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K-THEORY OF C^*-ALGEBRAS OF LOCALLY TRIVIAL CONTINUOUS FIELDS

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ABSTRACT. It is shown that the K-theory of the C^*-algebras of continuous fields on locally compact Hausdorff spaces with fibers elementary C^*-algebras is the same as the K-theory of the base spaces. We also consider the slightly generalized case. Furthermore, we give some applications of these results.

0. Introduction

We begin with the following:

Notation Let $C_0(X)$ denote the C^*-algebra of all continuous functions vanishing at infinity on a locally compact Hausdorff space X, and $C^b(X)$ be the C^*-algebra of all bounded continuous functions on X. Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on a locally compact Hausdorff space X with fibers $\mathfrak{A}_t$ C^*-algebras, consisting of (a certain family of) continuous operator fields vanishing at infinity on X, and $\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on X consisting of bounded continuous operator fields(See [2] and [3]). Let $K_*(\mathfrak{A})$ for $* = 0, 1$ be the K-groups of a C^*-algebra $\mathfrak{A}$(See [1] and [7]).

Our motivation for this study is the following:

PROBLEM. *Let $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on a locally compact Hausdorff space X with fibers $\mathfrak{A}_t$ C^*-algebras. Then how does one compute the K-groups of the C^*-algebra in terms of the space X and the fibers $\mathfrak{A}_t$? Similarly, what are the K-groups of $\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X})$?*

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Our philosophy for this problem is that those K-groups of the C^*-algebras of continuous fields could be written in terms of the K-groups of the base spaces and fibers. The general case seems to be still open. In this paper, as the main results based on that idea we consider the case where fibers are elementary C^*-algebras and the slightly generalized case where the C^*-algebras of continuous fields are locally trivial. For the proofs we use an argument based on inductive (or projective) limits of C^*-algebras associated with local triviality of the continuous fields since K-groups of C^*-algebras are continuous with respect to this inductive (or projective) process so that our argument is elementary but reasonable in a sense. Moreover, we give some applications of the results, which should be more interesting. Indeed, we give the K-theoretic proofs and understanding for calculating K-groups of the group C^*-algebra of the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ and the free product C^*-algebra $\mathbb{C} * \mathbb{C}$ both described as certain continuous field C^*-algebras.

1. The main results

Recall that a C^*-algebra is elementary if it is isomorphic to either the $n \times n$ matrix algebra $M_n(\mathbb{C})$ over $\mathbb{C}$ or $\mathbb{K}$ the C^*-algebra of all compact operators on a separable infinite dimensional Hilbert space(cf. [2]). Let $\mathfrak{A}$ be a C^*-algebra and $M_n(\mathfrak{A})$ the $n \times n$ matrix algebra over $\mathfrak{A}$. As a C^*-algebra, the identification: $\mathfrak{A} \ni a \longleftrightarrow a \oplus \cdots \oplus a \in M_n(\mathfrak{A})$ is standard, where $\oplus$ means the diagonal sum. However, it is found that when we forms continuous field C^*-algebras, we must distinguish $\mathfrak{A}$ from the diagonal subalgebra of $M_n(\mathfrak{A})$ (see the remark of Theorem 1.1). From this reason we say that $M_n(\mathbb{C})$ or $\mathbb{K}$ (in a usual sense) are acting on $\mathbb{C}^n$ or $l^2(\mathbb{C})$ the Hilbert space of all square summable sequences over $\mathbb{C}$ respectively.

THEOREM 1.1. *Let $\mathfrak{A} = \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a continuous field on a locally compact, paracompact Hausdorff space X with fibers $\mathfrak{A}_t$ elementary C^*-algebras acting on $\mathbb{C}^n$ or $l^2(\mathbb{C})$. Then for $* = 0, 1$,*

$$K_*(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) \cong K_*(C_0(X)).$$

PROOF. First note that $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \otimes M_n(\mathbb{C}) \cong \Gamma_0(X, \{\mathfrak{A}_t \otimes M_n(\mathbb{C})\}_{t\in X})$ for any $n \in \mathbb{N}$, which could be deduced from a result of [4] on open continuous maps from the primitive ideal spaces of C^*-algebras.

Moreover, we have

$$\begin{aligned}\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \otimes \mathbb{K} &\cong \overline{\cup_n \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \otimes M_n(\mathbb{C})} \\ &\cong \overline{\cup_n \Gamma_0(X, \{\mathfrak{A}_t \otimes M_n(\mathbb{C})\}_{t\in X})} \\ &\cong \Gamma_0(X, \{\mathfrak{A}_t \otimes \mathbb{K}\}_{t\in X}),\end{aligned}$$

where the overline means the closure. Since $\mathfrak{A}_t$ are elementary C^*-algebras,

$$\mathfrak{A}_t \otimes \mathbb{K} \cong \begin{cases} M_n(\mathbb{C}) \otimes \mathbb{K} = \overline{\cup_m M_n(\mathbb{C}) \otimes M_m(\mathbb{C})} = \overline{\cup_m M_{nm}(\mathbb{C})} = \mathbb{K}, \\ \mathbb{K} \otimes \mathbb{K} = \overline{\cup_{n,m} M_n(\mathbb{C}) \otimes M_m(\mathbb{C})} = \overline{\cup_{n,m} M_{nm}(\mathbb{C})} = \mathbb{K}. \end{cases}$$

Hence, $\Gamma_0(X, \{\mathfrak{A}_t \otimes \mathbb{K}\}_{t\in X}) \cong \Gamma_0(X, \{\mathbb{K}\}_{t\in X})$ with the fibers $\mathbb{K}$ acting on the same Hilbert space. Therefore, we obtain

$$\begin{aligned}K_*(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) &\cong K_*(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}) \otimes \mathbb{K}) \\ &\cong K_*(\Gamma_0(X, \{\mathfrak{A}_t \otimes \mathbb{K}\}_{t\in X})) \\ &\cong K_*(\Gamma_0(X, \{\mathbb{K}\}_{t\in X}))\end{aligned}$$

for $* = 0, 1$. Furthermore, note that if X is a paracompact Hausdorff space, the C^*-algebra $\Gamma_0(X, \{\mathbb{K}\}_{t\in X})$ with fibers $\mathbb{K}$ is locally trivial ([2, Theorem 10.8.8] and see also [5, Section 4.3]) in the sense that for any $t \in X$, there exists an open neighborhood U_t of t such that $\Gamma_0(U_t, \{\mathbb{K}\}_{s\in U_t}) \cong \Gamma_0(U_t, \{\mathbb{C}\}_{s\in U_t}) \otimes \mathbb{K}$, and $\Gamma_0(U_t, \{\mathbb{C}\}_{s\in U_t}) = C_0(U_t)$. Taking the unions $\cup_{j=1}^n U_{t_j}$ of (disjoint) such open neighborhood U_{t_j} for some $t_j \in X$ $(1 \le j \le n)$, we have that for $* = 0, 1$,

$$\begin{aligned}K_*(\Gamma_0(X, \{\mathbb{K}\}_{t\in X})) &\cong K_*(\lim_{n\to\infty} \Gamma_0(\cup_{j=1}^n U_{t_j}, \{\mathbb{K}\}_{t\in \cup_{j=1}^n U_{t_j}})) \\ &\cong \lim_{n\to\infty} K_*(\Gamma_0(\cup_{j=1}^n U_{t_j}, \{\mathbb{K}\}_{t\in \cup_{j=1}^n U_{t_j}})) \\ &\cong \lim_{n\to\infty} K_*(\Gamma_0(\cup_{j=1}^n U_{t_j}, \{\mathbb{C}\}_{t\in \cup_{j=1}^n U_{t_j}}) \otimes \mathbb{K}) \\ &\cong \lim_{n\to\infty} K_*(C_0(\cup_{j=1}^n U_{t_j})) \cong K_*(C_0(X)),\end{aligned}$$

where the first limit means an inductive limit of C^*-algebras, and the second and last isomorphisms follow from continuity of K-groups for inductive limits. Summing up, we obtain that $K_*(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) \cong K_*(C_0(X))$ for $* = 0, 1$. □

REMARK. Recall that the dimension dropping algebras D_n are defined to be the C^*-algebras of the elements f of $C([0,1], M_n(\mathbb{C}))$ the C^*-algebra of continuous $M_n(\mathbb{C})$-valued functions on the interval $[0,1]$ such that $f(0) = 0$ and $f(1) \in \mathbb{C}1_n$, where 1_n means the $n \times n$ identity matrix. It is known that $K_0(D_n) = 0$ and $K_1(D_n) = \mathbb{Z}_n$ (cf. [6, Chapter 13]). This is in part because of the following identification: $\mathbb{C} \ni 1 \longleftrightarrow 1 \oplus \cdots \oplus 1 = \sum_{j=1}^n p_j \in M_n(\mathbb{C})$, where p_j are rank 1 projections in $M_n(\mathbb{C})$ with the jj-th component 1 and other components zero. An important point to keep in mind is that the C^*-algebras D_n are not the continuous field C^*-algebras in Theorem 1. The reason comes from that we must distinguish $\mathbb{C}1_n$ with $\mathbb{C}$ if we consider continuous field C^*-algebras involving those fibers. In fact, note that elements of $\mathbb{C}1_n \otimes M_m(\mathbb{C})$ and $\mathbb{C} \otimes M_m(\mathbb{C})$ are respectively given by $\lambda 1_n \otimes (a_{ij}) = 1_n \otimes (\lambda a_{ij}) \in M_n(M_m(\mathbb{C}))$ and $\lambda \otimes (a_{ij}) = (\lambda a_{ij}) \in M_m(\mathbb{C})$ for $\lambda \in \mathbb{C}$ and $(a_{ij}) \in M_m(\mathbb{C})$, which should also be distinguished since the meanings of continuity at the fibers may be different.

By the same way as the proof above, we obtain

THEOREM 1.2. *Let $\mathfrak{A} = \Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X})$ for a locally compact Hausdorff space X and fibers $\mathfrak{A}_t$ elementary C^*-algebras acting on $\mathbb{C}^n$ or $l^2(\mathbb{C})$. Then for $* = 0, 1$,*

$$K_*(\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X})) \cong K_*(C^b(X)).$$

PROOF. Note that as in the proof of Theorem 1.1,

$$\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X}) \otimes \mathbb{K} \cong \Gamma^b(X, \{\mathfrak{A}_t \otimes \mathbb{K}\}_{t\in X}) \cong \Gamma^b(X, \{\mathbb{K}\}_{t\in X}),$$

and $C^b(X) \cong C^b(\varprojlim \cup K_j) \cong \varinjlim \oplus_j C^b(K_j)$, where each K_j is a compact subset of X contained in an open subset U_j of X such that $\Gamma_0(U_j, \{\mathfrak{A}_t\}_{t\in U_j}) \otimes \mathbb{K} \cong C_0(U_j) \otimes \mathbb{K}$, and $\varprojlim \cup K_j$ means the projective limit of unions of (disjoint) compact subsets K_j of X, and $\varinjlim \oplus_j C^b(K_j)$ is the inductive limit of direct sums of $C^b(K_j)$ where the associated homomorphisms (not necessarily injective): $\oplus_{j=1}^k C^b(K_j) \to \oplus_{j=1}^{k+1} C^b(K_j)$ for $k \geq 1$ are induced from certain extensions of generating elements of $C^b(\cup_{j=1}^k K_j)$ to $C^b(\cup_{j=1}^{k+1} K_j)$ (such extensions always exist but not necessarily canonical). Therefore, for $* = 0, 1$,

$$\begin{aligned} K_*(\Gamma^b(X, \{\mathbb{K}\}_{t\in X})) &\cong K_*(\Gamma^b(\varinjlim \cup_j K_j, \{\mathbb{K}\}_{t\in K_j})) \\ &\cong K_*(\varinjlim \oplus_j \Gamma^b(K_j, \{\mathbb{K}\}_{t\in K_j})) \\ &\cong \varinjlim \oplus_j K_*(\Gamma^b(K_j, \{\mathbb{K}\}_{t\in K_j})) \\ &\cong \varinjlim \oplus_j K_*(C^b(K_j) \otimes \mathbb{K}) \\ &\cong K_*(C^b(\varinjlim \cup_j K_j) \otimes \mathbb{K}) \\ &\cong K_*(C^b(X) \otimes \mathbb{K}) \cong K_*(C^b(X)). \end{aligned}$$

□

REMARK. Note that $C^b(X) \cong C(\beta X)$ where βX is the Stone-Čech compactification of X (cf. [7, 2.C]). The proof above is not using para-compactness of spaces as in the proof of Theorem 1.1. Replacing $\Gamma^b(\cdot)$, $C^b(\cdot)$ with $\Gamma_0(\cdot)$, $C_0(\cdot)$ respectively in the proof above, we can obtain the same conclusion as Theorem 1.1.

DEFINITION. We say that the C^*-algebra $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ of a continuous field on a locally compact Hausdorff space X with fibers $\mathfrak{A}_t$ is locally trivial (or the C^*-algebra of a locally trivial continuous field) if for any $s \in X$, there exists an open neighborhood U of s such that the restriction $\Gamma_0(U, \{\mathfrak{A}_t\}_{t\in U})$ of $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ to U is trivial, that is, isomorphic to $C_0(U) \otimes \mathfrak{A}_s$.

THEOREM 1.3. *Let $\mathfrak{A} = \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})$ be the C^*-algebra of a locally trivial continuous field on a locally compact Hausdorff space X with fibers the same $\mathfrak{A}_X = \mathfrak{A}_t$ for $t \in X$. Then*

$$K_*(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X})) \cong K_*(C_0(X) \otimes \mathfrak{A}_X)$$

for $* = 0, 1$. Moreover, $K_*(\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X})) \cong K_*(C^b(X) \otimes \mathfrak{A}_X)$ for $* = 0, 1$.

PROOF. By the assumption of being locally trivial, $\mathfrak{A}$ is an inductive limit of direct sums of the C^*-algebras of trivial continuous fields on open neighborhoods U_j of X, that is, $\mathfrak{A} = \varinjlim \oplus_j \Gamma_0(U_j, \{\mathfrak{A}_t\}_{t\in U_j})$ and $\Gamma_0(U_j, \{\mathfrak{A}_t\}_{t\in U_j}) \cong C_0(U_j) \otimes \mathfrak{A}_X$. Therefore, for $* = 0, 1$,

$$\begin{aligned} K_*(\mathfrak{A}) &\cong K_*(\varinjlim \oplus_j \Gamma_0(U_j, \{\mathfrak{A}_t\}_{t\in U_j})) \\ &\cong \varinjlim \oplus_j K_*(\Gamma_0(U_j, \{\mathfrak{A}_t\}_{t\in U_j})) \\ &\cong \varinjlim \oplus_j K_*(C_0(U_j) \otimes \mathfrak{A}_X) \\ &\cong K_*(C_0(\varinjlim \cup_j U_j) \otimes \mathfrak{A}_X) \cong K_*(C_0(X) \otimes \mathfrak{A}_X), \end{aligned}$$

where $\oplus_j$ means a finite direct sum, and $\varinjlim$ means inductive limit.

As for the second claim, we replace the inductive limit of unions of those open subsets U_j of X with the projective limit of unions of compact subsets K_j of X on which $\mathfrak{A}$ is locally splitting to the tensor products $C^b(K_j) \otimes \mathfrak{A}_X$. □

REMARK. Instead of the same $\mathfrak{A}_X$, we may assume that the fibers $\mathfrak{A}_t$ are stably isomorphic, that is, $\mathfrak{A}_t \otimes \mathbb{K} \cong \mathfrak{A}_s \otimes \mathbb{K}$ for any $t, s \in X$. In particular, we may choose the fibers as matrix algebras over $\mathbb{C}$ with their sizes different. Note that

$$K_0(C_0(X) \otimes \mathfrak{A}_X) \cong K_0(C_0(X)) \otimes K_0(\mathfrak{A}_X) \oplus K_1(C_0(X)) \otimes K_1(\mathfrak{A}_X),$$
$$K_1(C_0(X) \otimes \mathfrak{A}_X) \cong K_0(C_0(X)) \otimes K_1(\mathfrak{A}_X) \oplus K_1(C_0(X)) \otimes K_0(\mathfrak{A}_X)$$

when $\mathfrak{A}_X$ is in the bootstrap class and the K-groups of $C_0(X)$ and $\mathfrak{A}_X$ are torsion free (the Künneth formula) (cf. [7],9.3.3), and we may replace $C_0(X)$ with $C^b(X)$ in the formula.

2. The applications

Recall that the group C^*-algebra $C^*(\mathbb{Z}_2 * \mathbb{Z}_2)$ of the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to the C^*-algebra of the elements f of $C([0,1], M_2(\mathbb{C}))$ such that $f(0)$, $f(1)$ are diagonal. The group C^*-algebra is also the universal unital C^*-algebra generated by two projections. Moreover, it is known that $K_0(C^*(\mathbb{Z}_2 * \mathbb{Z}_2)) \cong \mathbb{Z}^3$ (cf. [1], 6.10.4) and $K_1(C^*(\mathbb{Z}_2 * \mathbb{Z}_2)) \cong 0$ since $C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \cong C(\mathbb{T}) \rtimes_\alpha \mathbb{Z}_2$ with the action α defined by $\alpha_1(f)(z) = f(\bar{z})$ for $f \in C(\mathbb{T})$, $z \in \mathbb{T}$ and $1 \in \mathbb{Z}_2$ ([1], 10.11.5). Using our results we can interpret this fact as follows.

THEOREM 2.1. *Let $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ be the continuous field C^*-algebra on $[0,1]$ with fibers $\mathfrak{A}_0 = \mathbb{C}^2$, $\mathfrak{A}_1 = \mathbb{C}^2$ and $\mathfrak{A}_t = M_2(\mathbb{C})$ for $t \in (0,1)$. Then*

$$K_0(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}^3, \quad K_1(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0, \quad \text{and}$$
$$K_*(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong K_*(C([0,1])) \oplus K_*(\mathbb{C}) \oplus K_*(\mathbb{C}) \quad \text{for } * = 0, 1$$

PROOF. We have the following short exact sequence E_1:

$$0 \to \Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1)} \cup \mathbb{C}) \to \Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]}) \to \mathbb{C} \to 0,$$

where $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1)} \cup \mathbb{C})$ is the closed ideal of the elements f of the middle C^*-algebra such that $f(1) = f_1(1) \oplus f_2(1) \in \mathbb{C}^2$ with $f_2(1) = 0$. Furthermore, the following sequence E_2:

$$0 \to \Gamma([0,1], \mathbb{C} \cup \{\mathfrak{A}_t\}_{t\in(0,1)} \cup \mathbb{C}) \to \Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1)} \cup \mathbb{C}) \to \mathbb{C} \to 0$$

is exact, where $\Gamma([0,1], \mathbb{C} \cup \{\mathfrak{A}_t\}_{t\in(0,1)} \cup \mathbb{C}) \equiv \mathfrak{J}$ is the closed ideal of the elements g of the middle C^*-algebra such that $g(0) = g_1(0) \oplus g_2(0) \in \mathbb{C}^2$ with $g_2(0) = 0$. Using theorem 1.1 we obtain

$$K_*(\mathfrak{J}) \cong K_*(C([0,1])) \cong \begin{cases} \mathbb{Z} & * = 0, \\ 0 & * = 1. \end{cases}$$

Note that $[0,1]$ is contractible so that $K_*(C([0,1])) \cong K_*(\mathbb{C})$ for $* = 0, 1$. Thus, the six-term exact sequence for the second sequence E_2 is given by

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1)} \cup \mathbb{C})) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1)} \cup \mathbb{C})) & \longleftarrow & 0. \end{array}$$

Hence, the K_0 and K_1-groups in the middles are isomorphic to $\mathbb{Z}^2$ and 0 respectively. Thus, the six-term exact sequence for the first sequence E_1 is given by

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longrightarrow & K_0(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1)} \cup \mathbb{C})) & \longleftarrow & 0, \end{array}$$

which implies the conclusion. □

REMARK. The continuous field C^*-algebra in the statement is regarded as the unitization $\mathfrak{J}^{++}$ of the closed ideal $\mathfrak{J}$ in the proof by adding continuous operator fields non-vanishing at the second components of the fibers $\mathbb{C}^2$ at 0 and 1. Thus the base space of the C^*-algebra is the compactification $[0,1]^{++}$ of $[0,1]$ with two points. Note also that the short exact sequences in the proof above are not splitting but their associated six-term sequences of K-groups are splitting into two short exact sequences of K_0 and K_1-groups. This allows us to compute the K-groups as desired.

Similarly as Theorem 2.1,

THEOREM 2.2. *Let $\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})$ be the continuous field C^*-algebra on $[0,1]$ with fibers $\mathfrak{A}_0=\mathbb{C}^n$, $\mathfrak{A}_1=\mathbb{C}^n$ and $\mathfrak{A}_t=M_n(\mathbb{C})$ for $t\in(0,1)$. Then*

$$K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}))\cong\mathbb{Z}^{2n-1},\quad K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}))\cong 0.$$

Consequently, for $*=0,1$ we have

$$K_*(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}))\cong K_*(C([0,1]))\oplus K_*(\mathbb{C}^{n-1})\oplus K_*(\mathbb{C}^{n-1}).$$

PROOF. Let $\mathfrak{I}_{kl}$ $(1\le k,l\le n)$ be the continuous field C^*-subalgebras of the C^*-algebra in the statement on $[0,1]$ with fibers $\mathfrak{A}_0=\mathbb{C}^k$, $\mathfrak{A}_1=\mathbb{C}^l$ and $\mathfrak{A}_t=M_n(\mathbb{C})$ for $t\in(0,1)$, where $\mathbb{C}^k$ and $\mathbb{C}^l$ are C^*-subalgebras of $\mathbb{C}^n$ in the canonical sense. Then we have inductively the exact sequences: $0\to\mathfrak{I}_{n,l-1}\to\mathfrak{I}_{nl}\to\mathbb{C}\to 0$ for $2\le l\le n$. Furthermore, we have $0\to\mathfrak{I}_{k-1,1}\to\mathfrak{I}_{k1}\to\mathbb{C}\to 0$ for $2\le k\le n$. Applying Theorem 1.1 to $\mathfrak{I}_{11}$ with the fibers $\mathbb{C}$ at $0,1$ and $M_n(\mathbb{C})$ for $(0,1)$, we obtain $K_0(\mathfrak{I}_{11})\cong\mathbb{Z}$ and $K_1(\mathfrak{I}_{11})\cong 0$. From the six-term exact sequence of K-groups, it follows that $K_0(\mathfrak{I}_{21})\cong\mathbb{Z}^2$ and $K_1(\mathfrak{I}_{21})\cong 0$. Continuing this process inductively we have $K_0(\mathfrak{I}_{k1})\cong\mathbb{Z}^k$ and $K_1(\mathfrak{I}_{k1})\cong 0$ for $1\le k\le n$. Furthermore, $K_0(\mathfrak{I}_{nl})\cong\mathbb{Z}^{n+l-1}$ and $K_1(\mathfrak{I}_{nl})\cong 0$ for $1\le l\le n$. Hence, the proof is complete. □

REMARK. The last formula in the statement above clearly explains why the first isomorphisms do hold. This also gives a partial answer to the Problem in the introduction.

More generally, extending the argument as in the proof of Theorem 2.2 we have

THEOREM 2.3. *Let $\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})$ be the continuous field C^*-algebra on $[0,1]$ with fibers $\mathfrak{A}_{s/m}=\mathbb{C}^n$ for $s=0,1,\cdots,m$ a positive integer and $\mathfrak{A}_t=M_n(\mathbb{C})$ for other $t\in[0,1]$. Then*

$$K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}))\cong\mathbb{Z}^{(m+1)(n-1)+1},\quad K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}))\cong 0$$

Consequently, for $*=0,1$ we have

$$K_*(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}))\cong K_*(C([0,1]))\oplus\sum_{s=0}^{m}K_*(\mathbb{C}^{n-1}).$$

REMARK. As a variation of Theorem 2.3 we can replace $\mathbb{C}_n$ with either $\oplus^n M_m(\mathbb{C})$ or $\oplus^n \mathbb{K}$ and $M_n(\mathbb{C})$ with either $M_n(M_m(\mathbb{C}))$ or $M_n(\mathbb{K})$ respectively. In this case we obtain the same conclusion as Theorem 2.3 since the K-groups of the fibers are the same.

Furthermore, recall that the free product C^*-algebra $\mathbb{C} * \mathbb{C}$ of $\mathbb{C}$ is isomorphic to the C^*-algebra of all the elements f of $C([0,1], M_2(\mathbb{C}))$ such that $f(0) = f_1(0) \oplus f_2(0) \in \mathbb{C}^2$ with $f_2(0) = 0$ and $f(1)$ diagonal. Also, $\mathbb{C} * \mathbb{C}$ is the universal C^*-algebra generated by two projections. Then we have the following exact sequence:

$$0 \to q\mathbb{C} \to \mathbb{C} * \mathbb{C} \to \mathbb{C} \to 0,$$

where the homomorphism from $\mathbb{C} * \mathbb{C}$ to $\mathbb{C}$ is induced from the identity map from $\mathbb{C}$ to $\mathbb{C}$, and $q\mathbb{C}$ is the kernel of this homomorphism. Also, $q\mathbb{C}$ is isomorphic to the C^*-algebra of all the elements f of $C([0,1], M_2(\mathbb{C}))$ such that $f(0) = 0$ and $f(1)$ diagonal(See [1], 10.11.13). By using the same strategy as in Theorem 2.1,

THEOREM 2.4. *Let $\mathbb{C} * \mathbb{C}$ be the free product C^*-algebra of $\mathbb{C}$. Then*

$$K_0(\mathbb{C} * \mathbb{C}) \cong \mathbb{Z}^2, \quad K_1(\mathbb{C} * \mathbb{C}) \cong 0.$$

Consequently, $K_*(\mathbb{C} * \mathbb{C}) \cong K_*(C([0,1])) \oplus K_*(\mathbb{C})$ for $* = 0, 1$. Furthermore, for $q\mathbb{C}$ the closed ideal of $\mathbb{C} * \mathbb{C}$ we have

$$\begin{cases} K_0(q\mathbb{C}) \cong \mathbb{Z} \cong K_0(C_0((0,1])) \oplus K_0(\mathbb{C}), \\ K_1(q\mathbb{C}) \cong 0 \cong K_1(C_0((0,1])) \oplus K_1(\mathbb{C}). \end{cases}$$

PROOF. The proof for $\mathbb{C} * \mathbb{C}$ is the same as that of Theorem 2.1. Note that we have the exact sequence: $0 \to \mathfrak{I} \to q\mathbb{C} \to \mathbb{C} \to 0$, where $\mathfrak{I} = \Gamma_0((0,1], \{M_2(\mathbb{C})\}_{t \in (0,1)} \cup \mathbb{C})$. By Theorem 1.1 we obtain

$$K_0(\mathfrak{I}) = K_0(C_0((0,1])), \quad K_1(\mathfrak{I}) = K_1(C_0((0,1])).$$

Furthermore, we have the split exact sequence: $0 \to C_0((0,1]) \to C([0,1]) \to \mathbb{C} \to 0$. Using the six-term exact sequence of K-groups we obtain $K_0(C_0((0,1])) = 0$ and $K_1(C_0((0,1])) = 0$. Therefore, it follows from the six-term exact sequence that $K_0(q\mathbb{C}) = \mathbb{Z}$ and $K_1(q\mathbb{C}) = 0$. □

REMARK. As variations of this theorem, we can obtain the extensive results as Theorems 2.2 and 2.3 with fibers at $0, 1$ or s/m taken as $\mathbb{C}^k \subset \mathbb{C}^n$ for $0 \leq k \leq n$.

Actually, we obtain the following:

THEOREM 2.5. *Let $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})$ be the continuous field C^*-algebra on $[0,1]$ with fibers $\mathfrak{A}_{j/m} = \mathbb{C}^{k_j} \subset \mathbb{C}^n$ for $j = 0, 1, \cdots, m$ a positive integer and $1 \leq k_j \leq n$, and $\mathfrak{A}_t = M_n(\mathbb{C})$ for other $t \in [0,1]$. Then*

$$K_0(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}^{1+\sum_{j=0}^{m}(k_j-1)}$$

$$K_1(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0.$$

Consequently, for $* = 0, 1$ we have

$$K_*(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong K_*(C([0,1])) \oplus \sum_{j=0}^{m} K_*(\mathbb{C}^{k_j-1}).$$

Furthermore, if $k_0 = 0$, then

$$K_0(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}^{\sum_{j=0}^{m}(k_j-1)}, \quad K_1(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0.$$

For the case $k_0 = 0$ and $k_m = 0$, if $k_j = 1$ for $1 \leq j \leq m-1$, then

$$\begin{cases} K_0(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0 \cong K_0(C_0((0,1))), \\ K_1(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z} \cong K_1(C_0((0,1))), \end{cases}$$

and if $k_j = 2$ and $k_{j'} = 1$ for other $1 \leq j' \leq m-1$, then for $* = 0, 1$,

$$K_*(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong K_*(C_0((0,1])) \cong 0,$$

and if otherwise,

$$K_0(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}^{-1+\sum_{j=0}^{m-1}(k_j-1)}$$

$$K_1(\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0.$$

PROOF. For the case $1 \leq k_j \leq n$ for $0 \leq j \leq m$, we use the strategy of Theorem 2.2. For the case $k_0 = 0$, note $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]}) \cong \Gamma_0((0,1], \{\mathfrak{A}_t\}_{t\in(0,1]})$. Thus, when $\mathfrak{A}_{j/m} = \mathbb{C}$ for $1 \leq j \leq m$, using Theorem 1.1 we have

$$K_*(\Gamma_0((0,1], \{\mathfrak{A}_t\}_{t\in(0,1]})) \cong K_*(C_0(0,1]) \cong 0$$

for $* = 0, 1$ (cf. The proof of Theorem 2.4). Then we continue the argument of Theorem 2.2 to obtain the conclusion. For the case $k_0 = 0 = k_m$, note $\Gamma([0,1], \{\mathfrak{A}_t\}_{t\in[0,1]}) \cong \Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)})$. Thus, when $\mathfrak{A}_{j/m} = \mathbb{C}$ for $1 \leq j \leq m-1$, using Theorem 1.1 we have

$$K_*(\Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)})) \cong K_*(C_0(0,1)) \cong K_{*+1}(\mathbb{C}),$$

for $* = 0, 1$, where $*+1$ (mod 1). If $k_j = 2$ and $k_{j'} = 1$ for other $1 \leq j' \leq m-1$, then we have the exact sequence: $0 \to \mathfrak{I} \to \Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)}) \to \mathbb{C} \to 0$, where $\mathfrak{I}$ is the continuous field C^*-algebra corresponding to $k_0 = 0$, $k_m = 0$ and $k_j = 1$ for $1 \leq j \leq m-1$. Then the six-term exact sequence is given by

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)})) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)})) & \longleftarrow & \mathbb{Z}. \end{array}$$

Thus, it follows that the K_0, K_1-groups of $\Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)}))$ are trivial. For other cases, we use the exact sequence: $0 \to \mathfrak{I} \to \Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)}) \to \mathbb{C}^l \to 0$ for some $2 \leq l \leq \sum_{j=0}^m (k_j - 1)$. Then the six-term exact sequence is given by

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)})) & \longrightarrow & \mathbb{Z}^l \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)})) & \longleftarrow & \mathbb{Z}. \end{array}$$

Hence, it follows that

$$\begin{cases} K_0(\Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)})) \cong \mathbb{Z}^{l-1}, \\ K_1(\Gamma_0((0,1), \{\mathfrak{A}_t\}_{t\in(0,1)})) \cong 0. \end{cases}$$

□

Finally, we consider the case involving the fiber $\mathbb{C}1_n$ in $M_n(\mathbb{C})$ as follows:

THEOREM 2.6. *Let $\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})$ be the continuous field C^*-algebra on $[0,1]$ with fibers $\mathfrak{A}_0 = \mathbb{C}^n$, $\mathfrak{A}_1 = \mathbb{C}1_n$ and $\mathfrak{A}_t = M_n(\mathbb{C})$ for $t \in (0,1)$. Then*

$$K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}^n, \quad K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0.$$

If the fiber $\mathfrak{A}_0 = \mathbb{C}^n$ is replaced with $\mathbb{C}1_n$, then

$$K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}, \quad K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0.$$

Furthermore, if the fibers are given by $\mathfrak{A}_0 = \mathbb{C}1_n$, $\mathfrak{A}_1 = \mathbb{C}1_n$, $\mathfrak{A}_{1/2} = \mathbb{C}^k$ $(1 \le k \le n)$ and $\mathfrak{A}_t = M_n(\mathbb{C})$ for $t \in (0,1/2)\cup(1/2,1)$, then

$$K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}^k, \quad K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0.$$

Moreover, if the fibers are given by $\mathfrak{A}_0 = \mathbb{C}^n$ or $\mathbb{C}1_n$, and $\mathfrak{A}_1 = \mathbb{C}1_n$, $\mathfrak{A}_{j/m} = \mathbb{C}^{k_j} \subset \mathbb{C}^n$ for $j = 1, \cdots, m-1$ a positive integer and $1 \le k_j \le n$, and $\mathfrak{A}_t = M_n(\mathbb{C})$ for other $t \in [0,1]$, then for $\mathfrak{A}_0 = \mathbb{C}^n$,

$$\begin{aligned} &K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \\ &\cong \mathbb{Z}^{\sum_{j=0}^{m-1}(k_j-1)}, \quad K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0, \end{aligned}$$

and for $\mathfrak{A}_0 = \mathbb{C}1_n$,

$$\begin{aligned} &K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \\ &\cong \mathbb{Z}^{1+\sum_{j=1}^{m-1}(k_j-1)}, \quad K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0. \end{aligned}$$

PROOF. We have the exact sequence: $0 \to \mathfrak{I} \to \Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}) \to \mathbb{C} \to 0$, where the closed ideal $\mathfrak{I}$ consists of the elements vanishing at 1. Using Theorem 2.5 we have the following six-term exact sequence:

$$\begin{array}{ccccc} \mathbb{Z}^{n-1} & \longrightarrow & K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longleftarrow & 0 \end{array}$$

for $n \ge 1$ with $\mathbb{Z}^0 = 0$. Hence we have

$$K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}^n, \quad K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0.$$

Also, we have the exact sequence: $0 \to \mathfrak{K} \to \Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}) \to \mathbb{C}^n \to 0$, where the closed ideal $\mathfrak{K}$ consists of the elements vanishing at 0. Then we have

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^n \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}_n. \end{array}$$

Therefore, $\mathbb{Z}^n$ in the middle is isomorphic to $(n\mathbb{Z}) \times \mathbb{Z}^{n-1} \cong \mathbb{Z}^n$.

For the second case, the sequence: $0 \to \mathfrak{H} \to \Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}) \to \mathbb{C}^2 \to 0$ is exact, where the closed ideal $\mathfrak{H}$ consists of the elements vanishing at 0 and 1. Using Theorem 2.5 we have

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longrightarrow & \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longleftarrow & \mathbb{Z}. \end{array}$$

Therefore, $K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong \mathbb{Z}$ and $K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) \cong 0$. Also, we have the exact sequence: $0 \to \mathfrak{I} \to \Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}) \to \mathbb{C}^2 \to 0$, where the closed ideal $\mathfrak{I}$ consists of the elements vanishing at 0. Then we have

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}_n. \end{array}$$

Thus, $\mathbb{Z}$ in the middle is isomorphic to the subgroup $n\mathbb{Z}$ of $\mathbb{Z}$.

For the third case, the sequence: $0 \to \mathfrak{H} \to \Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}) \to \mathbb{C}^2 \to 0$ is exact, where the closed ideal $\mathfrak{H}$ consists of the elements vanishing at 0 and 1. Using Theorem 2.5, if $k=1$, then we have

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longrightarrow & \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longleftarrow & \mathbb{Z} \end{array}$$

and if $k=2$, then

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longrightarrow & \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longleftarrow & 0 \end{array}$$

and if $k \geq 3$, then

$$\begin{array}{ccccc} \mathbb{Z}^{k-2} & \longrightarrow & K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longrightarrow & \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longleftarrow & 0 \end{array}$$

Therefore, we deduce the conclusion.

For the last case, the sequence: $0 \to \mathfrak{J} \to \Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]}) \to \mathfrak{D} \to 0$ with $\mathfrak{D} = \mathbb{C}$ or $\mathbb{C}^2$ is exact, where the closed ideal $\mathfrak{J}$ consists of the elements vanishing either at 1 or at 0, 1 respectively. Since the K-groups of $\mathfrak{J}$ are given by Theorem 2.5, the six-term exact sequence:

$$\begin{array}{ccccc} K_0(\mathfrak{J}) & \longrightarrow & K_0(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longrightarrow & K_0(\mathfrak{D}) \\ \uparrow & & & & \downarrow \\ 0 & \longleftarrow & K_1(\Gamma([0,1],\{\mathfrak{A}_t\}_{t\in[0,1]})) & \longleftarrow & 0 \end{array}$$

with $K_0(\mathfrak{D}) = \mathbb{Z}$ or $\mathbb{Z}^2$ respectively implies the conclusion. □

REMARK. By using the theorems in Section 1, more other variations such as the case with the base spaces non-compact such as the real line can be treated by the same way as those cases above.

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