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A NOTE ON NULL DESIGNS OF DUAL POLAR SPACES

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ABSTRACT. Null designs on the poset of dual polar spaces are considered. A poset of dual polar spaces is the set of isotropic subspaces of a finite vector space equipped with a nondegenerate bilinear form, ordered by inclusion. We show that the minimum number of isotropic subspaces to construct a nonzero null t-design is $\prod_{i=0}^{t}(1+q^i)$ for the types B_N, D_N, whereas for the case of type C_N, more isotropic subspaces are needed.

1. Introduction

Null designs are defined on ranked partially ordered sets. Let P be a finite ranked partially ordered set. Given two ranks of P, $t \leq k$, we can form a $0, 1$ matrix, called an adjacency matrix, with columns indexed by the elements of rank k and the rows indexed by the elements of rank t. The kernel of this adjacency matrix forms a space of very interesting objects called *null (t, k)-designs* [8]. The poset of dual polar spaces of a given type is the set of isotropic subspaces of a finite vector space equipped with the nondegenerate bilinear form of the corresponding type, ordered by inclusion. In this paper, we consider the space of null (t, k)-designs of posets of dual polar spaces of type B_N, C_N and D_N [2], [12], [13]. Note that the corresponding Chevalley group has a natural action on the space of null (t, k)-designs, that is, the space of null (t, k)-designs forms a representation of the corresponding Chevalley group. These representations are considered in [12].

We are especially interested in the number of nonzero entries, called the *support size*, of nonzero null t-designs. P. Frankl and J. Pach [7]

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proved that the minimum support size of non-zero null t-designs is 2^{t+1} for the Boolean algebras, and the minimal null designs are characterized for some special cases [4], [10]. It is proved that $\prod_{i=0}^{t}(1+q^i)$ is the minimum support size of non-zero null t-designs of the lattices of subspaces of a finite vector space in [6], when $k = t+1$, and the minimal null designs are characterized in [5]. Moreover, there is a general theorem which gives a lower bound for the minimum support size of non-zero null t-designs [4]. In this article, we apply known theorems on the support size of non-zero null t-designs to the poset of dual polar spaces of type B_N, C_N and D_N. In Section 2, we state some known theorems on the number of elements needed to construct a nonzero null design. In Section 3, we apply the results in Section 2 to the posets of dual polar spaces of type B_N, C_N and D_N.

2. Preliminaries

In this section, we give basic definitions and state some known theorems.

For a finite set X, we let $\mathbb{R}[X] = \left\{\sum_{x\in X} c_x x \; : \; c_x \in \mathbb{R}\right\}$ denote the vector space over the real field $\mathbb{R}$ with a basis X. If P is a finite ranked partially ordered set, then we let X_i^P be the set of elements of rank i of P and define the linear map $d_{i,j}^P : \mathbb{R}[X_i^P] \to \mathbb{R}[X_j^P]$, $j \le i$, as follows;

$$\text{for} \quad x \in X_i^P, \quad d_{i,j}^P(x) = \sum_{\substack{y \le x \\ y \in X_j^P}} y .$$

For integers $0 \le t < k$, *null* (t,k)*-design* of a finite ranked poset P is an element of the kernel of $d_{k,t}^P$. We will use $N_P(t,k)$ for the vector space of null (t,k)-designs of P.

For a finite ranked poset P, we say that P satisfies the *downmap condition*, if the following condition is satisfied;

$$\text{for all } t < k, \; d_{k,t}^P(x) = 0 \;\text{ implies }\; d_{k,t'}^P(x) = 0 \;\text{ if } t' \le t.$$

For an element $\omega \in \mathbb{R}[X]$ and $x \in X$, $c_\omega(x)$ is the coefficient of x in ω and the *support of* ω is the set $\mathrm{Supp}(\omega) = \{x \in X \mid c_\omega(x) \neq 0\}$. A *minimal null design* is a nonzero null design with the minimum support size. In the following propositions, we suppose that P is a finite ranked meet semilattice with the downmap condition, where μ_P denote the Möbius function defined on P, and for $x, y \in P$, $x \wedge y$ is the *meet* of x and y. We refer to [11] for the definitions of the meet semilattice and the Möbius function. The following propositions provide general rules to find the minimum support size of nonzero null t-designs and to characterise the minimum null t-designs, whose proofs are in [6].

PROPOSITION 1. *If* $\omega \in N_P(t, t+1)$, $\omega \neq 0$, *then*

$$|Supp(\omega)| \geq \min_{y \in X^P_{t+1}} \left(\sum_{z \leq y} |\mu_P(z, y)| \right) .$$

PROPOSITION 2. *If the lower bound in Proposition 1 gives the tight bound, then the coefficients of a nonzero minimal null design in* $N_P(t, t+1)$ *are* $\pm c$ *or* 0 *for some nonzero constant* $c \in \mathbb{R}$. *Moreover, if* $\omega \in N_P(t, t+1)$ *is a minimal null design, then for each* $y \in Supp(\omega)$ *and* $z \leq y$, *there must be exactly* $|\mu_P(z, y)|$ *many* $x \in Supp(\omega)$ *such that* $x \wedge y = z$ *and* $c_\omega(x) = sign(\mu_P(z, y))c_\omega(y)$.

3. Null designs of dual polar spaces

In this section, we apply the propositions stated in Section 2 to the lattice of subspaces of a finite vector space and to posets of dual polar spaces. We first define posets of isotropic spaces with respect to nondegenerate bilinear form of types B_N, C_N, and D_N (see [1],[3]). Let q be a power of a prime number, and each vector space has the Galois field $\mathbb{F}_q$ as its base field.

DEFINITION 3. (1) A $(2N+1)$-dimensional vector space V_N over $\mathbb{F}_q$ is of type B_N, if it has a basis $\{e_1, \dots, e_N, e_{-1}, \dots, e_{-N}, e_0\}$ with a symmetric bilinear form B, whose Gram matrix is

$$\begin{bmatrix} \mathbf{0} & I_N & 0 \\ I_N & \mathbf{0} & \vdots \\ 0 & \dots & 1 \end{bmatrix} .$$

(2) A $2N$-dimensional vector space V_N over $\mathbb{F}_q$ is of type C_N, if it has a basis $\{e_1, \dots, e_N, e_{-1}, \dots, e_{-N}\}$ with a skew symmetric bilinear form B, whose Gram matrix is

$$\begin{bmatrix} \mathbf{0} & I_N \\ -I_N & \mathbf{0} \end{bmatrix} .$$

(3) A $2N$-dimensional vector space V_N over $\mathbb{F}_q$ is of type D_N, if it has a basis $\{e_1, \dots, e_N, e_{-1}, \dots, e_{-N}\}$ with a symmetric bilinear form B,

whose Gram matrix is

$$\begin{bmatrix} \mathbf{0} & I_N \\ I_N & \mathbf{0} \end{bmatrix}$$

DEFINITION 4. For each type of vector spaces V_N defined in Definition 3, define a poset as the set of isotropic subspaces of V_N with respect to the given bilinear form B, ordered by inclusion. Let us call these posets P_{B_N}, P_{C_N} and P_{D_N} depending on the type of given vector space V_N.

Note that, by Witt's theorem, each poset in Definition 4 has maximal rank N and that a subspace of an isotropic subspace of V_N is again an isotropic subspace of V_N. Therefore, P_{B_N}, P_{C_N} and P_{D_N} are meet semilattices. We let $L_n(q)$ denote the lattice of subspaces of an n-dimensional vector space V over $\mathbb{F}_q$. Note that $\mu_P(z,y) = (-1)^{i-j} q^{\binom{i-j}{2}}$ when $P = L_n(q)$ and $z \in P_j$, $y \in P_i$ (see [11]). It is known that the signed sum of maximal isotropic subspaces of V_N of type D_N forms a null $(N-1,N)$-design, where the sign of each isotropic space is defined as $\mathrm{sign}(y) = (-1)^{\dim(y \wedge y_0)}$ for some fixed isotropic space y_0 (see [9]). If we apply Propositions 1 and 2 to $L_n(q)$, we obtain the following result that serves as the fundamental case for the posets of dual polar spaces. The proofs are in [5] and [6].

PROPOSITION 5. *If P is the subspace lattice of an n-dimensional vector space V, i.e. $P = L_n(q)$,*

(1) the minimum of the support size of non-zero elements of $N_P(t,t+1)$ is $\prod_{i=0}^{t}(1+q^i)$,

(2) if ω is a minimal null design in $N_P(t,t+1)$, then ω is a multiple of the signed sum of maximal isotropic subspaces of some $2(t+1)$-dimensional subspace of V equipped with the symmetric bilinear form of type D_{t+1}.

Observe that for a given isotropic subspace x of V_N of types B_N, C_N and D_N, the interval $[\langle 0\rangle, x]$ is exactly same as the interval $[\langle 0\rangle, x]$ in $L_q(n)$, where $n = 2N+1$ or $n = 2N$ depending on the type. Hence, the value of Möbius functions on P_{B_N}, P_{C_N} and P_{D_N} equals to the value of Möbius function on $L_n(q)$, and the lower bounds given in Proposition 1 are $\prod_{i=0}^{t}(1+q^i)$. We now show that this bound is tight for P_{B_N}, P_{D_N}, but it is not tight for P_{C_N}.

THEOREM 6. *If $P = P_{B_N}, P_{D_N}$, the minimum of the support size of non-zero elements of $N_P(t,t+1)$ is $\prod_{i=0}^{t}(1+q^i)$.*

PROOF. For types B_N and D_N, corresponding Gram matrices contain $\begin{bmatrix} \mathbf{0} & I_{t+1} \\ I_{t+1} & \mathbf{0} \end{bmatrix}$, as a submatrix (note that $t < N$). Hence, by Proposition 5, the signed sum of maximal ($(t+1)$-dimensional) isotropic subspaces of a $(2t+2)$-dimensional subspace of V_N is a null t-design with support size $\prod_{i=0}^{t}(1+q^i)$. Thus, for the spaces of types B_N and D_N, $\prod_{i=0}^{t}(1+q^i)$ is the tight lower bound for the support size of nonzero null $(t,t+1)$-designs. □

We, however, can prove that $\prod_{i=0}^{t}(1+q^i)$ is not the tight bound for some cases of type C_N. For l linearly independent vectors $u_1, u_2, \ldots, u_l$, $\langle u_1, u_2, \ldots, u_l\rangle$ is the l-dimensional space spanned by $u_i, i = 1, \ldots, l$. For two elements $x, y \in L_n(q)$, we let $x \vee y$ be the *join* of x and y, that is the smallest space that contains x and y. The *meet* of x and y, denoted by $x \wedge y$, is the intersection of x and y.

THEOREM 7. *Let $t+1 = N$, $N > 1$, then for $\omega \in N_{P_{C_N}}(t, t+1)$, which is non-zero,*

$$|Supp(\omega)| > \prod_{i=0}^{t}(1+q^i).$$

PROOF. Throughout the proof, we let $P = P_{C_N}$. Let us assume that there is a non-zero element ω of $N_P(t, t+1)$, whose support size $\prod_{i=0}^{t}(1+q^i)$. We apply Proposition 2 to ω throughout the proof to get a contradiction. Without loss of generality, we may assume that $x_0 = \langle e_1, \ldots, e_{t+1}\rangle \in P_{t+1}$ is in Supp(ω) and $c_\omega(x_0) = +1$. Let $z_0 = \langle e_1, \ldots, e_{t-1}\rangle \in P_{t-1}$ and $y_1 = z_0 \vee \langle e_t\rangle$, $y_2 = z_0 \vee \langle e_{t+1}\rangle$ and $y_3 = z_0 \vee \langle e_t, e_{t+1}\rangle$ be elements in P_t. Then, since $y_i \le x_0$ for each $i = 1, 2, 3$, by Proposition 2, there must be unique x_i in Supp(ω) such that $x_0 \wedge x_i = y_i$ and $c_\omega(x_i) = -1$. We also let $A = \{x \in \text{Supp}(\omega) \mid x_0 \wedge x = z_0\}$, then by Proposition 2, $|A| = q$ and $c_\omega(x) = +1$ for all $x \in A$. Choose two vectors w_1, w_2 so that $x_i = y_i \vee \langle w_i\rangle$ for $i = 1, 2$, and $x_1^A \in A$, then $x_i \wedge x_1^A$, $i = 1, 2$, is t-dimensional since both x_i and x_1^A contain $z_0 \in P_{t-1}$, and $c_\omega(x_i) = -c_\omega(x_1^A)$. Hence $x_1^A \wedge x_1 = z_0 \vee \langle v_1\rangle$ for some $v_1 = w_1 + \alpha e_t$, and $x_1^A \wedge x_2 = z_0 \vee \langle v_2\rangle$ for some $v_2 = w_2 + \beta e_{t+1}$, $\alpha, \beta \in \mathbb{F}_q$ (note that $x_1^A \wedge x_i$ can not be y_i for $i = 1, 2$). Observe that $x_i = y_i \vee \langle w_i\rangle = y_i \vee \langle v_i\rangle$ for $i = 1, 2$, and $x_1^A = z_0 \vee \langle v_1, v_2\rangle$.

Since $t+1 = N$, x_1 and x_2 are maximal isotropic spaces and $x_1 \vee \langle v_2\rangle$ and $x_2 \vee \langle v_1\rangle$ are not isotropic spaces. Therefore, $B(e_t, v_2)$ and $B(e_{t+1}, v_1)$ are non-zero and without loss of generality, we assume that $B(e_t, v_2) = B(e_{t+1}, v_1) = 1$. Let us consider another element x_2^A of A, then by the same argument as above, $x_2^A = z_0 \vee \langle v_1', v_2' \rangle$ where $v_1' = v_1 + \alpha' e_t$ and $v_2' = v_2 + \beta' e_{t+1}$, for $\alpha', \beta' \in \mathbb{F}_q$. Since x_2^A is isotropic, $B(v_1', v_2') = B(v_1 + \alpha' e_t, v_2 + \beta' e_{t+1}) = \beta' B(v_1, e_{t+1}) + \alpha' B(e_t, v_2) = -\beta' + \alpha'$ should be 0. Hence, we have $\alpha' = \beta'$. Above observation implies that $A = \{z_0 \vee \langle v_1 + \gamma e_t, v_2 + \gamma e_{t+1} \rangle \mid \gamma \in \mathbb{F}_q\}$, since $|A| = q$.

We now consider $x_3 \in \mathrm{Supp}(\omega)$ and $x_3^A = z_0 \vee \langle v_1 + e_t, v_2 + e_{t+1} \rangle \in A$. Since $x_3 \wedge x_1^A$ is t-dimensional, $x_3 = y_3 \vee \langle v_3 \rangle$, where $v_3 = \alpha_1 v_1 + \alpha_2 v_2$, $\alpha_1, \alpha_2 \in \mathbb{F}_q$. x_3 is an isotropic space, so we have $B(e_t + e_{t+1}, v_3) = \alpha_2 + \alpha_1 = 0$ and, we can say that $x_3 = z_0 \vee \langle e_t + e_{t+1}, v_1 - v_2 \rangle$. Moreover, since $x_3 \wedge x_3^A$ is t-dimensional, $\langle v_1 + e_t, v_2 + e_{t+1} \rangle \wedge \langle e_t + e_{t+1}, v_1 - v_2 \rangle$ should be 1-dimensional but it is $\langle 0 \rangle$, so we have a contradiction. This completes the proof. □

REMARK 8. Note that the skew symmetry of the bilinear form of type C_N plays the central role in the proof of Theorem 7.

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