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# ON $(\alpha, \beta)$-SKEW-COMMUTING AND $(\alpha, \beta)$-SKEW-CENTRALIZING MAPS IN RINGS WITH LEFT IDENTITY

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ABSTRACT. Let $R$ be a ring with left identity. Let $G : R \times R \to R$ be a symmetric biadditive mapping and $g$ the trace of $G$. Let $\alpha : R \to R$ be an endomorphism and $\beta : R \to R$ an epimorphism. In this paper we show the following: (i) Let $R$ be 2-torsion-free. If $g$ is $(\alpha, \beta)$-skew-commuting on $R$, then we have $G = 0$. (ii) If $g$ is $(\beta, \beta)$-skew-centralizing on $R$, then $g$ is $(\beta, \beta)$-commuting on $R$. (iii) Let $n \geq 2$. Let $R$ be $(n+1)!$-torsion- free. If $g$ is $n$-$(\alpha, \beta)$-skew-commuting on $R$, then we have $G = 0$. (iv) Let $R$ be 6-torsion-free. If $g$ is 2-$(\alpha, \beta)$-commuting on $R$, then $g$ is $(\alpha, \beta)$-commuting on $R$.

## 1. Preliminaries

Throughout, all rings $R$ will be associative, and the center of a ring will be denoted by $Z$. Let $\alpha$, $\beta$, $\theta$, $\varphi$ be additive mappings of $R$ into $R$ and let $x$, $y \in R$. As usual, the commutator $yx - xy$ will be denoted by $[y, x]$, and for convenience, the product $yx + xy$, $y\alpha(x) + \beta(x)y$, and $y\alpha(x) - \beta(x)y$ by $\langle y, x\rangle$, $\langle y, x\rangle_{(\alpha,\beta)}$ and $[y, x]_{(\alpha,\beta)}$, respectively. We will use extensively the following basic properties: for any $x$, $y$, $z \in R$, $[xy, z] = x[y, z] + [x, z]y$, $[x, y + z]_{(\alpha,\beta)} = [x, y]_{(\alpha,\beta)} + [x, z]_{(\alpha,\beta)}$, $\langle x, y + z\rangle_{(\alpha,\beta)} = \langle x, y\rangle_{(\alpha,\beta)} + \langle x, z\rangle_{(\alpha,\beta)}$, $[x + y, z]_{(\alpha,\beta)} = [x, z]_{(\alpha,\beta)} + [y, z]_{(\alpha,\beta)}$, $\langle x + y, z\rangle_{(\alpha,\beta)} = \langle x, z\rangle_{(\alpha,\beta)} + \langle y, z\rangle_{(\alpha,\beta)}$.

Let $f$ be a mapping from $R$ into $R$, and $S$ a nonempty subset of $R$. Then $f$ is called $(\alpha, \beta)$-skew-commuting (resp. $(\alpha, \beta)$-skew-centralizing) on $S$ if $\langle f(x), x\rangle_{(\alpha,\beta)} = 0$ (resp. $\langle f(x), x\rangle_{(\alpha,\beta)} \in Z$) for all $x \in S$. Similarly $f$ is said to be $(\alpha, \beta)$-commuting on $S$ if $[f(x), x]_{(\alpha,\beta)} = 0$ for

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all $x \in S$. If we let $\alpha = \beta = 1$ (the identity map on $R$), then $f$ is called simply skew-commuting, skew-centralizing and commuting on $S$, respectively.

As a simple example, let

$$R = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} : w,\, x,\, y,\, z \in I \right\}$$

be a ring and

$$S = \left\{ \begin{pmatrix} w & x \\ 0 & 0 \end{pmatrix} : w,\, x \in I \right\} \subset R,$$

where $I$ is the set of integers.

Let $\alpha,\, \beta : R \to R$ be mappings defined by

$$\alpha \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} -w & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \beta \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & -x \\ 0 & 0 \end{pmatrix}.$$

Let us define the mapping $f : R \to R$ by

$$f \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $f$ is $(\alpha, \beta)$-skew-commuting on $S$ but not skew-commuting on $S$.

A mapping $G : R \times R \to R$ is said to be symmetric if $G(x, y) = G(y, x)$ for all $x, y \in R$. A mapping $g : R \to R$ defined by $g(x) = G(x, x)$ for all $x \in R$, where $G : R \times R \to R$ is a symmetric mapping, is called the trace of $G$. It is obvious that, in case when $G : R \times R \to R$ is a symmetric mapping which is also biadditive (i.e., additive in both arguments), the trace $g$ of $G$ satisfies the relation $g(x + y) = g(x) + g(y) + 2G(x, y)$ for all $x, y \in R$.

The study of (skew-)centralizing and (skew-)commuting mappings has been investigated by many authors (see, e.g., Brešar [3], Vukman [5] and references therein). In this connection, Bell and Lucier [1] obtained some results concerning skew-commuting, and skew-centralizing additive maps in which the condition of primeness is replaced by the existence of a left identity element.

We here investigate symmetric biadditive maps with the generalized skew-commuting and skew-centralizing traces, that is, $(\alpha, \beta)$-skew-commuting and $(\alpha, \beta)$-skew-centralizing ones, in rings with left identity.

## 2. Results

We begin with the following result.

THEOREM 1. *Let $R$ be a 2-torsion-free ring with left identity $e$. Let $\alpha : R \to R$ be an endomorphism and $\beta : R \to R$ an epimorphism. Let $G : R \times R \to R$ be a symmetric biadditive mapping and $g$ the trace of $G$. If $g$ is $(\alpha, \beta)$-skew-commuting on $R$, then we have $G = 0$.*

PROOF. We are given that

$$\text{(1)} \qquad \langle g(x), x\rangle_{(\alpha,\beta)} = g(x)\alpha(x) + \beta(x)g(x) = 0 \text{ for all } x \in R.$$

First, observe that $\beta(e)$ is also a left identity of $R$ since $\beta$ is onto. From this and (1), it follows that

$$\text{(2)} \qquad \langle g(e), e\rangle_{(\alpha,\beta)} = g(e)\alpha(e) + g(e) = 0;$$

and right-multiplying by $\alpha(e)$ gives $2g(e)\alpha(e) = 0 = g(e)\alpha(e)$. Hence, by (2), we get $g(e) = 0$.

Let us replace $x$ by $x + e$ in (1). We then have, for all $x \in R$,

$$\text{(3)} \qquad \langle g(x), e\rangle_{(\alpha,\beta)} + 2\langle G(x,e), x\rangle_{(\alpha,\beta)} + 2\langle G(x,e), e\rangle_{(\alpha,\beta)} = 0.$$

Substituting $-x$ for $x$ in (3) and comparing (3) with the result, we obtain

$$\text{(4)} \qquad \langle G(x,e), e\rangle_{(\alpha,\beta)} = G(x,e)\alpha(e) + G(x,e) = 0 \text{ for all } x \in R$$

since $g$ is an even function and $R$ is 2-torsion free. Right multiplication of (4) by $\alpha(e)$ gives $2G(x,e)\alpha(e) = 0 = G(x,e)\alpha(e)$, and so, by (4), we have $G(x,e) = 0$ for all $x \in R$.

Therefore we arrive at

$$g(x+e) = g(x) + g(e) + 2G(x,e) = g(x) \text{ for all } x \in R.$$

Since $g$ is $(\alpha, \beta)$-skew-commuting on $R$, the relation $g(x+e)\alpha(x+e) + \beta(x+e)g(x+e) = 0$ becomes $g(x)\alpha(x)+g(x)\alpha(e)+\beta(x)g(x)+\beta(e)g(x) = 0$, and thus we obtain

$$\text{(5)} \qquad g(x)\alpha(e) + g(x) = 0 \text{ for all } x \in R.$$

Right-multiplying by $\alpha(e)$ in (5), we get $2g(x)\alpha(e) = 0 = g(x)\alpha(e)$, and hence the relation (5) implies $g(x) = 0$ for all $x \in R$ which gives the conclusion. $\square$

The next result is to improve the Bell and Lucier's result [1, Theorem 2].

COROLLARY 2. *Let $R$ be a 2-torsion-free ring with left identity $e$. Let $\alpha$, $\theta : R \to R$ be endomorphisms and $\beta$, $\varphi : R \to R$ epimorphisms. If $f$ is an additive map on $R$ such that the mapping $x \mapsto \langle f(x), x\rangle_{(\alpha,\beta)}$ is $(\theta,\varphi)$-skew-commuting on $R$, then we have $f = 0$.*

PROOF. Defining a mapping $G : R \times R \to R$ by

$$G(x,y) = \langle f(x), y\rangle_{(\alpha,\beta)} + \langle f(y), x\rangle_{(\alpha,\beta)} \text{ for all } x,\ y \in R;$$

and a mapping $g : R \to R$ by $g(x) = G(x,x)$ for all $x \in R$, it is obvious that $G$ is symmetric and biadditive, and that $g$ is the trace of $G$. The hypothesis that the mapping $x \mapsto \langle f(x), x\rangle_{(\alpha,\beta)}$ is $(\theta,\varphi)$-skew-commuting on $R$ is equivalent to the fact that $g$ is $(\theta,\varphi)$-skew-commuting on $R$, and so Theorem 1 tells us that $g = 0$, that is, $f$ is $(\alpha,\beta)$-skew-commuting on $R$, from which it follows that

$$f(e)\alpha(e) + \beta(e)f(e) = f(e)\alpha(e) + f(e) = 0; \tag{6}$$

and right-multiplying by $\alpha(e)$ gives $2f(e)\alpha(e) = 0 = f(e)\alpha(e)$. By (6), we get $f(e) = 0$ and so $f(x+e) = f(x)$ for all $x \in R$.
The condition that $f(x+e)\alpha(x+e) + \beta(x+e)f(x+e) = 0$ now makes $f(x)\alpha(x) + f(x)\alpha(e) + \beta(x)f(x) + f(x) = 0$, and it follows that

$$f(x)\alpha(e) + f(x) = 0 \text{ for all } x,\ y \in R. \tag{7}$$

Right-multiplying by $\alpha(e)$, we get $2f(x)\alpha(e) = 0 = f(x)\alpha(e)$, so by (7) we have $f(x) = 0$ for all $x \in R$. $\square$

We continue our investigation with the next result.

THEOREM 3. *Let $R$ be a 2-torsion-free ring with left identity $e$. Let $\beta : R \to R$ be an epimorphism. Let $G : R \times R \to R$ be a symmetric biadditive mapping and $g$ the trace of $G$. If $g$ is $(\beta,\beta)$-skew-centralizing on $R$, then $g$ $(\beta,\beta)$-commuting on $R$.*

PROOF. Suppose that

$$\langle g(x), x\rangle_{(\beta,\beta)} = g(x)\beta(x) + \beta(x)g(x) \in Z \quad \text{for all } x \in R. \tag{8}$$

Since $\beta(e)$ is a left identity of $R$ by the ontoness of $\beta$, our assumption implies

$$g(e)\beta(e) + \beta(e)g(e) = g(e)\beta(e) + g(e) \in Z. \tag{9}$$

Commuting with $\beta(e)$ gives $g(e) = g(e)\beta(e)$; and by (9) $2g(e) \in Z$, hence $g(e) \in Z$.
Let us replace $x$ by $x+e$ in (8). Then we get, for all $x \in R$,

$$\begin{aligned} &g(x)\beta(e) + 2\beta(x)g(e) + 2G(x,e)\beta(x) \\ &+ 2G(x,e)\beta(e) + g(x) + 2\beta(x)G(x,e) + 2G(x,e) \in Z. \end{aligned} \tag{10}$$

Substituting $-x$ for $x$ in (10) and comparing (10) with the result, we obtain

$$\beta(x)g(e) + G(x,e)\beta(e) + G(x,e) \in Z \quad \text{for all} \quad x \in R \tag{11}$$

because $g$ is even and $R$ is 2-torsion free.

Since $g(e) \in Z$ and $\beta(e)$ is a left identity of $R$, commuting with $\beta(e)$ in (11) gives

$$[G(x,e), \beta(e)] = 0 \quad \text{for all} \quad x \in R; \tag{12}$$

and thus, by (12), we have $G(x,e) = G(x,e)\beta(e)$ for all $x \in R$.

Now (11) comes to

$$\beta(x)g(e) + 2G(x,e) \in Z \quad \text{for all} \quad x \in R; \tag{13}$$

and commuting with $\beta(x)$ in (13) gives

$$2[G(x,e), \beta(x)] = 0 = [G(x,e), \beta(x)] \quad \text{for all} \quad x \in R \tag{14}$$

which, by the ontoness of $\beta$, gives $G(x,e) \in Z$ for all $x \in R$.

In view of $G(x,e) = G(x,e)\beta(e)$ and $G(x,e) \in Z$, the relation (10) can be rewritten in the form

$$g(x)\beta(e) + g(x) + 2\beta(x)g(e) + 4\beta(x)G(x,e) \in Z \quad \text{for all} \quad x \in R. \tag{15}$$

Commuting with $\beta(e)$ in (15) and then using the fact that $[y, \beta(e)]z = 0$ for all $y, z \in R$ yield

$$(16) \qquad [g(x), \beta(e)]\beta(e) + [g(x), \beta(e)] = 0 \ \text{ for all } \ x \in R;$$

and right-multiplying by $\beta(e)$ gives

$$2[g(x), \beta(e)]\beta(e) = 0 = [g(x), \beta(e)]\beta(e)$$

and so it follows from (16) that $g(x) = g(x)\beta(e)$ for all $x \in R$.

Consequently, we see that the relation (15) becomes

$$(17) \qquad g(x) + \beta(x)g(e) + 2\beta(x)G(x, e) \in Z \ \text{ for all } \ x \in R.$$

since $R$ is 2-torsion free.

Commuting with $\beta(x)$ in (17), we have, for all $x \in R$,

$$[g(x), \beta(x)] = 0 \ \text{ for all } \ x \in R$$

which completes the proof. □

Let $\alpha, \beta : R \to R$ be endomorphisms. By analogy with the definition of $n$-commutativity introduced in [2] and [4], for $n \geq 2$ we define a mapping $f : R \to R$ to be $n$-$(\alpha, \beta)$-skew-commuting (resp. $n$-$(\alpha, \beta)$-skew-centralizing) on the subset $S$ if $\langle f(x), x^n \rangle_{(\alpha,\beta)} = 0$ (resp. $\langle f(x), x^n \rangle_{(\alpha,\beta)} \in Z$) for all $x \in S$, and $f$ is said to be $n$-$(\alpha, \beta)$-commuting on $S$ if $[f(x), x^n]_{(\alpha,\beta)} = 0$ for all $x \in S$. Of course, in case when $\alpha = \beta = 1$ (the identity map on $R$), $f$ is simply called $n$-skew-commuting, $n$-skew-centralizing and $n$-commuting on $S$, respectively.

Here we extend the results on $(\alpha, \beta)$-skew-commuting maps to $n$-$(\alpha, \beta)$-skew-commuting ones.

THEOREM 4. *Let $n \geq 2$. Let $R$ be a $(n+1)!$-torsion-free ring with left identity $e$. Let $\alpha : R \to R$ be an endomorphism and $\beta : R \to R$ an epimorphism. Let $G : R \times R \to R$ be a symmetric biadditive mapping and $g$ the trace of $G$. If $g$ is $n$-$(\alpha, \beta)$-skew-commuting on $R$, then we have $G = 0$.*

PROOF. Assume that

$$(18) \qquad \langle g(x), x^n \rangle_{(\alpha,\beta)} = 0 \ \text{ for all } \ x \in R.$$

Note that $g(e) = 0$ by the same argument used in the proof of Theorem 1.

Let $t$ be any positive integer. Replacing $x$ by $x + te$ in (18) and using $g(x + te) = g(x) + t^2 g(e) + 2tG(x, e)$ for all $x \in R$, we obtain

$$tP_1(x, e) + t^2 P_2(x, e) + \cdots + t^{n+1} P_{n+1}(x, e) = 0 \quad \text{for all} \quad x \in R,$$

where $P_k(x, e)$ is the sum of terms involving $x$ and $e$ such that $P_k(x, te) = t^k P_k(x, e)$, $k = 1, 2, \cdots, n + 1$.

Replacing $t$ by $1, 2, \cdots, n+1$ in turn, and expressing the resulting system of $n+1$ homogeneous equations with the variables $P_1(x, e), P_2(x, e), \cdots, P_{n+1}(x, e)$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ n+1 & (n+1)^2 & \ldots & (n+1)^{n+1} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than $n + 1$, and since $R$ is $(n + 1)!$-torsion free, it follows immediately that for each $k = 1, 2, \cdots, n + 1$,

$$P_k(x, e) = 0 \quad \text{for all} \quad x \in R.$$

In particular, we have, for all $x \in R$,

$$P_{n+1}(x, e) = 2\langle G(x, e), e^n \rangle_{(\alpha,\beta)} = 0 \tag{19}$$

and

$$\begin{aligned} P_n(x, e) =& \langle g(x), e^n \rangle_{(\alpha,\beta)} + 2\langle G(x, e), xe^{n-1} \rangle_{(\alpha,\beta)} \\ & + 2\langle G(x, e), exe^{n-2} \rangle_{(\alpha,\beta)} + 2\langle G(x, e), e^2xe^{n-3} \rangle_{(\alpha,\beta)} \\ & + \cdots + 2\langle G(x, e), e^{n-2}xe \rangle_{(\alpha,\beta)} + 2\langle G(x, e), e^{n-1}x \rangle_{(\alpha,\beta)} = 0. \end{aligned} \tag{20}$$

By (19), we obtain that, for all $x \in R$,

$$2\{G(x, e)\alpha(e) + \beta(e)G(x, e)\} = 0 = G(x, e)\alpha(e) + G(x, e); \tag{21}$$

and right-multiplying by $\alpha(e)$ and using (21), we get $G(x, e) = 0$ for all $x \in R$. Hence this forces (20) to

$$\langle g(x), e^n \rangle_{(\alpha,\beta)} = g(x)\alpha(e) + \beta(e)g(x) = g(x)\alpha(e) + g(x) = 0 \tag{22}$$

for all $x \in R$. Multiplying by $\alpha(e)$ on the right and utilizing (22), we conclude that $g(x) = 0$ for all $x \in R$. This completes the proof. □

COROLLARY 5. *Let $n \geq 2$. Let $R$ be a $(n+1)!$-torsion-free ring with left identity $e$. Let $\alpha : R \to R$ be an endomorphism and $\beta : R \to R$ an epimorphism such that $\alpha$ is $(\beta, \beta)$-commuting on $R$. If $f$ is an additive map on $R$ which is $n$-$(\alpha, \beta)$-skew-centralizing on $R$, then $f$ is $(\beta, \beta)$-commuting on $R$.*

PROOF. Since $f(x)\alpha(x)^n + \beta(x)^n f(x) \in Z$ for all $x \in R$, we have

$$[f(x)\alpha(x)^n + \beta(x)^n f(x),\, \beta(x)] = 0 \quad \text{for all} \quad x \in R;$$

whence $[f(x),\, \beta(x)]\alpha(x)^n + f(x)[\alpha(x)^n,\, \beta(x)] + \beta(x)^n[f(x),\, \beta(x)] = 0$ which reduces to

$$(23) \qquad [f(x),\, \beta(x)]\alpha(x)^n + \beta(x)^n[f(x),\, \beta(x)] = 0 \quad \text{for all} \quad x \in R$$

because $\alpha$ is $(\beta, \beta)$-commuting on $R$, i.e., $[\alpha(x),\, \beta(x)] = 0$ for all $x \in R$.

We introduce the mapping $G : R \times R \to R$ defined by

$$G(x, y) = [f(x),\, \beta(y)] + [f(y),\, \beta(x)] \quad \text{for all } x, y \in R,$$

and the mapping $g : R \to R$ by $g(x) = G(x, x)$ for all $x \in R$, it is obvious that $G$ is symmetric and biadditive, and that $g$ is the trace of $G$.

Now the relation (23) is equivalent to the fact that $g$ is $n$-$(\alpha, \beta)$-skew-commuting, and so it follows from Theorem 4 that $g(x) = 2[f(x), \beta(x)] = 0$ for all $x \in R$. Since $R$ is 2-torsion-free, we obtain the conclusion of the theorem. □

We provide the following example supporting the notion of $(\alpha, \beta)$-skew-commutativity.

EXAMPLE. Let

$$R = \left\{ \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ y & z & w \end{pmatrix} : w,\, x,\, y,\, z \in \mathbb{C} \right\},$$

where $\mathbb{C}$ is the set of complex numbers. Then $R$ is a noncommutative associative ring with left identity as the unit matrix under the usual matrix addition and multiplication. The mapping $\alpha : R \to R$ defined by

$$\alpha \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ y & z & w \end{pmatrix} = \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ 0 & z & w \end{pmatrix}$$

is an endomorphism and the mapping $\beta : R \to R$ defined by

$$\beta \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ y & z & w \end{pmatrix} = \begin{pmatrix} w & 0 & 0 \\ -x & w & 0 \\ y & -z & w \end{pmatrix}$$

is an epimorphism. We define a mapping $f : R \to R$ by

$$f \begin{pmatrix} w & 0 & 0 \\ x & w & 0 \\ y & z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ y & 0 & 0 \end{pmatrix}.$$

It is obvious that $f$ is additive.

Now, defining a mapping $G : R \times R \to R$ by

$$G(X, Y) = [f(X), Y] + [f(Y), X] \quad \text{for all } X,\, Y \in R,$$

we can easily check that $G$ is symmetric and biadditive, and that the map $g$ on $R$ defined by $g(X) = G(X, X)$ is the $n$-$(\alpha, \beta)$-skew-commuting trace of $G$ $(n \geq 1)$. It is trivial to see that $G = 0$.

On the other hand, putting

$$Z = \left\{ \begin{pmatrix} w & 0 & 0 \\ 0 & w & 0 \\ y & 0 & w \end{pmatrix} : w,\, y \in \mathbb{C} \right\},$$

it is immediate to see that $Z$ is the center of $R$. Defining a mapping $G : R \times R \to R$ by

$$G(X, Y) = \langle f(X), Y \rangle + \langle f(Y), X \rangle \quad \text{for all } X,\, Y \in R,$$

$G$ is also symmetric and biadditive, and the map $g$ on $R$ defined by $g(X) = G(X, X)$ is the $(\beta, \beta)$-skew-centralizing trace of $G$. It is clear that $g$ is $(\beta, \beta)$-commuting on $R$.

We now close our investigation with the following result.

THEOREM 6. *Let $R$ be a 6-torsion-free ring with left identity $e$. Let $\alpha : R \to R$ be an endomorphism and $\beta : R \to R$ an epimorphism. Let $G : R \times R \to R$ be a symmetric biadditive mapping and $g$ the trace of $G$. If $g$ is 2-$(\alpha, \beta)$-commuting on $R$, then $g$ is $(\alpha, \beta)$-commuting on $R$.*

Proof. Let us define a mapping $h : R \to R$ by $h(x) = [g(x), x]_{(\alpha,\beta)}$ for all $x \in R$. Our assumption can now be written in the form

$$\langle h(x),\, x\rangle_{(\alpha,\beta)} = [g(x), x^2]_{(\alpha,\beta)} = 0 \text{ for all } x \in R. \tag{24}$$

Since $\beta(e)$ is also a left identity of $R$ by the ontoness of $\beta$, it follows that

$$h(e)\alpha(e) + \beta(e)h(e) = h(e)\alpha(e) + h(e) = 0 \text{ for all } x \in R; \tag{25}$$

and right-multiplying by $\alpha(e)$ gives $2h(e)\alpha(e) = 0 = h(e)\alpha(e)$. Hence, by (25), we get $h(e) = [g(e), e]_{(\alpha,\beta)} = 0$. Note that $h$ is odd and for all $x \in R$,

$$\begin{aligned} h(x+e) =& h(x) + [g(e), x]_{(\alpha,\beta)} + 2[G(x,e), e]_{(\alpha,\beta)} \\ &+ [g(x), e]_{(\alpha,\beta)} + 2[G(x,e), x]_{(\alpha,\beta)}. \end{aligned} \tag{26}$$

We claim that $h(x+e) = h(x)$ for all $x \in R$.
Replacing $x$ by $x+e$ in (24) and using (26), we have, for all $x \in R$,

(27)

$$\begin{aligned} 0 =& \langle h(x+e),\, x+e\rangle_{(\alpha,\beta)} \\ =& h(x)\alpha(e) + [g(e), x]_{(\alpha,\beta)}\alpha(x) + [g(e), x]_{(\alpha,\beta)}\alpha(e) \\ &+ 2[G(x,e), e]_{(\alpha,\beta)}\alpha(x) + 2[G(x,e), e]_{(\alpha,\beta)}\alpha(e) + [g(x), e]_{(\alpha,\beta)}\alpha(x) \\ &+ [g(x), e]_{(\alpha,\beta)}\alpha(e) + 2[G(x,e), x]_{(\alpha,\beta)}\alpha(x) + 2[G(x,e), x]_{(\alpha,\beta)}\alpha(e) \\ &+ h(x) + \beta(x)[g(e), x]_{(\alpha,\beta)} + [g(e), x]_{(\alpha,\beta)} + 2\beta(x)[G(x,e), e]_{(\alpha,\beta)} \\ &+ 2[G(x,e), e]_{(\alpha,\beta)} + \beta(x)[g(x), e]_{(\alpha,\beta)} + [g(x), e]_{(\alpha,\beta)} \\ &+ 2\beta(x)[G(x,e), x]_{(\alpha,\beta)} + 2[G(x,e), x]_{(\alpha,\beta)}. \end{aligned}$$

Substituting $-x$ for $x$ in (27) and comparing (27) with the result, we get, for all $x \in R$,

(28)

$$\begin{aligned} &[g(e), x]_{(\alpha,\beta)}\alpha(x) + 2[G(x,e), e]_{(\alpha,\beta)}\alpha(x) + [g(x), e]_{(\alpha,\beta)}\alpha(e) \\ &+ 2[G(x,e), x]_{(\alpha,\beta)}\alpha(e) + \beta(x)[g(e), x]_{(\alpha,\beta)} + 2\beta(x)[G(x,e), e]_{(\alpha,\beta)} \\ &+ [g(x), e]_{(\alpha,\beta)} + 2[G(x,e), x]_{(\alpha,\beta)} = 0; \end{aligned}$$

and right multiplication of (28) by $\alpha(e)$ gives, for all $x \in R$,

$$(29)\quad \begin{aligned} 0 =&[g(e),x]_{(\alpha,\beta)}\alpha(x)\alpha(e) + 2[G(x,e),e]_{(\alpha,\beta)}\alpha(x)\alpha(e) \\ &+ 2[g(x),e]_{(\alpha,\beta)}\alpha(e) + 4[G(x,e),x]_{(\alpha,\beta)}\alpha(e) \\ &+ \beta(x)[g(e),x]_{(\alpha,\beta)}\alpha(e) + 2\beta(x)[G(x,e),e]_{(\alpha,\beta)}\alpha(e). \end{aligned}$$

Let us put $x+e$ instead of $x$ in (29) and utilize (29). Then we obtain

$$6[g(e),x]_{(\alpha,\beta)}\alpha(e) + 12[G(x,e),e]_{(\alpha,\beta)}\alpha(e) = 0,$$

and so

$$(30)\quad [g(e),x]_{(\alpha,\beta)}\alpha(e) + 2[G(x,e),e]_{(\alpha,\beta)}\alpha(e) = 0 \quad \text{for all} \quad x \in R;$$

and this relation (30) yields, for all $x \in R$,

$$(31)\quad \begin{aligned} &[g(e),x]_{(\alpha,\beta)}\alpha(x) + 2[G(x,e),e]_{(\alpha,\beta)}\alpha(x) \\ &= [g(e),x]_{(\alpha,\beta)}\alpha(ex) + 2[G(x,e),\ e]_{(\alpha,\beta)}\alpha(ex) \\ &= \{[g(e),x]_{(\alpha,\beta)}\alpha(e) + 2[G(x,e),e]_{(\alpha,\beta)}\alpha(e)\}\alpha(x) = 0. \end{aligned}$$

Hence the relation (29) becomes

$$2[g(x),e]_{(\alpha,\beta)}\alpha(e) + 4[G(x,e),x]_{(\alpha,\beta)}\alpha(e) = 0,$$

which gives

$$(32)\quad [g(x),e]_{(\alpha,\beta)}\alpha(e) + 2[G(x,e),x]_{(\alpha,\beta)}\alpha(e) = 0 \;\text{ for all }\; x \in R.$$

According to (31) and (32), we therefore can be written (28) in the form

$$(33)\quad \begin{aligned} &\beta(x)[g(e),x]_{(\alpha,\beta)} + 2\beta(x)[G(x,e),e]_{(\alpha,\beta)} \\ &+ [g(x),e]_{(\alpha,\beta)} + 2[G(x,e),x]_{(\alpha,\beta)} = 0 \;\text{ for all }\; x \in R. \end{aligned}$$

Finally, replacing $x$ by $x+e$ in (33) and applying (33) to the result, we obtain

$$3[g(e),x]_{(\alpha,\beta)} + 6[G(x,e),e]_{(\alpha,\beta)} = 0,$$

which implies that

$$[g(e), x]_{(\alpha,\beta)} + 2[G(x,e), e]_{(\alpha,\beta)} = 0 \quad \text{for all} \quad x \in R; \tag{34}$$

and the relation (33) with (34) yields

$$[g(x), e]_{(\alpha,\beta)} + 2[G(x,e), x]_{(\alpha,\beta)} = 0 \quad \text{for all} \quad x \in R. \tag{35}$$

By applying (34) and (35) to (26), we now obtain that $h(x+e) = h(x)$ for all $x \in R$, as claimed.

Since $\langle h(x), x\rangle_{(\alpha,\beta)} = 0$ for all $x \in R$, the relation $h(x+e)\alpha(x+e) + \beta(x+e)h(x+e) = 0$ becomes $h(x)(\alpha(x)+\alpha(e)) + (\beta(x)+\beta(e))h(x) = 0$, and it follows that

$$h(x)\alpha(e) + h(x) = 0 \quad \text{for all} \quad x \in R. \tag{36}$$

Right-multiplying by $\alpha(e)$ in (36), we get $2h(x)\alpha(e) = 0 = h(x)\alpha(e)$, and hence the relation (36) yields $h(x) = 0$ for all $x \in R$ which gives the conclusion. □

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