Commun. Korean Math. Soc. **20** (2005), No. 1, pp. 1–14

ORDER-CONGRUENCES ON S-POSETS

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Abstract. The aim of this paper is to study order-congruences on a S-poset A and to characterize the order-congruences by the concepts of pseudooreders on A and quasi-chains module a congruence ρ. Some homomorphism theorems of S-posets are given which is similar to the one of ordered semigroups. Finally, It is shown that there exists the non-trivial order-congruence on a S-poset by an example.

1. Introduction and preliminaries

Many kinds of partially ordered algebras have appeared in the literature so far, for example, partially ordered groups, semigroups, rings and fields, etc. Recently, Fakhruddin in [3, 8] has been studied the category of posets acted on by a pomonoid S (the category of S-posets), absolute flatness and amalgams of S-posets. Similar to the theory of regular congruences on ordered semigroups [1, 2, 4, 5, 6], order-congruences on S-posets play an important role in studying the structures of S-posets. In this paper we introduce the concept of pseudoorders on A, discuss the relationship between order-congruences on A by means of the concept of quasi-chains modulo ρ, which is similar to the one of ordered semigroups [2].

We shall use the notation and terminology of [6] and [9] in the sequel. Throughout this paper, S always denotes an ordered semigroup. A nonempty subset I of S is called an *ideal* of S if 1) $IS \subseteq I, SI \subseteq I$, 2) $a \in I, S \ni b \leq a$ implies $b \in I$ (see e.g.[1]). Let B be poset. A nonempty subset A of B is called *convex* if $a, b \in A$, for $c \in S$ such that

Received May 26, 2003.

2000 Mathematics Subject Classification: 20M30, 06F05.

Key words and phrases: S-poset, order-congruence, quasi-chain module ρ, pseudo order.

Project supported by the GuangDong Provincial Natural Science Foundation of China and the NSF of the Education Department of Guangdong Province of China.

$a \leq c \leq b$, then $c \in A$. A is called *strongly convex* if $A = (A] := \{b \in B \mid (\exists a \in A)\ b \leq a\}$.

Let $(S, \leq_S)$ be an ordered semigroup which is not necessarily commutative and let $(A, \leq)$ a poset. A is called a left *S-poset* [3] (the adjective "left" would be omitted in the sequel) if S acts on A in such way:

1) $(\forall a, b \in A)(\forall s \in S)\ a \leq_A b \Rightarrow sa \leq_A sb$.
2) $(\forall s, t \in S)(\forall a \in A)\ s \leq_S t \Rightarrow sa \leq_A ta$.
3) $(\forall s, t \in S)(\forall a \in A)\ (st)a = s(ta)$,

where sa stands for the result of the action of s on a. Let A be a S-poset, B a nonempty subset of A. B is called a *S-subposet* of A if for any $b \in B, s \in S$, $sb \in B$, denoted by $B \leq A$. Obviously, for $a \in A$, Sa is a S-subposet of A, called cyclic S-subposet. It is easily seen that an ordered semigroup S is a S-poset with respect to the multiplication of S, an ideal I of S is a S-subposet of S. It is clear that any S-poset A is a S-act.

Let $(A, \leq_A)$, $(B, \leq_B)$ be S-posets, $f : A \to B$ a mapping from A into B. f is called *isotone* if $x, y \in A, x \leq_A y$ implies $f(x) \leq_B f(y)$. f is called *reverse isotone* if $x, y \in A$, $f(x) \leq_B f(y)$ implies $x \leq_A y$. f is called a *homomorphism* if it is isotone and satisfies $f(sx) = sf(x)$ for all $s \in S, x \in A$. f is called an *isomorphism* if it is a homomorphism, onto, and reverse isotone. The S-poset A and B are called *isomorphic*, in symbol $A \cong B$, if there exists an isomorphism between them.

By a congruence on a S-poset $(A, \leq_A)$ we mean an equivalent relation ρ on A such that if $a, b \in A$, $(a, b) \in \rho$ implies $(sa, sb) \in \rho$ for any $s \in S$. The set $C(A)$ of all congruences on a S-poset $(A, \leq_A)$ is a complete lattice with respect to the intersection of set-theoretic and the union (also is called *transitive product*) defined as follows:

$$(a,b) \in \prod_{\alpha \in \Gamma} \rho_\alpha \quad \Leftrightarrow \quad \exists c_\circ = a, c_1, \ldots, c_n = b \in S$$
$$\text{such that } (c_j, c_{j+1}) \in \rho_{\alpha_j} \text{ for some } \rho_{\alpha_j} \in \{\rho_\alpha\}_{\alpha \in \Gamma}.$$

ι arising from equality is the minimum element of $C(A)$, and π which identifies all elements of S is the greatest element of $C(A)$.

For a congruence ρ on a S-poset $(A, \leq_A)$, A/ρ is called a *quotient S-poset* if there exists an order $\preceq$ on A/ρ such that the poset $(A/\rho, \preceq)$ is a S-poset, where S acts on A/ρ in the usual multiplication [7] defined by $s(x)_\rho := (sx)_\rho$. A relation σ on a S-poset $(A, \leq_A)$ is called *pseudoorder* if $\leq_A \subseteq \sigma$, $\sigma \circ \sigma \subseteq \sigma$, and σ is compatible with the S-action.

2. Pseudoorders and characterizations

DEFINITION 1. Let $(A, \leq_A)$ be a S-poset, ρ a congruence on A. ρ is called an *order-congruence* if there exists an order "$\preceq$" on A/ρ such that:
1) $(A/\rho, \preceq)$ is a S-poset (where the S-action on A/ρ is defined as $s(x)_\rho := (sx)_\rho$).
2) The mapping

$$\varphi : A \to A/\rho \mid x \to (x)_\rho$$

is isotone (Then φ is a homomorphism).

It is clear that the congruences ι and π in the lattice $C(A)$ of a S-poset A are order-congruences. Furthermore, we have the following statements of a S-poset A.

THEOREM 2. *Let $(A, \leq)$ be a S-poset, $B \leq A$ and $(B] = B$. Let λ_B be a Rees congruence on the S-act A determined by B. We define a relation "$\preceq$" on A/λ_B $(= \{\{x\} \mid x \in A \backslash B\} \cup \{B\})$ as follows:*

$$\preceq := \{(B, \{x\}) \mid x \in A \backslash B\} \cup \{(\{x\}, \{y\}) \mid x, y \in A \backslash B, x \leq y\} \cup \{(B, B)\}.$$

Then $(A/\lambda_B, \preceq)$ is a S-poset, and the Rees congruence λ_B on A is an order-congruence.

PROOF. 1) It is not difficult to verify that $(A/\lambda_B, \preceq)$ is a poset. Let $(x)_{\lambda_B}, (y)_{\lambda_B} \in A/\lambda_B$, $(x)_{\lambda_B} \preceq (y)_{\lambda_B}$. Since $(x)_{\lambda_B} \in A/\lambda_B$, we have $(x)_{\lambda_B} = \{x\}, x \in A \backslash B$ or $(x)_{\lambda_B} = B$.

α) Let $(x)_{\lambda_B} = B$. Then $x \in B$. Since $B \leq A$, we have $sx \in B$ for any $s \in S$. Thus $s(x)_{\lambda_B} = B \preceq s(y)_{\lambda_B}$.

β) Let $(x)_{\lambda_B} = \{x\}$. By Definition of "$\preceq$", since $(x)_{\lambda_B} \preceq (y)_{\lambda_B}$, we have $(y)_{\lambda_B} = \{y\}$ and $x \leq_A y$. Thus $sx \leq_A sy$ for any $s \in S$.

i) If $sx \in B$, then

$$s(x)_{\lambda_B} = (sx)_{\lambda_B} = B \preceq s(y)_{\lambda_B}.$$

ii) If $sx \in A \backslash B$, then $sy \in A \backslash B$ since $(B] = B$. Thus $\{sx\} \preceq \{sy\}$, that is, $s(x)_{\lambda_B} \preceq s(y)_{\lambda_B}$. Moreover, let $s_1, s_2 \in S, s_1 \leq_S s_2$. Similar to discuss as above, we have $s_1(a)_{\lambda_B} \leq_A s_2(a)_{\lambda_B}$ for any $a \in A$. Therefore, $(A/\lambda_B, \preceq)$ is a S-poset.

2) λ_B is an order-congruence. In fact: Let

$$\phi : A \longrightarrow A/\lambda_B \mid x \to (x)_{\lambda_B}.$$

Then $\varphi(sa) = s\varphi(a)$ for any $s \in S, a \in A$. And if $x \leq y$, we consider three cases:

α) If $x \in B$, then $(x)_{\lambda_B} = B \preceq (y)_{\lambda_B}$.
β) If $y \in B$, Then $x \in B$. Thus $(x)_{\lambda_B} = (y)_{\lambda_B} = B$.
γ) If $x, y \in S \backslash B$, then $(x)_{\lambda_B} = \{x\} \preceq \{y\} = (y)_{\lambda_B}$. □

By Theorem 2, we have λ_B is an order-congruence on a S-poset $(A, \leq_A)$, moreover we have

THEOREM 3. *Let B be a strongly convex S-subposet of a S-poset $(A, \leq_A)$. Let*

$$\mathcal{A} := \{J \mid J \text{ a strongly convex } S\text{-subposet of } A \text{ containing } B\},$$

and let $\mathcal{B}$ be the set of all strongly convex S-subposets of the S-poset $(A/\lambda_B, \preceq)$ (where the relation "$\preceq$" is defined as one in Theorem 2). For $J \in \mathcal{A}$, we define a mapping θ as follows:

$$\theta: \ \mathcal{A} \to \mathcal{B} | J \mapsto (J)_{\lambda_B}.$$

Then the θ is $(1-1)$, onto, and inclusion-preserving.

PROOF. 1) Let $J \in \mathcal{A}$. Then $(J)_{\lambda_B} \in \mathcal{B}$. In fact, $((J)_{\lambda_B}, \preceq)$ is a poset. For any $s \in S, (a)_{\lambda_B} \in (J)_{\lambda_B}$ $a \in J$, since J is a S-subposet of A, we have $s(a)_{\lambda_B} = (sa)_{\lambda_B} \in (J)_{\lambda_B}$. If $(x)_{\lambda_B} \preceq (y)_{\lambda_B}$ and $(y)_{\lambda_B} \in (J)_{\lambda_B}$. Then there exists $j \in J$ such that $(y)_{\lambda_B} = (j)_{\lambda_B}$. We consider the cases as follows:

α) If $x \in B$, then $x \in J$. Clearly, $(x)_{\lambda_B} \in (J)_{\lambda_B}$.

β) If $x \notin B$, then $(x)_{\lambda_B} = \{x\}$. Since $(x)_{\lambda_B} \preceq (j)_{\lambda_B}$, we have $(j)_{\lambda_B} = \{j\}$. Thus $x \leq j$. Since J is strongly convex, we have $x \in J$, i.e. $(x)_{\lambda_B} \in (J)_{\lambda_B}$. We summarize the situation in the fact that $(J)_{\lambda_B}$ is a strongly convex S-subposet of A/λ_B.

2) θ is (1–1). Let $J_1, J_2 \in \mathcal{A}$, $J_1 \neq J_2$. Then there exists $j_1 \in J_1 \backslash J_2$ or $j_2 \in J_2 \backslash J_1$. If $j_1 \in J_1 \backslash J_2$, then $(j_1)_{\lambda_B} \notin (J_2)_{\lambda_B}$. In fact: If $(j_1)_{\lambda_B} = (x)_{\lambda_B}$ for some $x \in J_2$, since $j_1 \notin B$ we have $(x)_{\lambda_B} = \{j_1\}$, thus $x = j_1$. Impossible. Thus $(J_1)_{\lambda_B} \neq (J_2)_{\lambda_B}$. For $j_2 \in J_2 \backslash J_1$, by the same way, we have $(J_1)_{\lambda_B} \neq (J_2)_{\lambda_B}$.

3) θ is onto. Let K be a strongly convex S-poset of $(A/\lambda_B, \preceq)$. There exists a strongly convex S-subposet J of A containing B such that $K = (J)_{\lambda_B}$. Indeed: let

$$J = \{x \in A \mid (x)_{\lambda_B} \in K\}.$$

It is known that J is a S-subposet of A. Furthermore, J is strongly convex and $B \subset J$. In fact: Let $y \in S$, $y \leq x \in J$. Then $(y)_{\lambda_B} \preceq (x)_{\lambda_B} \in K$. Thus we have $(y)_{\lambda_B} \in K$, i.e. $y \in J$. For $\forall x \in B$, $(x)_{\lambda_B} = \{B\} \preceq (k)_{\lambda_B} (\forall (k)_{\lambda_B} \in K)$, we have $(x)_{\lambda_B} \in K$. Thus $x \in J$, i.e. $B \subseteq J$. By

definition of the set J, clearly, $K = (J)_{\lambda_B}$.
4) θ is inclusion-preserving. Straightforward. $\square$

In order to obtain the relationship between order-congruences and pseudoorders on A, the following lemma is essential.

LEMMA 4. *Let $(A, \leq_A)$ be a S-poset, $\sigma \subseteq A \times A$. The following are equivalent:*
1) σ is a pseudoorder on A.
2) There exist a S-poset $(C, \preceq)$ and a homomorphism $\varphi : A \to C$ such that

$$\overrightarrow{ker}\varphi = \{(a,b) \in S \times S \mid \varphi(a) \preceq \varphi(b)\} = \sigma,$$

where $\overrightarrow{ker}\varphi$ is called the directed kernel of φ.

PROOF. 1) $\Longrightarrow$ 2) If σ is a pseudoorder on A, we denoted by $\bar{\sigma}$ the congruence on A defined by

$$\bar{\sigma} := \{(a,b) \mid (a,b) \in \sigma, (b,a) \in \sigma\} (= \sigma \cap \sigma^{-1}).$$

The set $A/\bar{\sigma} := \{(a)_{\bar{\sigma}} \mid a \in A\}$ with the S-action $s(a)_{\bar{\sigma}} := (sa)_{\bar{\sigma}}$ for $s \in S$, $a \in A$ and the order

$$\preceq_{\bar{\sigma}} := \{((a)_{\bar{\sigma}}, (b)_{\bar{\sigma}}) \mid (a,b) \in \sigma\}$$

is a S-poset. Let $C = (A/\bar{\sigma}, \preceq_{\bar{\sigma}})$ and φ be the mapping of A onto $A/\bar{\sigma}$ defined by $\varphi : A \longrightarrow A/\bar{\sigma} \mid a \to (a)_{\bar{\sigma}}$. Then φ is the homomorphism of A onto $A/\bar{\sigma}$ and $\overrightarrow{ker}\varphi = \sigma$.

2) $\Longrightarrow$ 1) If $(A, \leq_A)$ and $(C, \preceq)$ are S-posets and $\varphi : A \longrightarrow C$ a homomorphism, then $\overrightarrow{ker}\varphi$ is a pseudoorder on A. In fact: Let $(a,b) \in \leq_A$. Then $\varphi(a) \preceq \varphi(b)$. Thus $(a,b) \in \overrightarrow{ker}\varphi$, $\leq_A \subseteq \overrightarrow{ker}\varphi$. Moreover, let $(a,b) \in \overrightarrow{ker}\varphi \circ \overrightarrow{ker}\varphi$. Then there exists $c \in A$ such that $(a,d), (d,b) \in \overrightarrow{ker}\varphi$. Thus $\varphi(a) \preceq \varphi(d) \preceq \varphi(b)$. Therefore $\varphi(a) \preceq \varphi(b)$, i.e. $(a,b) \in \overrightarrow{ker}\varphi$. If $(a,b) \in \overrightarrow{ker}\varphi$, then $\varphi(a) \preceq \varphi(b)$. Since C is a S-poset, we have

$$s\varphi(a) = \varphi(sa) \preceq s\varphi(b) = \varphi(sb).$$

Then $(sa, sb) \in \overrightarrow{ker}\varphi$. $\square$

THEOREM 5. *Let A be a S-poset, ρ a congruence on A. The following are equivalent:*
1) ρ is an order-congruence.
2) there exists a pseudoorder σ on A such that $\rho = \sigma \cap \sigma^{-1}$.

3) *there exists a S-poset C and a homomorphism $\varphi : A \to C$ such that $\rho = \overrightarrow{ker}\varphi$.*

PROOF. 1) $\Longrightarrow$ 2) Let ρ be an order-congruence on A. Then there exists an order relation "$\preceq$" on the quotient poset A/ρ such that $(A/\rho, \preceq)$ is a S-poset, and $\varphi : A \to A/\rho$ is a homomorphism. Let $\sigma = \overrightarrow{ker}\varphi$. By Lemma 4, σ is a pseudoorder on A and it is easy to check that $\rho = \sigma \cap \sigma^{-1}$.

2) $\Longrightarrow$ 3) For a pseudoorder σ on A, by Lemma 4, there exists a S-poset C and a homomorphism $\varphi : A \to C$ such that $\sigma = \overrightarrow{ker}\varphi$. Then

$$\ker\varphi = \overrightarrow{ker}\varphi \cap (\overrightarrow{ker}\varphi)^{-1} = \sigma \cap \sigma^{-1} = \rho.$$

3) $\Longrightarrow$ 1) By hypothesis and Lemma 4, $\overrightarrow{ker}\varphi$ is a pseudoorder on A, then $\rho = \overrightarrow{ker}\varphi \cap \overrightarrow{ker}\varphi^{-1}$, is a congruence on A. By the proof of Lemma 4, ρ is an order-congruence on A. □

For an order-congruence ρ on A, since the order "$\preceq$" such that $(A/\rho, \preceq)$ is a S-poset is not unique in general, we have the pseudoorder σ containing ρ such that $\rho = \sigma \cap \sigma^{-1}$ is not unique. If σ is a pseudoorder on A, then $\rho = \sigma \cap \sigma^{-1}$ is the greatest order-congruence on A contained in σ. In fact, if ρ_1 is an order-congruence on A contained in σ, then $\rho_1 \cap \rho_1^{-1} = \rho_1 \subseteq \sigma \cap \sigma^{-1} = \rho$.

THEOREM 6. *Let ρ be an order-congruence on a S-poset $(A, \leq)$. Then the least pseudoorder σ containing ρ is the transitive closure of relations $\leq \circ\rho$, that is,*

$$\sigma = \bigcup_{n=1}^{\infty} (\leq \circ\rho)^n.$$

PROOF. 1) Let $\sigma_1 = \bigcup_{n=1}^{\infty} (\leq \circ\rho)^n$. Clearly, $\rho, \leq \subseteq \leq \circ\rho \subseteq \sigma_1$. Then σ_1 is transitive.

2) If $(a, b) \in \sigma_1, \forall c \in S$, then there is $n \in \mathbf{N}$ such that $(a, b) \in (\leq \circ\rho)^n$ i.e. $\exists a_1, b_1, a_2, b_2, ..., a_n \in A$ such that

$$a \leq a_1\rho b_1 \leq a_2\rho b_2 \leq \cdots \leq a_n\rho b.$$

Thus,

$$ca \leq ca_1\rho cb_1 \leq ca_2\rho cb_2 \leq \cdots \leq ca_n\rho cb.$$

Clearly, $(ca, cb) \in (\leq \circ\rho)^n \subseteq \sigma_1$. It implies that $\bigcup_{n=1}^{\infty} (\leq \circ\rho)^n$ is a pseudoorder on A containing ρ. Furthermore, since σ is transitive, and $\rho \subseteq \sigma, \leq \subseteq \sigma$, we have $\bigcup_{n=1}^{\infty} (\leq \circ \rho)^n \subseteq \sigma$. By hypothesis, then $\sigma = \bigcup_{n=1}^{\infty} (\leq \circ \rho)^n$. □

Now, we will give out other characterization of order-congruences on a S-poset A. In order to facilitate the proving of the main results, we first introduce the following concept.

DEFINITION 7. Let $(A, \leq)$ be a S-poset, ρ a congruence on A. A sequence in A $(x, a_1, b_1, a_2, b_2, ..., a_n, y)$ is called a *quasi-chain modulo* ρ if

$$x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_n \rho y,$$

and n is called *length* of this quasi-chain modulo ρ, x, y is called the *initial* and *terminal elements* respectively. A quasi-chain modulo ρ is called *close* if its initial and terminal elements are equal, i.e. $x = y$.

We denote by ${}_\rho C_{xy}$ all quasi-chains modulo ρ with the initial x and terminal y in the sequel.

After Definition 7, we have

LEMMA 8. *Let $(A, \leq)$ be a S-poset, ρ a congruence on A. Then*

1) *There exists a quasi-chain modulo ρ with length n in ${}_\rho C_{xy}$ if and only if $(x, y) \in (\leq \circ \rho)^n$.*

2) *If a sequence in A $(x, a_1, b_1, a_2, b_2, ..., a_n, y)$ is a quasi-chain modulo ρ, so are $(cx, ca_1, cb_1, ca_2, cb_2, ..., ca_n, cy)$ $\forall c \in S$.*

3) *If x, y are contained in a close quasi-chain modulo ρ. Then there exists $m \in \mathbf{N}$ such that $(x, y) \in (\leq \circ \rho)^m$.*

PROOF. 1) and 2) are easy, we only prove 3). Let $(a_\circ, a_1, b_1, a_2, b_2, ..., a_n, a_\circ)$ is a close quasi-chain modulo ρ containing x, y, we consider the following cases:

A) $x = a_i, y = a_j$;

B) $x = a_i, y = b_j$;

C) $x = b_i, y = a_j$;

D) $x = b_i, y = b_j$.

We now prove the case A), and this is the way we prove the cases B), C) and D).

α) If $i \leq j$, $x = a_i \leq a_i \rho b_i \leq \cdots \leq a_j \rho a_j (= y)$. Then we have $(x, y) \in (\leq \circ \rho)^{j-i}$.

β) If $i \geq j$, $x = a_i \leq a_i \rho b_i \leq \cdots \leq a_n \rho a_\circ \leq a_1 \rho b_1 \leq \cdots \leq a_j \rho a_j (= y)$. Then we have $(x, y) \in (\leq \circ \rho)^{n-i+j}$. □

LEMMA 9. *Let A be a S-poset, ρ a congruence on A. If $(x, y) \in \rho$, $(z, k) \in \rho$, then ${}_\rho C_{xk} \neq \phi$ if and only if ${}_\rho C_{yz} \neq \phi$.*

PROOF. ($\Longrightarrow$) If ${}_\rho C_{xk} \neq \phi$, by Lemma 2.8, there exists $n \in \mathbf{N}$ such that $(x, k) \in (\leq \circ \rho)^n$. Since $(x, y) \in \rho$, $(z, k) \in \rho$, we have

$$y \leq y \rho x (\leq \circ \rho)^n k \leq k \rho z,$$

i.e. $(y, z) \in (\leq \circ \rho)^{n+2}$. By Lemma 8, we have ${}_\rho C_{yz} \neq \phi$.
($\Longleftarrow$) Similar to the proof of necessity, we omit. □

Now we are ready to describe the order-congruences.

THEOREM 10. *A congruence on a S-poset $(A, \leq)$ is an order-congruence if and only if for any $x \in A$, every close quasi-chain modulo ρ in ${}_\rho C_{xx}$ is contained in a single ρ-class.*

PROOF. ($\Longrightarrow$) Since ρ is an order-congruence, then there exists an order $\preceq$ on the quotient set A/ρ such that $(A/\rho, \preceq)$ is a S-poset and $\varphi : A \to A/\rho$ is a homomorphism. For any $x \in A$, and every close quasi-chain modulo ρ in ${}_\rho C_{xx}$ $(x, a_1, b_1, ..., a_n, x)$, we have

$$x \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_n \rho x.$$

Then,

$$\varphi(x) \preceq \varphi(a_1) = \varphi(b_1) \preceq \varphi(a_2) = \varphi(b_2) \preceq \cdots \preceq \varphi(a_n) = \varphi(x).$$

It implies that $\varphi(x) = \varphi(a_1) = \varphi(b_1) = ... = \varphi(a_n)$. Consequently, we have

$$(x, a_1, b_1, \cdots, a_n, x)$$

is contained in a single ρ-class.
($\Longleftarrow$) Conversely, we define a relation "$\preceq$" on the quotient set A/ρ as follows:

$$\preceq := \{((x)_\rho, (y)_\rho) \mid {}_\rho C_{xy} \neq \phi\}.$$

1) $\preceq$ is well-defined. For $x_1, y_1 \in A$, if $(x)_\rho = (x_1)_\rho$, $(y)_\rho = (y_1)_\rho$. By Lemma 9, if ${}_\rho C_{xy} \neq \phi$, then ${}_\rho C_{x_1 y_1} \neq \phi$.

2) $\preceq$ is an ordered relation on A/ρ.

α) $\preceq$ is reflexive. In fact, since for any $x \in A$, $x \leq x \rho x$, then we have ${}_\rho C_{xx} \neq \phi$.

β) $\preceq$ is transitive. In fact: If $((x)_\rho, (y)_\rho) \in \preceq$, $((y)_\rho, (z)_\rho) \in \preceq$. We have ${}_\rho C_{xy} \neq \phi$, ${}_\rho C_{yz} \neq \phi$. By Lemma 8, there exist $m, n \in \mathbf{N}$ such that $(x, y) \in (\leq \circ\rho)^m$, $(y, z) \in (\leq \circ\rho)^n$. Then we have

$$(x, z) \in (\leq \circ\rho)^m \circ (\leq \circ\rho)^n = (\leq \circ\rho)^{m+n},$$

i.e. ${}_\rho C_{xz} \neq \phi$.

γ) $\preceq$ is anti-symmetric. In fact, if $((x)_\rho, (y)_\rho) \in \preceq$, $((y)_\rho, (x)_\rho) \in \preceq$, then ${}_\rho C_{xy} \neq \phi$, ${}_\rho C_{yx} \neq \phi$. By β), ${}_\rho C_{xx} \neq \phi$, i.e. there exists a close quasi-chain modulo ρ in ${}_\rho C_{xx}$ containing x and y. By hypothesis, $(x)_\rho = (y)_\rho$.

3) $(A/\rho, \preceq)$ is a S-poset. We only need verify that $\preceq$ is compatible with S-action. Indeed: Let $((x)_\rho, (y)_\rho) \in \preceq$. Then ${}_\rho C_{xy} \neq \phi$. By Lemma 8 2), for $c \in S$, we have ${}_\rho C_{(cx)(cy)} \neq \phi$, i.e. $((cx)_\rho, (cy)_\rho) \in \preceq$. Let $s_1, s_2 \in S$ and $s_1 \leq s_2$. Then $s_1 x \leq s_2 x$ for any $x \in A$. Thus $(s_1 x, s_2 x) \in \leq \circ\rho$. We have ${}_\rho C_{(s_1 x)(s_2 x)} \neq \phi$, i.e. $((s_1 x)_\rho, (s_2 x)_\rho) \in \preceq$.

4) $\varphi : A \to A/\rho | x \mapsto (x)_\rho$ is a homomorphism. It is easily seen that

$$\varphi(sy) = (sy)_\rho = s(y)_\rho = s\varphi(y).$$

If $x \leq y$. Then $(x, y) \in \leq \circ\rho$, we have ${}_\rho C_{xy} \neq \phi$, i.e. $(x)_\rho \preceq (y)_\rho$. □

By Theorem 10, we have

COROLLARY 11. *If ρ is an order-congruence on a S-poset $(A, \leq)$, then every ρ-class in A is convex.*

PROOF. If $x \leq y \leq z$ with $(x)_\rho = (z)_\rho$, since

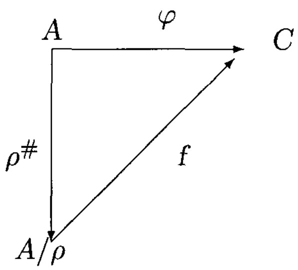
$$x \leq y\rho y \leq z\rho x,$$

i.e. (x, y, y, z, x) is a close quasi-chain modulo ρ, by Theorem 10, we have $(x)_\rho = (y)_\rho = (z)_\rho$. □

3. Homomorphisms

Two isomorphism theorems of semigroups, ordered semigroups and S-acts based on congruences have been given in [9], [2], [6] and [7]. In case of ordered semigroups, pseudoorders play the role congruences which are “bigger”than the congruences. In this section, we give out two isomorphism theorems of S-posets by pseudoorders in the S-posets.

THEOREM 12. *Let $(A, \leq_A)$, $(C, \leq_C)$ be S-posets, $\varphi : A \to C$ a homomorphism. Then: If λ is a pseudoorder on A such that $\lambda \subseteq \overrightarrow{ker}\varphi$, then there exists the unique homomorphism $f : A/\rho \longrightarrow C \mid (a)_\rho \to \varphi(a)$ such that the diagram*



commutes, where $\rho = \lambda \cap \lambda^{-1}$. *Moreover,* $Im(\varphi) = Im(f)$. *Conversely, if* λ *is a pseudoorder on* A *for which there exists a homomorphism* $f : (A/\rho, \preceq) \to (C, \leq_C)(\rho = \lambda \cap \lambda^{-1})$ *such that the above diagram commutes, then* $\lambda \subseteq \overrightarrow{ker}\varphi$.

PROOF. 1) f is well-defined. If $(a)_\rho = (b)_\rho$, then $(a,b) \in \lambda$. Since $\lambda \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(a), \varphi(b)) \in \leq_C$. Furthermore, since $(b,a) \in \lambda \subseteq \overrightarrow{ker}\varphi$, we have $(\varphi(b), \varphi(a)) \in \leq_C$. Therefore $\varphi(a) = \varphi(b)$.

2) f is a homomorphism and $\varphi = f \circ \rho^{\#}$. In fact: By Theorem 5, there exists an order "$\preceq$" on A/ρ such that $(A/\rho, \preceq)$ is a S-poset and the natural mapping $\rho^{\#}$ is a homomorphism.

$$\begin{aligned} (a)_\rho \preceq (b)_\rho &\Leftrightarrow (a,b) \in \lambda \subseteq \overrightarrow{ker}\varphi \\ &\Rightarrow \varphi(a) \leq_C \varphi(b) \\ &\Leftrightarrow f((a)_\rho) \leq_C f((b)_\rho). \\ f(s(a)_\rho) &= f((sa)_\rho) = \varphi(sa) = s\varphi(a) = sf((a)_\rho) \\ f \circ \rho^{\#}(a) &= f((a)_\rho) = \varphi(a). \end{aligned}$$

Let g is a homomorphism of a S-poset $(A/\rho, \preceq)$ into $(C, \leq_C)$ such that $g \circ \rho^{\#} = \varphi$. Then

$$f((a)_\rho) = \varphi(a) = (g \circ \rho^{\#})(a) = g((a)_\rho).$$

Moreover, $ran(f) = \{f((a)_\rho) \mid a \in A\} = \{\varphi(a) \mid a \in A\} = ran(\varphi)$.

Conversely, by hypothesis,

$$\begin{aligned} (a,b) \in \lambda &\Longrightarrow (a)_\rho \preceq (b)_\rho \Longrightarrow f((a)_\rho) \leq_C f((b)_\rho) \\ &\Longrightarrow (f \circ \rho^{\#})(a) \leq_C (f \circ \rho^{\#})(b) \\ &\Longrightarrow \varphi(a) \leq_C \varphi(b) \Longrightarrow (a,b) \in \overrightarrow{ker}\varphi, \end{aligned}$$

where the order $\preceq$ on A/ρ is defined as in proof of Lemma 2.4, that is,

$$\preceq := \{((x)_\rho, (y)_\rho) \mid (x,y) \in \sigma\}.$$

□

COROLLARY 13. *Let* $(A, \leq_A)$, $(C, \leq_C)$ *be* S*-poset,* $\varphi : A \mapsto C$ *is a homomorphism. Then* $A/Ker\varphi \cong ran(\varphi)$.

PROOF. By Theorems, 5, 12, let $\lambda = \vec{ker}\varphi$, and $\rho = \vec{ker}\varphi \cap \vec{ker}\varphi^{-1}$. Then ρ is an order-congruence and $f : A/\rho \mapsto C \mid (a)_\rho \to \varphi(a)$ is a homomorphism. f is reverse isotone. Indeed: Let $\varphi(a) \leq_C \varphi(b)$. Then $(a, b) \in \vec{ker}\varphi$. By Lemma 4, $((a)_\rho, (b)_\rho) \in \preceq \Leftrightarrow (a, b) \in \vec{ker}\varphi$. Then $((a)_\rho, (b)_\rho) \in \preceq$. Since $\rho = Ker\varphi$, we have $S/Ker\varphi \cong ran(\varphi)$. □

Let A be a S-poset, ρ, σ pseudoorders on A and $\rho \subseteq \sigma$. We define a pseudoorder on the S-poset $(A/\bar{\rho}, \leq_{\bar{\rho}})$ denoted by σ/ρ as follows:

$$\sigma/\rho := \{((a)_{\bar{\rho}}, (b)_{\bar{\rho}}) \mid (a, b) \in \sigma\},$$

where $\leq_{\bar{\rho}} := \{((a)_{\bar{\rho}}, (b)_{\bar{\rho}}) \mid (a, b) \in \rho\}, \bar{\rho} = \rho \cap \rho^{-1}$.

THEOREM 14. *Let* A *be a* S*-poset,* ρ *and* σ *pseudoorders on* A *and* $\rho \subseteq \sigma$. *Then* $A/\bar{\rho}/\overline{\sigma/\rho} \cong A/\bar{\sigma}$.

PROOF. Since σ/ρ is a pseudooder on $A/\bar{\rho}$, we have the mapping $\varphi : A/\bar{\rho} \mapsto A/\bar{\sigma} \mid (a)_{\bar{\rho}} \to (a)_{\bar{\sigma}}$ is a homomorphism. In fact:

1) φ is well-defined. Since $(a)_{\bar{\rho}} = (b)_{\bar{\rho}}$, we have $(a)_{\bar{\sigma}} = (b)_{\bar{\sigma}}$.

2) φ is homomorphism. Obviously, φ is onto. Since

$$(a)_{\bar{\rho}} \leq_{\bar{\rho}} (b)_{\bar{\rho}} \Rightarrow (a, b) \in \rho \Rightarrow (a, b) \in \sigma \Rightarrow (a)_{\bar{\sigma}} \leq_{\bar{\sigma}} (b)_{\bar{\sigma}},$$

we have φ is isotone.

3) Let $\vec{ker}\varphi := \{((a)_{\bar{\rho}}, (b)_{\bar{\rho}}) \mid \varphi((a)_{\bar{\rho}}) \leq_{\bar{\sigma}} \varphi((b)_{\bar{\rho}})\}$. Then

$$\begin{aligned} ((a)_{\bar{\rho}}, (b)_{\bar{\rho}}) \in \vec{ker}\varphi &\iff (a)_{\bar{\sigma}} \leq_{\bar{\sigma}} (b)_{\bar{\sigma}} \iff (a, b) \in \sigma \\ &\iff ((a)_{\bar{\rho}}, (b)_{\bar{\rho}}) \in \sigma/\rho. \end{aligned}$$

Thus $Ker\varphi = \vec{ker}\varphi \cap (\vec{ker}\varphi)^{-1} = \sigma/\rho \cap (\sigma/\rho)^{-1} = \overline{\sigma/\rho}$. By Corollary 13, we have $A/\bar{\rho}/\overline{\sigma/\rho} \cong A/\bar{\sigma}$. □

We now approach an example of this section to illustrate there exists the order-congruence in a S-poset.

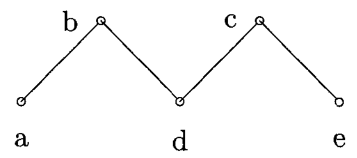
EXAMPLE 1. We consider the ordered semigroup $S = \{a, b, c, d, e\}$ [7] defined by multiplication and the order below:

$\cdot$	a	b	c	d	e
a	b	b	d	d	d
b	b	b	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	c

$$\begin{aligned}\leq: \;&= \{(a,a),(b,b),(c,c),(d,d),(e,e),\\ &\quad (a,b),(d,b),(d,c),(e,c)\}.\end{aligned}$$

We give the covering relation "$\prec$" and the figure of S.

$$\prec = \{(a,b),(d,b),(d,c),(e,c)\}.$$

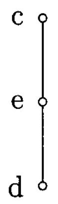


We consider the partially ordered set $A = \{c,d,e\}$ defined by the order below:

$$\leq_A := \{(c,c),(d,d),(e,e),(d,e),(d,c),(e,c).$$

We give the covering relation "$\prec$" and the figure of A.

$$\prec_A = \{(d,e),(e,c)\}.$$



Then $(A, \leq_A)$ is a S-poset about S-action on A as above multiplication table.

Let σ_1, σ_2 be congruences on A defined as follows:

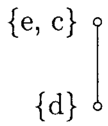
$$\begin{aligned}\sigma_1 &= \{(d,d),(c,c),(e,e),(d,c),(c,d)\}\\ \sigma_2 &= \{(d,d),(c,c),(e,e),(e,c),(c,e)\}\end{aligned}$$

Then $A/\sigma_1 = \{\{d,c\},\{e\}\}$, $A/\sigma_2 = \{\{e,c\},\{d\}\}$. Moreover,

σ_1 is not an order-congruence on A. In fact: If σ_1 is an order-congruence on A, then there exists an order "$\preceq_{A/\sigma_1}$" on A/σ_1 such that $(A/\sigma_1, \preceq_{A/\sigma_1})$ is a S-poset and $\varphi:\ A \to A/\sigma_1$ is isotone. Since $d <_A e$, we have $(d)_{\sigma_1} \preceq_{A/\sigma_1} (e)_{\sigma_1}$. Since $e <_A c$, we have $(e)_{\sigma_1} \preceq_{A/\sigma_1} (c)_{\sigma_1} = (d)_{\sigma_1}$. Then $(e)_{\sigma_1} = (d)_{\sigma_1}$. Impossible.

σ_2 is an order-congruence on A. In fact: We define an order on A/σ_2 as follows:

$$\preceq_{A/\sigma_2} := \{(\{d\}, \{d\}), (\{e,c\}, \{e,c\}), (\{d\}, \{e,c\})\}$$



Then $(A/\sigma_2, \preceq_{A/\sigma_2})$ is a S-poset. It is easily seen that the natural mapping $\varphi: A \longrightarrow A/\sigma_2 \mid a \to (a)_{\sigma_2}$ is a homomorphism.

References

[1] S. M. Fakhruddin, *Absolute flatness and amalgrams in pomonoids,* Semigroup Forum **33** (1986), 15–22.

[2] ———, *On the category of S-posets,* Acta Sci. Math. **52** (1988), 85–92.

[3] J. M. Howie, An Introduction to Semigroup Theory, Academic Press, 1976.

[4] N. Kehayopulu and M. Tsingelis, *On subdirectly irreducible ordered semigroups,* Semigroup Forum **50** (1995), 161–177.

[5] ———, *Pseudoorder in ordered semigroups*, Semigroup Forum **50** (1995), 389–392.

[6] X. Y. Xie, *An Introduction to Ordered Semigroup Theory*, Beijing, Science Press, 2001.

[7] ———, *Contributions to theory of congruences on ordered semigroups,* Doctoral Dissertation, Lanzhou University, 1995.

[8] X. Y. Xie and M. F. Wu, *On congruences on ordered semigroups,* Math. Japonica, **45** (1997), no. 1, 81–85.

[9] Z. Liu, *S-systems Theory of Semigroups*, Beijing, Science Press, 1999.

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