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WEIGHTED COMPOSITION OPERATORS FROM BERGMAN SPACES INTO WEIGHTED BLOCH SPACES

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ABSTRACT. In this paper we study bounded and compact weighted composition operator, induced by a fixed analytic function and an analytic self-map of the open unit disk, from Bergman space into weighted Bloch space. As a corollary, obtain the characterization of composition operator from Bergman space into weighted Bloch space.

1. Introduction

Let D be the open unit disk in the complex plane and $\varphi : D \to D$ be an analytic self map, the composition operator C_φ with symbol φ is defined by $C_\varphi(f) = f(\varphi(z))$ for f analytic on D. It is a well know consequence of Littlewood's subordination principle that φ induces through composition a bounded linear operator on the classical Hardy and Bergman spaces. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces, the book [2] contains plenty of information. Problems of this kind were studied recently for composition operators between Bloch type spaces and Hardy and Besov spaces [9], between Bloch spaces and Dirichlet space [7], to mention only some related work. Let u be a fixed function on D, we can define a linear operator uC_φ on the space of analytic functions on D, called a weighted composition operator, by $uC_\varphi f = u \cdot (f \circ \varphi)$ for a function f analytic on D. We can regard this operator as a generalization of multiplication operator and a composition operator.

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Let dA denote Lebesgue area measure in the unit disk D normalized so that $A(D) = 1$. If $0 < p < +\infty$, $-1 < \alpha < \infty$, the weighted Bergman space A^p_α is the set of all analytic functions f on the unit disk D such that

$$\|f\|^p_{A^p_\alpha} = \int_D |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

When $\alpha = 0$, A^p is called Bergman space, we denote $\|f\|_{A^p}$ by $\|f\|_p$. It is clear that $A^q \subset A^p$, if $0 < p < q < \infty$. Note that $\|f\|_p$ is a true norm if and only if $1 \leq p < \infty$. When $0 < p < 1$, A^p is an F-space with respect to the translation-invariant metric defined by $d_p(f,g) = \|f-g\|^p_p$.

The growth of functions in weighted Bergman spaces is essential in our study, the following sharp estimate (see Lemma 3.2 of [4]) will be useful.

LEMMA 1.1. *Let $f \in A^p_\alpha$, then for every point z in D we have*

$$|f(z)| \leq \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{2+\alpha}{p}}}.$$

An analytic function f on D is said to belong to the Bloch space $\mathcal{B}$ if

$$B(f) = \sup_{z \in D}(1-|z|^2)|f'(z)| < \infty.$$

The expression $B(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}} = |f(0)| + B(f)$. It makes $\mathcal{B}$ into a conformally invariant Banach space.

An analytic function f on D is said to belong to weighted Bloch space $\mathcal{B}_{log}$ if

$$\|f\|_{\mathcal{B}_{log}} = \sup_{a \in D}(1-|z|^2)\log\frac{2}{1-|z|^2}|f'(z)| < \infty.$$

The expression $\|f\|_{\mathcal{B}_{log}}$ defines a seminorm while the natural norm is given by $\|f\|_{log} = |f(0)| + \|f\|_{\mathcal{B}_{log}}$. It makes $\mathcal{B}_{log}$ into a Banach space. In [10], Zhu proved that for $f \in H(D)$, $f\mathcal{B} \subset \mathcal{B}$ if and only if $f \in H^\infty \bigcap \mathcal{B}_{log}$. In [1], K.R.M.Attele proved that for $f \in L^2_a(D)$, the Hankel operator $L^1_a \to L^1$ is bounded if and only if $f \in \mathcal{B}_{log}$.

In [6], Perez-Gonzalez and Xiao studied composition operator from Hardy space into Bloch space. In [5], Ohno studied the weighted composition operators between H^∞ and the Bloch space. In [8], Yoneda studied the boundedness and compactness of composition operator on $\mathcal{B}_{log}$. In this paper, we study the boundedness and compactness of weighted composition operators from Bergman space into weighted Bloch space.

2. Main Theorem and Proof

In this section we will state and prove the main theorems of this paper.

THEOREM 2.1. *Let φ be an analytic self-map of D and u be an analytic function on the unit disk D and $1 \le p < \infty$ such that*

$$N = \sup_{z\in D} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}} \log \frac{2}{1-|z|^2}|u'(z)| < \infty. \tag{1}$$

Then $uC_\varphi : A^p \to \mathcal{B}_{log}$ is bounded if and only if the following (i) and (ii) are satisfied:

(i) $u \in \mathcal{B}_{log}$;

(ii)

$$M = \sup_{z\in D} \frac{1-|z|^2}{(1-|\varphi(z)|^2)^{1+2/p}} \log \frac{2}{1-|z|^2}|u(z)\varphi'(z)| < \infty \tag{2}$$

PROOF. Suppose that $u \in \mathcal{B}_{log}$ and (2) holds. It follows from a theorem of Hardy-Littwood and Flett [3] that, whenever $f \in A^p$, then its derivative $f' \in A^p_p$, and there exists a positive constant c_p such that $\|f'\|_{A^p_p} \le c_p\|f\|_p$. By Lemma 1.1 we get

$$|f'(z)| \le \frac{\|f'\|_{A^p_p}}{(1-|z|^2)^{(2+p)/p}} \le \frac{c_p\|f\|_p}{(1-|z|^2)^{(2+p)/p}}$$

independently of f in A^p. Then for arbitrary z in D we have

$$\begin{aligned}
& (1-|z|^2)\log\frac{2}{1-|z|^2}|(uC_\varphi f)'(z)| \\
= \;& (1-|z|^2)\log\frac{2}{1-|z|^2}|u'(z)f(\varphi(z)) + u(z)(f\circ\varphi)'(z)| \\
\le \;& (1-|z|^2)\log\frac{2}{1-|z|^2}|u'(z)||f(\varphi(z))| \\
& +(1-|z|^2)\log\frac{2}{1-|z|^2}|f'(\varphi(z))||u(z)\varphi'(z)|
\end{aligned}$$

$$\begin{aligned}
&\leq (1-|z|^2)\log\frac{2}{1-|z|^2}|u'(z)|\frac{\|f\|_p}{(1-|\varphi(z)|^2)^{2/p}}\\
&\quad +c_p(1-|z|^2)\log\frac{2}{1-|z|^2}|u(z)\varphi'(z)|\frac{\| f \|_p}{(1-|\varphi(z)|^2)^{1+2/p}}\\
&\leq \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}\log\frac{2}{1-|z|^2}|u'(z)|\|f\|_p\\
&\quad +c_p\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}\log\frac{2}{1-|z|^2}|u(z)\varphi'(z)| \| f \|_p\\
&< (N+c_pM) \| f \|_p
\end{aligned}$$

Consequently, $uC_\varphi f\in\mathcal{B}_{log}$. In addition to this, Lemma 1.1 yields

$$|(uC_\varphi f(0)|\leq\frac{|u(0)| \| f \|_p}{(1-|\varphi(0)|^2)^{2/p}}.$$

The last two inequalities show that $\| uC_\varphi f \|_{log}\leq const\cdot \| f \|_p$. Hence $uC_\varphi : A^p\to\mathcal{B}_{log}$ is bounded.

Conversely, suppose that $uC_\varphi : A^p\to\mathcal{B}_{log}$ is bounded. Then it is evident that $u\in\mathcal{B}_{log}$, and

$$\sup_{z\in D}(1-|z|^2)\log\frac{2}{1-|z|^2}|u(z)\varphi'(z)|<\infty. \tag{3}$$

For $\lambda\in D$, let

$$f(z)=\left(\frac{1-|\varphi(\lambda)|^2}{(1-\overline{\varphi(\lambda)}z)^2}\right)^{2/p}.$$

Then $f\in A^p$ and $\| f \|_p\leq 1$

$$\begin{aligned}
&\| uC_\varphi \|\geq\| uC_\varphi f \|_{\mathcal{B}_{log}}\\
&\geq \Big|\frac{4}{p}\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}\log\frac{2}{1-|\lambda|^2}|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|\\
&\quad -\frac{(1-|\lambda|^2)}{(1-|\varphi(\lambda)|^2)^{2/p}}\log\frac{2}{1-|\lambda|^2}|u'(\lambda)|\Big|
\end{aligned}$$

Since

$$\sup_{z\in D}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}\log\frac{2}{1-|z|^2}|u'(z)|<\infty,$$

$$\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}\log\frac{2}{1-|\lambda|^2}|u(\lambda)\overline{\varphi(\lambda)}\varphi'(\lambda)|<\infty. \tag{4}$$

Thus, for a fixed $\delta, 0 < \delta < 1$, by (4)
(5)

$$\sup\{\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}\log\frac{2}{1-|\lambda|^2}|u(\lambda)\varphi'(\lambda)| : \lambda \in D, |\varphi(\lambda)| > \delta\} < \infty.$$

For $\lambda \in D$ such that $|\varphi(\lambda)| \leq \delta$, we have

$$\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}\log\frac{2}{1-|\lambda|^2}|u(\lambda)\varphi'(\lambda)|$$
$$\leq \frac{1}{(1-\delta^2)^{1+2/p}}(1-|\lambda|^2)\log\frac{2}{1-|\lambda|^2}|u(\lambda)\varphi'(\lambda)|$$

and so by (3)
(6)

$$\sup\{\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}\log\frac{2}{1-|\lambda|^2}|u(\lambda)\varphi'(\lambda)| : \lambda \in D, |\varphi(\lambda)| \leq \delta\} < \infty.$$

Consequently by (5) and (6), we have

$$\sup_{\lambda\in D}\frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)^{1+2/p}}\log\frac{2}{1-|\lambda|^2}|u(\lambda)\varphi'(\lambda)| < \infty.$$

We finish the proof. □

THEOREM 2.2. *Let φ be an analytic self-map of D and u be an analytic function on the unit disk D and $1 \leq p < \infty$ such that*

$$N = \sup_{z\in D}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}\log\frac{2}{1-|z|^2}|u'(z)| < \infty.$$

Suppose that uC_φ exists as a bounded operator from A^p into $\mathcal{B}_{log}$, then $uC_\varphi : A^p \to \mathcal{B}_{log}$ is compact if and only if the following (i) and (ii) are satisfied:

(i)

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}\log\frac{2}{1-|z|^2}|u'(z)| = 0,$$

(ii)

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}\log\frac{2}{1-|z|^2}|u(z)\varphi'(z)| = 0.$$

PROOF. Assume (i) and (ii) hold, in order to prove that uC_φ is compact, it suffices to show that if $\{f_n\}$ is a bounded sequence in A^p that converges to 0 uniformly on compact subsets of D, then $\| uC_\varphi f_n \|_{log} \to 0$. This criterion for compactness follows by standard arguments similar to those outlined in proposition 3.11 of [2], for example. Let $\{f_n\}$ be a sequence in A^p with $\|f_n\|_p \le 1$ and $f_n \to 0$ uniformly on compact subsets of D. By the assumption, for any $\epsilon > 0$, there is a constant $\delta, 0 < \delta < 1$, such that $\delta < |\varphi(z)| < 1$ implies

$$\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}\log\frac{2}{1-|z|^2}|u'(z)| < \epsilon/2$$

and

$$\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}\log\frac{2}{1-|z|^2}|u(z)\varphi'(z)| < \epsilon/2.$$

Let $K = \{w \in D : |w| \le \delta\}$. Note that K is a compact subsect of D, then

$$\begin{aligned}
\| uC_\varphi f_n \|_{log} &= \sup_{z\in D}(1-|z|^2)\log\frac{2}{1-|z|^2}|(uC_\varphi f_n)'(z)| \\
&\le \sup_{z\in D}(1-|z|^2)\log\frac{2}{1-|z|^2}|u'(z)f_n(\varphi(z))| \\
&\quad + \sup_{z\in D}(1-|z|^2)\log\frac{2}{1-|z|^2}|u(z)f_n'(\varphi(z))\varphi'(z)| \\
&\le \sup_{\{z\in D:\varphi(z)\in K\}}(1-|z|^2)\log\frac{2}{1-|z|^2}|u'(z)f_n(\varphi(z))| \\
&\quad + \sup_{\{z\in D:\varphi(z)\in K\}}(1-|z|^2)\log\frac{2}{1-|z|^2}|u(z)\varphi'(z)||f_n'(\varphi(z))| + \epsilon \\
&\le \| u \|_{log}\sup_{w\in K}|f_n(w)| + M\sup_{w\in K}(1-|w|^2)^{1+2/p}|f_n'(w)| + \epsilon,
\end{aligned}$$

where

$$M = \sup\{\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}\log\frac{2}{1-|z|^2}|u(z)\varphi'(z)| : z \in D\}.$$

As $n \to \infty$, $\| uC_\varphi f_n \|_{log} \to 0$. Consequently, $uC_\varphi : A^p \to \mathcal{B}_{log}$ is compact.

Conversely, suppose $uC_\varphi : A^p \to \mathcal{B}_{log}$ is compact. Let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Let

$$f_n(z) = (\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^2})^{\frac{2}{p}}.$$

Then $f_n \in A^p$ and $\|f_n\| \le 1$ and f_n converges to 0 uniformly on compact subsets of D. Since uC_φ is compact, we have $\| uC_\varphi f_n \|_{log} \to 0$ as $n \to \infty$. Thus

$$0 \leftarrow \| uC_\varphi f_n \|_{log}$$

$$\geq \sup_{z\in D}(1-|z|^2)\log\frac{2}{1-|z|^2}|(uC_\varphi f_n)'(z)|$$
$$\geq \Big|\frac{(1-|z_n|^2)}{(1-|\varphi(z_n)|^2)^{2/p}}\log\frac{2}{1-|z_n|^2}|u'(z_n)|$$
$$-\frac{4}{p}\frac{(1-|z_n|^2)}{(1-|\varphi(z_n)|^2)^{1+\frac{2}{p}}}\log\frac{2}{1-|z_n|^2}|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|\Big|.$$

So we get

$$\begin{aligned}&\lim_{|\varphi(z_n)|\to 1}\frac{(1-|z_n|^2)}{(1-|\varphi(z_n)|^2)^{2/p}}\log\frac{2}{1-|z_n|^2}|u'(z_n)|\\ &=\lim_{|\varphi(z_n)|\to 1}\frac{4}{p}\frac{(1-|z_n|^2)}{(1-|\varphi(z_n)|^2)^{1+2/p}}\log\frac{2}{1-|z_n|^2}|u(z_n)\varphi'(z_n)|.\end{aligned}\tag{7}$$

Next let

$$g_n(z)=\frac{1-|\varphi(z_n)|^2}{(1-\overline{\varphi(z_n)}z)^{1+\frac{2}{p}}}-\Big(\frac{1}{1-\overline{\varphi(z_n)}z}\Big)^{2/p},$$

for a sequence $\{z_n\}$ in D such that $|\varphi(z_n)|\to 1$, then $g_n(z)$ is a bounded sequence in A^p and $g_n(z)\to 0$ uniformly on every compact subset of D, $g_n(\varphi(z_n))=0$ and

$$g'(\varphi(z_n))=\frac{\overline{\varphi(z_n)}}{(1-|\varphi(z_n)|^2)^{1+2/p}}.$$

Then

$$\begin{aligned}0 &\leftarrow \| uC_\varphi g_n \|_{log}\\ &\geq \sup_{z\in D}(1-|z|^2)\log\frac{2}{1-|z|^2}|(uC_\varphi g_n)'(z)|\\ &\geq \frac{1-|z_n|^2}{(1-|\varphi(z_n)|^2)^{1+2/p}}\log\frac{2}{1-|z_n|^2}|u(z_n)\overline{\varphi(z_n)}\varphi'(z_n)|.\end{aligned}$$

Thus we can get

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}}\log\frac{2}{1-|z|^2}|u(z)\varphi'(z)|=0,$$

and so by (7), we have

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{2/p}}\log\frac{2}{1-|z|^2}|u'(z)|=0.$$

□

From the last two theorems, we can easily obtain the following two theorems:

THEOREM 2.3. *Let φ be an analytic self-map of D and $1 \le p < \infty$. Then $C_\varphi : A^p \to \mathcal{B}_{log}$ is bounded if and only if the following is satisfied:*

$$\sup_{z\in D} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}} \log \frac{2}{1-|z|^2} |\varphi'(z)| < \infty.$$

THEOREM 2.4. *Let φ be an analytic self-map of D and $1 \le p < \infty$. Suppose $C_\varphi : A^p \to \mathcal{B}_{log}$ is bounded, then $C_\varphi : A^p \to \mathcal{B}_{log}$ is compact if and only if the following is satisfied:*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2/p}} \log \frac{2}{1-|z|^2} |\varphi'(z)| = 0.$$

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