and quasi-Baer rings*, Duke Math. J. **37** (1970), 127–138.

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