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CONNECTEDNESS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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ABSTRACT. We introduce the notion of (r,s)-connected sets in intuitionistic fuzzy topological spaces and investigate some properties of them. In particular, we show that every (r,s)-component in an intuitionistic fuzzy topological space is (r,s)-component in the stratification of it.

1. Introduction and preliminaries

Chang [4] introduced the notion of a fuzzy topology. Later, Lowen [16] redefined it which is now known as a stratified fuzzy topology. Pu and Liu [19] studied fuzzy connectedness in fuzzy topological spaces. It has been developed in many directions [6, 17, 26]. Šostak [23] introduced the notion of a smooth topology as an extension of Chang and Lowen's fuzzy topology and developed the theory of smooth topological spaces in [24,25]. After that, several authors [5, 9-13, 18, 20] have reintroduced the same definition and studied smooth topological spaces being unaware of Šostak's work. They referred to the fuzzy topology in the sense of Chang and Lowen as the topology on fuzzy subsets.

On the other hand, Atanassov [1] introduced the idea of intuitionistic fuzzy sets. Recently, much work has been done with these concepts [1, 2, 3]. Çoker [7,8] introduced the notion of intuitionistic fuzzy topological spaces in a Chang'sense using intuitionistic fuzzy sets. Samanta and Mondal [21, 22] introduced the notion of an intuitionistic gradation of openness as an extension of a smooth topology in a Šostak'sense.

In this paper, we introduce the notion of (r,s)-connected fuzzy sets in intuitionistic fuzzy topological spaces and investigate some properties

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of them. We show that every (r,s)-component in an intuitionistic fuzzy topological space is (r,s)-component in the stratification of it.

In this paper, let X be a nonempty set, $I = [0,1]$, $I_0 = (0,1]$ and $I_1 = [0,1)$. For $\alpha \in I$, $\overline{\alpha}(x) = \alpha$ for all $x \in X$. The family of all fuzzy sets on X denoted by I^X.

DEFINITION 1.1. ([22]) An *intuitionistic gradation of openness* (IGO, for short) on X is an ordered pair $(\mathcal{T}, \mathcal{T}^*)$ of mappings from I^X to I such that

(IGO1) $\mathcal{T}(\lambda) + \mathcal{T}^*(\lambda) \leq 1$, for all $\lambda \in I^X$,

(IGO2) $\mathcal{T}(\overline{0}) = \mathcal{T}(\overline{1}) = 1$ and $\mathcal{T}^*(\overline{0}) = \mathcal{T}^*(\overline{1}) = 0$,

(IGO3) $\mathcal{T}(\lambda_1 \wedge \lambda_2) \geq \mathcal{T}(\lambda_1) \wedge \mathcal{T}(\lambda_2)$ and $\mathcal{T}^*(\lambda_1 \wedge \lambda_2) \leq \mathcal{T}^*(\lambda_1) \vee \mathcal{T}^*(\lambda_2)$, for each $\lambda_i \in I^X, i = 1, 2$,

(IGO4) $\mathcal{T}(\bigvee_{i \in \Delta} \lambda_i) \geq \bigwedge_{i \in \Delta} \mathcal{T}(\lambda_i)$ and $\mathcal{T}^*(\bigvee_{i \in \Delta} \lambda_i) \leq \bigvee_{i \in \Delta} \mathcal{T}^*(\lambda_i)$ for each $\lambda_i \in I^X, i \in \Delta$.

The triplet $(X, \mathcal{T}, \mathcal{T}^*)$ is called an *intuitionistic fuzzy topological space* (ifts, for short). $\mathcal{T}$ and $\mathcal{T}^*$ may be interpreted as gradation of openness and gradation of nonopenness, respectively.

An ifts $(X, \mathcal{T}, \mathcal{T}^*)$ is called *stratified* if

(IS) $\mathcal{T}(\overline{\alpha}) = 1$ and $\mathcal{T}^*(\overline{\alpha}) = 0$ for each $\alpha \in I$.

Let $(\mathcal{U}, \mathcal{U}^*)$ and $(\mathcal{T}, \mathcal{T}^*)$ be IGO's on X. We say $(\mathcal{U}, \mathcal{U}^*)$ is *finer than* $(\mathcal{T}, \mathcal{T}^*)$ ($(\mathcal{T}, \mathcal{T}^*)$ is *coarser than* $(\mathcal{U}, \mathcal{U}^*)$) if $\mathcal{T}(\lambda) \leq \mathcal{U}(\lambda)$ and $\mathcal{T}^*(\lambda) \geq \mathcal{U}^*(\lambda)$ for all $\lambda \in I^X$.

DEFINITION 1.2. ([14]) Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts. A function $\mathcal{C} : I^X \times I_0 \times I_1 \to I^X$ is called an *intuitionistic closure operator* if for $\lambda, \mu \in I^X$, $r \in I_0$ and $s \in I_1$ with $r + s \leq 1$, it satisfies the following conditions:

(C1) $\mathcal{C}(\overline{0}, r, s) = \overline{0}$.

(C2) $\lambda \leq \mathcal{C}(\lambda, r, s)$.

(C3) $\mathcal{C}(\lambda, r, s) \vee \mathcal{C}(\mu, r, s) = \mathcal{C}(\lambda \vee \mu, r, s)$.

(C4) $\mathcal{C}(\lambda, r, s) \leq \mathcal{C}(\lambda, r_1, s_1)$ if $r \leq r_1$, $s \geq s_1$ with $r_1 + s_1 \leq 1$.

(C5) $\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) = \mathcal{C}(\lambda, r, s)$.

THEOREM 1.3. ([14]) *Let $\mathcal{C}$ be an intuitionistic closure operator on X. Define the functions $\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}} : I^X \to I$ by*

$$\mathcal{T}_{\mathcal{C}}(\lambda) = \bigvee \{r \in I_0 \mid \mathcal{C}(\overline{1} - \lambda, r, s) = \overline{1} - \lambda\},$$

$$\mathcal{T}^*_{\mathcal{C}}(\lambda) = \bigwedge \{s \in I_1 \mid \mathcal{C}(\overline{1} - \lambda, r, s) = \overline{1} - \lambda\}.$$

Then, $(\mathcal{T}_{\mathcal{C}}, \mathcal{T}^*_{\mathcal{C}})$ is an IGO on X.

THEOREM 1.4. ([14]) *Let* $(X, \mathcal{T}, \mathcal{T}^*)$ *be an ifts. Then for each* $r \in I_0$, $s \in I_1$, $\lambda \in I^X$ *we define an operator* $C_{\mathcal{T},\mathcal{T}^*} : I^X \times I_0 \times I_1 \to I^X$ *as follows*

$$C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \bigwedge\{\mu \in I^X \mid \lambda \leq \mu, \mathcal{T}(\overline{1} - \mu) \geq r, \mathcal{T}^*(\overline{1} - \mu) \leq s\}.$$

Then (1) $C_{\mathcal{T},\mathcal{T}^*}$, *is an intuitionistic closure operator.*

(2) $\mathcal{T}_{C_{\mathcal{T},\mathcal{T}^*}} = \mathcal{T}$ *and* $\mathcal{T}^*_{C_{\mathcal{T},\mathcal{T}^*}} = \mathcal{T}^*$.

PROOF. (1) It is proved in [14].

(2) Let $\mathcal{T}(\mu) = r$ and $\mathcal{T}^*(\mu) = s$. Then $C_{\mathcal{T},\mathcal{T}^*}(\overline{1} - \mu, r, s) = \overline{1} - \mu$. Hence $\mathcal{T}_{C_{\mathcal{T},\mathcal{T}^*}} \geq \mathcal{T}$ and $\mathcal{T}^*_{C_{\mathcal{T},\mathcal{T}^*}} \leq \mathcal{T}^*$.

Suppose $\mathcal{T}_{C_{\mathcal{T},\mathcal{T}^*}} \not\leq \mathcal{T}$. Then there exists μ with $C_{\mathcal{T},\mathcal{T}^*}(\overline{1} - \mu, r, s) = \overline{1} - \mu$ such that

$$\mathcal{T}_{C_{\mathcal{T},\mathcal{T}^*}}(\mu) \geq r > \mathcal{T}(\mu).$$

But, by the definition of $C_{\mathcal{T},\mathcal{T}^*}$, $\mathcal{T}(\mu) \geq r$. It is a contradiction. Other case is similarly proved. □

DEFINITION 1.5. ([22]) Let $(X, \mathcal{T}, \mathcal{T}^*)$ and $(Y, \mathcal{U}, \mathcal{U}^*)$ be ifts's and $f : X \to Y$ a function. Then, f is called *intuitionistic continuous* if $\mathcal{U}(\lambda) \leq \mathcal{T}(f^{-1}(\lambda))$ and $\mathcal{U}^*(\lambda) \geq \mathcal{T}^*(f^{-1}(\lambda))$ for all $\lambda \in I^Y$.

THEOREM 1.6. *Let* $(X, \mathcal{T}, \mathcal{T}^*)$ *and* $(Y, \mathcal{U}, \mathcal{U}^*)$ *be ifts's and* $f : (X, \mathcal{T}, \mathcal{T}^*) \to (Y, \mathcal{U}, \mathcal{U}^*)$ *a function. Then the following statements are equivalent, for each* $\lambda \in I^X, \mu \in I^Y, r \in I_0, s \in I_1$.

(1) f *is intuitionistic continuous.*

(2) $f(C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)) \leq C_{\mathcal{U},\mathcal{U}^*}(f(\lambda), r, s)$.

(3) $C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s) \leq f^{-1}(C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s))$.

PROOF. (1)$\Rightarrow$ (2) Let f be intuitionistic continuous. Then $\mathcal{T}(\overline{1} -$

$f^{-1}(\mu)) \geq \mathcal{U}(\overline{1} - \mu)$ and $\mathcal{T}^*(\overline{1} - f^{-1}(\mu)) \leq \mathcal{U}^*(\overline{1} - \mu)$. Hence

$$\begin{aligned}
& C_{\mathcal{U},\mathcal{U}^*}(f(\lambda), r, s) \\
&= \bigwedge\{\mu \in I^Y \mid f(\lambda) \leq \mu, \mathcal{U}(\overline{1} - \mu) \geq r, \mathcal{U}^*(\overline{1} - \mu) \leq s\} \\
&\geq \bigwedge\{\mu \in I^Y \mid \lambda \leq f^{-1}(\mu), \mathcal{T}(\overline{1} - f^{-1}(\mu)) \geq r, \\
&\quad \mathcal{T}^*(\overline{1} - f^{-1}(\mu)) \leq s\} \\
&\geq \bigwedge\{f(f^{-1}(\mu)) \in I^Y \mid \lambda \leq f^{-1}(\mu), \mathcal{T}(\overline{1} - f^{-1}(\mu)) \geq r, \\
&\quad \mathcal{T}^*(\overline{1} - f^{-1}(\mu)) \leq s\} \\
&\geq f(\bigwedge\{f^{-1}(\mu) \in I^Y \mid \lambda \leq f^{-1}(\mu), \mathcal{T}(\overline{1} - f^{-1}(\mu)) \geq r, \\
&\quad \mathcal{T}^*(\overline{1} - f^{-1}(\mu)) \leq s\}) \\
&\geq f(C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)).
\end{aligned}$$

(2)$\Rightarrow$ (3) For all $\mu \in I^Y$, put $\lambda = f^{-1}(\mu)$. By (2),

$$f(C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s)) \leq C_{\mathcal{U},\mathcal{U}^*}(f(f^{-1}(\mu)), r, s) \leq C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s).$$

Thus $C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s) \leq f^{-1}(C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s))$.

(3)$\Rightarrow$ (1) It follows from $C_{\mathcal{U},\mathcal{U}^*}(\mu, r, s) = \mu$ implies $C_{\mathcal{T},\mathcal{T}^*}(f^{-1}(\mu), r, s) = f^{-1}(\mu)$. □

2. Connectedness in intuitionistic fuzzy topological spaces

DEFINITION 2.1. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts. For $\lambda, \mu, \rho \in I^X$, λ and μ are called (r,s)-*separated* if for $r \in I_0$ and $s \in I_1$,

$$C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \wedge \mu = C_{\mathcal{T},\mathcal{T}^*}(\mu, r, s) \wedge \lambda = \overline{0}.$$

A fuzzy set ρ is called (r,s)-*connected* if there not exist (r,s)-separated $\lambda, \mu \in I^X - \{\overline{0}\}$ such that $\rho = \lambda \vee \mu$. A fuzzy set ρ is called *connected* if it is (r,s)-connected for all $r \in I_0$ and $s \in I_1$. A triplet $(X, \mathcal{T}, \mathcal{T}^*)$ is called (r,s)-*connected* if $\overline{1}$ is (r,s)-connected.

REMARK 2.2. Let λ and μ be (r,s)-separated. For each $\rho \in I^X$ and $r_1 \leq r$, $s_1 \geq s$, since $C_{\mathcal{T},\mathcal{T}^*}(\rho, r_1, s_1) \leq C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s)$, λ and μ are (r_1, s_1)-separated. Furthermore, from this fact, if ρ is (r_1, s_1)-connected for $r_1 \leq r$, $s_1 \geq s$, ρ is (r,s)-connected.

EXAMPLE 2.3. Let $X = \{x, y\}$ be a set. We define an IGO $(\mathcal{T}, \mathcal{T}^*)$ on X as follows: for each $\lambda \in I^X$,

$$\mathcal{T}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{3} & \text{if } \lambda = \chi_{\{x\}}, \\ \frac{1}{2} & \text{if } \lambda = \chi_{\{y\}}, \\ 0 & \text{otherwise,} \end{cases} \qquad \mathcal{T}^*(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{2}{3} & \text{if } \lambda = \chi_{\{x\}}, \\ \frac{1}{2} & \text{if } \lambda = \chi_{\{y\}}, \\ 1 & \text{otherwise.} \end{cases}$$

We can obtain

$$C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \begin{cases} \overline{0} & \text{if } \lambda = \overline{0},\ r \in I_0, s \in I_1, \\ \chi_{\{x\}} & \text{if } \overline{0} \neq \lambda \leq \chi_{\{x\}},\ r \leq \frac{1}{2}, s \geq \frac{1}{2}, \\ \chi_{\{y\}} & \text{if } \overline{0} \neq \lambda \leq \chi_{\{y\}},\ r \leq \frac{1}{3}, s \geq \frac{2}{3}, \\ \overline{1} & \text{otherwise.} \end{cases}$$

If $r \leq \frac{1}{3}, s \geq \frac{2}{3}$, then $(\chi_{\{x\}} = C_{\mathcal{T},\mathcal{T}^*}(\chi_{\{x\}}, r, s)) \wedge \chi_{\{y\}} = \overline{0}$ and $\chi_{\{x\}} \wedge (\chi_{\{y\}} = C_{\mathcal{T},\mathcal{T}^*}(\chi_{\{y\}}, r, s)) = \overline{0}$. Thus, $\overline{1}_X = \chi_{\{x\}} \vee \chi_{\{y\}}$ is not (r,s)-connected for $r \leq \frac{1}{3}$ and $s \geq \frac{2}{3}$. If $r > \frac{1}{3}$ and $s < \frac{2}{3}$, $(X, \mathcal{T}, \mathcal{T}^*)$ is (r,s)-connected.

THEOREM 2.4. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts. The following statements are equivalent.*

(1) $(X, \mathcal{T}, \mathcal{T}^*)$ is (r,s)-connected.

(2) *If $\lambda \vee \mu = \overline{1}$ and $\lambda \wedge \mu = \overline{0}$ for $(\mathcal{T}(\lambda) \geq r,\ \mathcal{T}^*(\lambda) \leq s)$ and $(\mathcal{T}(\mu) \geq r,\ \mathcal{T}^*(\mu) \leq s)$, then $\lambda = \overline{0}$ or $\mu = \overline{0}$.*

(3) *If $\lambda \vee \mu = \overline{1}$ and $\lambda \wedge \mu = \overline{0}$ for $(\mathcal{T}(\overline{1} - \lambda) \geq r,\ \mathcal{T}^*(\overline{1} - \lambda) \leq s)$ and $(\mathcal{T}(\overline{1} - \mu) \geq r,\ \mathcal{T}^*(\overline{1} - \mu) \leq s)$, then $\lambda = \overline{0}$ or $\mu = \overline{0}$.*

PROOF. (1) $\Rightarrow$ (2) Suppose that there exist $\lambda, \mu \in I^X - \{\overline{0}\}$ such that for $(\mathcal{T}(\lambda) \geq r,\ \mathcal{T}^*(\lambda) \leq s)$ and $(\mathcal{T}(\mu) \geq r,\ \mathcal{T}^*(\mu) \leq s)$, $\lambda \vee \mu = \overline{1},\ \lambda \wedge \mu = \overline{0}$. It implies

$$(\overline{1} - \lambda) \wedge (\overline{1} - \mu) = \overline{0}, \ \ (\overline{1} - \lambda) \vee (\overline{1} - \mu) = \overline{1}.$$

Since $C_{\mathcal{T},\mathcal{T}^*}(\overline{1} - \lambda, r, s) = \overline{1} - \lambda$ and $C_{\mathcal{T},\mathcal{T}^*}(\overline{1} - \mu, r, s) = \overline{1} - \mu$ from Theorem 1.4, $\overline{1} - \lambda$ and $\overline{1} - \mu$ are (r,s)-separated. Suppose $\lambda = \overline{1}$. Then $\mu = \lambda \wedge \mu = \overline{0}$. It is a contradiction. Thus, $\overline{1} - \lambda \in I^X - \{\overline{0}\}$. Similarly, $\overline{1} - \mu \in I^X - \{\overline{0}\}$. Furthermore, $(\overline{1} - \lambda) \vee (\overline{1} - \mu) = \overline{1}$. Hence $\overline{1}$ is not (r,s)-connected.

(2) $\Rightarrow$ (3) By the De Morgan's law, it is easily proved.

(3) $\Rightarrow$ (1) If $(X,\mathcal{T},\mathcal{T}^*)$ is not (r,s)-connected, then there exist (r,s)-separated $\lambda,\mu \in I^X - \{\overline{0}\}$ such that $\lambda \vee \mu = \overline{1}$. Since $\lambda \wedge \mu \le C_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s) \wedge \mu = \overline{0}$, we have $\lambda \wedge \mu = \overline{0}$. So, $(\overline{1}-\lambda) \wedge (\overline{1}-\mu) = \overline{0}$ implies $\overline{1}-\mu \le \lambda$. Furthermore, $C_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s) \wedge \mu = \overline{0}$ implies $C_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s) \le \overline{1}-\mu$. Hence $C_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s) \le \lambda$. By Definition 1.2(C2), we have $C_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s) = \lambda$. From Theorem 1.4, we have $\mathcal{T}(\overline{1}-\lambda) \ge r$ and $\mathcal{T}^*(\overline{1}-\lambda) \le s$. Similarly, we have $\mathcal{T}(\overline{1}-\mu) \ge r$ and $\mathcal{T}^*(\overline{1}-\mu) \le s$. It does not satisfy the condition of (3). □

LEMMA 2.5. *Let $(X,\mathcal{T},\mathcal{T}^*)$ be an ifts and $\lambda,\mu,\rho \in I^X$. If μ and ρ are (r,s)-separated, $\lambda \wedge \mu$ and $\lambda \wedge \rho$ are (r,s)-separated.*

PROOF. Let μ and ρ be (r,s)-separated. So,

$$C_{\mathcal{T},\mathcal{T}^*}(\lambda \wedge \mu, r, s) \wedge (\lambda \wedge \rho) \le C_{\mathcal{T},\mathcal{T}^*}(\mu, r, s) \wedge \rho = \overline{0}.$$

Similarly, $(\lambda \wedge \mu) \wedge C_{\mathcal{T},\mathcal{T}^*}(\lambda \wedge \rho, r, s) = \overline{0}$. Hence $\lambda \wedge \mu$ and $\lambda \wedge \rho$ are (r,s)-separated. □

THEOREM 2.6. *Let $(X,\mathcal{T},\mathcal{T}^*)$ be an ifts and $\lambda \in I^X$. The following statements are equivalent.*

(1) λ is (r,s)-connected.

(2) If μ and ρ are (r,s)-separated such that $\lambda \le \mu \vee \rho$, then $\lambda \wedge \mu = \overline{0}$ or $\lambda \wedge \rho = \overline{0}$.

(3) If μ and ρ are (r,s)-separated such that $\lambda \le \mu \vee \rho$, then $\lambda \le \mu$ or $\lambda \le \rho$.

PROOF. (1) $\Rightarrow$ (2) Let μ and ρ be (r,s)-separated such that $\lambda \le \mu \vee \rho$. By Lemma 2.5, $\lambda \wedge \mu$ and $\lambda \wedge \rho$ are (r,s)-separated. Since λ is (r,s)-connected and $\lambda = \lambda \wedge (\mu \vee \rho) = (\lambda \wedge \mu) \vee (\lambda \wedge \rho)$, then $\lambda \wedge \mu = \overline{0}$ or $\lambda \wedge \rho = \overline{0}$.

(2) $\Rightarrow$ (3) It easily proved from the following statements. If $\lambda \wedge \mu = \overline{0}$, then

$$\lambda = \lambda \wedge (\mu \vee \rho) = (\lambda \wedge \mu) \vee (\lambda \wedge \rho) = \overline{0} \vee (\lambda \wedge \rho) = \lambda \wedge \rho.$$

Hence $\lambda \le \rho$. If $\lambda \wedge \rho = \overline{0}$, similarly, $\lambda \le \mu$.

(3) $\Rightarrow$ (1) Let μ and ρ be (r,s)-separated such that $\lambda = \mu \vee \rho$. By (3), $\lambda \le \mu$ or $\lambda \le \rho$. If $\lambda \le \mu$ and μ and ρ are (r,s)-separated, then

$$\rho = \rho \wedge \lambda \le \rho \wedge \mu \le \rho \wedge C_{\mathcal{T}}(\mu, r, s) = \overline{0}.$$

Hence $\rho = \overline{0}$. If $\lambda \le \rho$, similarly $\mu = \overline{0}$. □

THEOREM 2.7. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts and $\lambda, \mu \in I^X$.*

(1) If λ is (r,s)-connected and $\lambda \le \mu \le C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$, then μ is (r,s)-connected.

(2) If λ and μ are (r,s)-connected fuzzy sets which are not (r,s)-separated, then $\lambda \vee \mu$ is (r,s)-connected.

PROOF. (1) Let ν and ρ be (r,s)-separated such that $\mu = \nu \vee \rho$. Put $\nu_1 = \lambda \wedge \nu$ and $\rho_1 = \lambda \wedge \rho$. Then ν_1 and ρ_1 are (r,s)-separated such that $\lambda = \nu_1 \vee \rho_1$. Since λ is (r,s)-connected, $\nu_1 = \overline{0}$ or $\rho_1 = \overline{0}$. If $\nu_1 = \overline{0}$, then $\lambda = \rho_1 = \lambda \wedge \rho \Rightarrow \lambda \le \rho$. It implies

$$\mu \le C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) \le C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s).$$

Hence $\nu = \nu \wedge \mu \le \nu \wedge C_{\mathcal{T},\mathcal{T}^*}(\rho, r, s) = \overline{0}$. If $\rho_1 = \overline{0}$, similarly, $\rho = \overline{0}$. Thus, μ is (r,s)-connected.

(2) Let ν and ρ be (r,s)-separated such that $\lambda \vee \mu = \nu \vee \rho$. Since λ is (r,s)-connected, by Theorem 2.6(3), $\lambda \le \nu$ or $\lambda \le \rho$. Say $\lambda \le \nu$. Suppose $\mu \le \rho$. Since $(\lambda \vee \mu) \wedge \nu = \lambda$ and $(\lambda \vee \mu) \wedge \rho = \mu$, by Lemma 2.5, λ and μ are (r,s)-separated. It is a contradiction. Hence $\mu \le \nu$. Thus $\lambda \vee \mu \le \nu$, by Theorem 2.6(3), $\lambda \vee \mu$ is (r,s)-connected. □

THEOREM 2.8. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts. Let $\mathcal{A} = \{\lambda_i \mid i \in \Gamma\}$ be a family of (r,s)-connected fuzzy sets in $(X, \mathcal{T}, \mathcal{T}^*)$ such that no two members of $\mathcal{A}$ are (r,s)-separated. Then $\bigvee_{i\in\Gamma} \lambda_i$ is (r,s)-connected.*

PROOF. Put $\lambda = \bigvee_{i\in\Gamma} \lambda_i$. Let μ and ρ be (r,s)-separated such that $\lambda = \mu \vee \rho$. Since any two member $\lambda_i, \lambda_j \in \mathcal{A}$ are not (r,s)-separated, by Theorem 2.7 (2), $\lambda_i \vee \lambda_j$ is (r,s)-connected. From Theorem 2.6(3), $\lambda_i \vee \lambda_j \le \mu$ or $\lambda_i \vee \lambda_j \le \rho$, say $\lambda_i \vee \lambda_j \le \mu$. It implies $\lambda \le \mu$. Hence λ is (r,s)-connected. □

The following corollary is obvious from Theorem 2.8.

COROLLARY 2.9. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts. Let $\{\lambda_i \mid i \in \Gamma\}$ be a family of (r,s)-connected fuzzy sets in $(X, \mathcal{T}, \mathcal{T}^*)$. If $\bigwedge_{i\in\Gamma} \lambda_i \ne \overline{0}$, then $\bigvee_{i\in\Gamma} \lambda_i$ is (r,s)-connected.*

THEOREM 2.10. *Let $(X, \mathcal{T}_1, \mathcal{T}_1^*)$ and $(Y, \mathcal{T}_2, \mathcal{T}_2^*)$ be ifts's. If $f : (X, \mathcal{T}_1, \mathcal{T}_1^*) \to (Y, \mathcal{T}_2, \mathcal{T}_2^*)$ is intuitionistic continuous and λ is (r,s)-connected, then $f(\lambda)$ is (r,s)-connected.*

PROOF. Let μ and ρ be (r,s)-separated such that $f(\lambda) = \mu \vee \rho$. We have

$$\lambda \leq f^{-1}(f(\lambda)) = f^{-1}(\mu \vee \rho) = f^{-1}(\mu) \vee f^{-1}(\rho).$$

Since f is intuitionistic continuous, by Theorem 1.6(3),

$$C_{\mathcal{T}_1,\mathcal{T}_1^*}(f^{-1}(\mu), r, s) \leq f^{-1}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\mu, r, s)).$$

Hence

$$\begin{aligned} C_{\mathcal{T}_1,\mathcal{T}_1^*}(f^{-1}(\mu), r, s) \wedge f^{-1}(\rho) &\leq f^{-1}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\mu, r, s)) \wedge f^{-1}(\rho) \\ &= f^{-1}(C_{\mathcal{T}_2,\mathcal{T}_2^*}(\mu, r, s) \wedge \rho) \\ &= f^{-1}(\overline{0}) = \overline{0}. \end{aligned}$$

Similarly, we have $f^{-1}(\mu) \wedge C_{\mathcal{T}_1,\mathcal{T}_1^*}(f^{-1}(\rho), r, s) = \overline{0}$. So, $f^{-1}(\mu)$ and $f^{-1}(\rho)$ are (r,s)-separated. Since λ is (r,s)-connected, by Theorem 2.6(3), $\lambda \leq f^{-1}(\mu)$ or $\lambda \leq f^{-1}(\rho)$, say $\lambda \leq f^{-1}(\mu)$. Then $f(\lambda) \leq f(f^{-1}(\mu)) \leq \mu$. Therefore, $f(\lambda)$ is (r,s)-connected. □

DEFINITION 2.11. Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts. A fuzzy set λ is a (r,s)-*component* in $(X, \mathcal{T}, \mathcal{T}^*)$ if λ is a maximal (r,s)-connected fuzzy set in $(X, \mathcal{T}, \mathcal{T}^*)$, i.e. if $\mu \geq \lambda$ and μ is (r,s)-connected, then $\mu = \lambda$.

THEOREM 2.12. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts.*

(1) *If λ is (r,s)-component, then $C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \lambda$.*

(2) *If λ_1 and λ_2 are (r,s)-components in $(X, \mathcal{T}, \mathcal{T}^*)$ such that $\lambda_1 \wedge \lambda_2 = \overline{0}$, then λ_1 and λ_2 are (r,s)-separated.*

(3) *Each fuzzy point x_t is connected.*

(4) *Every (r,s)-component is a crisp set.*

PROOF. (1) Since λ is (r,s)-connected and $\lambda \leq C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$, by Theorem 2.7(1), $C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s)$ is (r,s)-connected. Since λ is (r,s)-component, $C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \lambda$.

(2) By (1), it is trivial.

(3) Let λ and μ be (r,s)-separated such that $x_t = \lambda \vee \mu$. Then $x_t = \lambda$ or $x_t = \mu$. If $x_t = \lambda$, then

$$\mu = \mu \wedge (\lambda \vee \mu) = \mu \wedge x_t = \mu \wedge \lambda \leq \mu \wedge C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \overline{0}.$$

Similarly, if $x_t = \mu$, then $\lambda = \overline{0}$. Hence x_t is connected.

(4) Let λ be a (r,s)-component with $x \in supp(\lambda) = \{x \in X \mid \lambda(x) > 0\}$ and μ a (r,s)-component containing x_1. Since $\lambda \wedge \mu \geq x_{\lambda(x)} \wedge x_1 = x_{\lambda(x)} \neq \overline{0}$, by Corollary 2.9, $\lambda \vee \mu$ is (r,s)-connected. Thus $\lambda = \mu = \lambda \vee \mu$ is (r,s)-component. So, $x \in supp(\lambda)$ implies $\lambda(x) = 1$, that is, λ is a crisp set. □

3. Stratification of intuitionistic fuzzy topological spaces

THEOREM 3.1. *Let $(X, \mathcal{T}, \mathcal{T}^*)$ be an ifts. Define the functions $\mathcal{T}_{st}, \mathcal{T}^*_{st}$ $I^X \to I$ as follows: for each $\lambda \in I^X$,*

$$\mathcal{T}_{st}(\lambda) = \bigvee \{ \bigwedge_{j\in J} \mathcal{T}(\lambda_j) \mid \lambda = \bigvee_{j\in J} (\lambda_j \wedge \overline{\alpha_j})\}$$

where the first $\bigvee$ is taken over all families $\{\lambda_j \mid j \in J\}$ with $\lambda = \bigvee_{j\in J}(\lambda_j \wedge \overline{\alpha_j})$,

$$\mathcal{T}^*_{st}(\lambda) = \bigwedge \{ \bigvee_{j\in J} \mathcal{T}^*(\lambda_j) \mid \lambda = \bigvee_{j\in J} (\lambda_j \wedge \overline{\alpha_j})\}$$

where the first $\bigwedge$ is taken over all families $\{\lambda_j \mid j \in J\}$ with $\lambda = \bigvee_{j\in J}(\lambda_j \wedge \overline{\alpha_j})$. Then $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is the coarsest stratified IGO on X which is finer than $(\mathcal{T}, \mathcal{T}^*)$.

PROOF. First, we will show that $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is a stratified IGO on X.
(IGO1) Suppose there exists $\lambda \in I^X$ such that

$$\mathcal{T}_{st}(\lambda) + \mathcal{T}^*_{st}(\lambda) > 1.$$

There exist $r \in I$ and a family $\{\lambda_j \mid j \in J\}$ with $\lambda = \bigvee_{j\in J}(\lambda_j \wedge \overline{\alpha_j})$ such that

$$\mathcal{T}_{st}(\lambda) \geq \bigwedge_{j\in J} \mathcal{T}(\lambda_j) > r > 1 - \mathcal{T}^*_{st}(\lambda).$$

Since $\mathcal{T}(\lambda_j) > r$ for each $j \in J$, there exist r_j and s_j such that

$$\mathcal{T}(\lambda_j) \geq r_j > r, \ \ \mathcal{T}^*(\lambda_j) \leq s_j \leq 1 - r_j.$$

Hence

$$\mathcal{T}^*_{st}(\lambda) \leq \bigvee_{j\in J} \mathcal{T}^*(\lambda_j) \leq \bigvee_{j\in J} s_j \leq \bigvee_{j\in J} (1 - r_j) \leq 1 - r.$$

It is contradiction.
(IGO2) and (IS). For each $\alpha \in I$, there exists a family $\{\overline{1}\}$ with $\overline{\alpha} = \overline{\alpha} \wedge \overline{1}$, we have $\mathcal{T}_{st}(\overline{\alpha}) \geq \mathcal{T}(\overline{1}) = 1$ and $\mathcal{T}^*_{st}(\overline{\alpha}) \leq \mathcal{T}^*(\overline{1}) = 0$. Hence $\mathcal{T}_{st}(\overline{\alpha}) = 1$ and $\mathcal{T}^*_{st}(\overline{\alpha}) = 0$.

(IGO3) Suppose there exist $\mu, \nu \in I^X$ and $r \in I_0$, $s \in I_1$ with

$$\mathcal{T}_{st}(\mu \wedge \nu) < r < \mathcal{T}_{st}(\mu) \wedge \mathcal{T}_{st}(\nu) \text{ and } \mathcal{T}^*_{st}(\mu \wedge \nu) > s > \mathcal{T}^*_{st}(\mu) \vee \mathcal{T}^*_{st}(\nu).$$

Since ($\mathcal{T}_{st}(\mu) > r$ and $\mathcal{T}_{st}(\nu) > r$) and ($\mathcal{T}^*_{st}(\mu) < s$ and $\mathcal{T}^*_{st}(\nu) < s$), by the definition of $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$, there exist two families $\{\mu_j \mid j \in J\}$ with $\mu = \bigvee_{j \in J}(\mu_j \wedge \overline{\alpha_j})$ and $\{\nu_k \mid k \in K\}$ with $\nu = \bigvee_{k \in K}(\nu_k \wedge \overline{\alpha_k})$ such that

$$\mathcal{T}_{st}(\mu) \geq \bigwedge_{j \in J} \mathcal{T}(\mu_j) > r \text{ and } \mathcal{T}_{st}(\nu) \geq \bigwedge_{k \in K} \mathcal{T}(\nu_k) > r$$

$$\mathcal{T}^*_{st}(\mu) \leq \bigvee_{j \in J} \mathcal{T}^*(\mu_j) < s \text{ and } \mathcal{T}^*_{st}(\nu) \leq \bigvee_{k \in K} \mathcal{T}^*(\nu_k) < s.$$

Since I is completely distributive lattice, we have

$$\begin{aligned}
\mu \wedge \nu &= \Big(\bigvee_{j \in J} (\mu_j \wedge \overline{\alpha_j})\Big) \wedge \Big(\bigvee_{k \in K} (\nu_k \wedge \overline{\alpha_k})\Big) \\
&= \bigvee_{j,k} (\mu_j \wedge \nu_k) \wedge (\overline{\alpha_j} \wedge \overline{\alpha_k}) \\
&= \bigvee_{j,k} (\mu_j \wedge \nu_k) \wedge \overline{\alpha_{jk}}. \qquad (\overline{\alpha_{jk}} = \overline{\alpha_j} \wedge \overline{\alpha_k})
\end{aligned}$$

Moreover, since $\mathcal{T}(\mu_j \wedge \nu_k) \geq \mathcal{T}(\mu_j) \wedge \mathcal{T}(\nu_k)$ and $\mathcal{T}^*(\mu_j \wedge \nu_k) \leq \mathcal{T}^*(\mu_j) \vee \mathcal{T}^*(\nu_k)$, we have

$$\begin{aligned}
\mathcal{T}_{st}(\mu \wedge \nu) &\geq \bigwedge_{j,k} \mathcal{T}(\mu_j \wedge \nu_k) \\
&\geq \bigwedge_{j,k} \big(\mathcal{T}(\mu_j) \wedge \mathcal{T}(\nu_k)\big) \\
&= \Big(\bigwedge_{j \in J} \mathcal{T}(\mu_j)\Big) \wedge \Big(\bigwedge_{k \in K} \mathcal{T}(\nu_k)\Big) > r,
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}^*_{st}(\mu \wedge \nu) &\leq \bigvee_{j,k} \mathcal{T}^*(\mu_j \wedge \nu_k) \\
&\leq \bigvee_{j,k} \big(\mathcal{T}^*(\mu_j) \vee \mathcal{T}^*(\nu_k)\big) \\
&= \Big(\bigvee_{j \in J} \mathcal{T}^*(\mu_j)\Big) \vee \Big(\bigvee_{k \in K} \mathcal{T}^*(\nu_k)\Big) < s.
\end{aligned}$$

It is a contradiction. Hence, for all $\mu, \nu \in I^X$,

$$\mathcal{T}_{st}(\mu \wedge \nu) \geq \mathcal{T}_{st}(\mu) \wedge \mathcal{T}_{st}(\nu) \text{ and } \mathcal{T}^*_{st}(\mu \wedge \nu) \leq \mathcal{T}^*_{st}(\mu) \vee \mathcal{T}^*_{st}(\nu).$$

(IGO4) Suppose there exists a family $\{\mu_i \in I^X \mid i \in \Gamma\}$ and $r \in I_0$, $s \in I_1$ with

$$\mathcal{T}_{st}(\bigvee_{i\in\Gamma} \mu_j) < r < \bigwedge_{i\in\Gamma} \mathcal{T}_{st}(\mu_i),\ \mathcal{T}^*_{st}(\bigvee_{i\in\Gamma} \mu_j) > s > \bigvee_{i\in\Gamma} \mathcal{T}^*_{st}(\mu_i).$$

Since $\mathcal{T}_{st}(\mu_i) > r$ and $\mathcal{T}^*_{st}(\mu_i) < s$ for each $i \in \Gamma$, there exists a family $\{\mu_{ij} \mid j \in J_i\}$ with $\mu_i = \bigvee_{j\in J_i}(\mu_{ij} \wedge \overline{\alpha_j})$ such that

$$\mathcal{T}_{st}(\mu_i) \geq \bigwedge_{j\in J_i} \mathcal{T}(\mu_{ij}) > r \text{ and } \mathcal{T}^*_{st}(\mu_i) \leq \bigvee_{j\in J_i} \mathcal{T}^*(\mu_{ij}) < s.$$

Since $\bigvee_{i\in\Gamma} \mu_i = \bigvee_{i\in\Gamma} \left(\bigvee_{j\in J_i}(\mu_{ij} \wedge \overline{\alpha_j})\right) = \bigvee_{i,j}(\mu_{ij} \wedge \overline{\alpha_j})$, we have

$$\mathcal{T}_{st}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i,j} \mathcal{T}(\mu_{ij}) = \bigwedge_{i\in\Gamma} \Big(\bigwedge_{j\in J_i} \mathcal{T}(\mu_{ij})\Big) \geq r,$$

$$\mathcal{T}^*_{st}(\bigvee_{i\in\Gamma} \mu_i) \leq \bigvee_{i,j} \mathcal{T}^*(\mu_{ij}) = \bigvee_{i\in\Gamma} \Big(\bigvee_{j\in J_i} \mathcal{T}^*(\mu_{ij})\Big) \leq s.$$

It is a contradiction. Hence, for any $\{\mu_i\}_{i\in\Gamma} \subset I^X$,

$$\mathcal{T}_{st}(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \mathcal{T}_{st}(\mu_i) \text{ and } \mathcal{T}^*_{st}(\bigvee_{i\in\Gamma} \mu_i) \leq \bigvee_{i\in\Gamma} \mathcal{T}^*_{st}(\mu_i).$$

Second, for each $\lambda \in I^X$, there exists a family $\{\overline{1}\}$ with $\lambda = \overline{1} \wedge \lambda$ such that $\mathcal{T}_{st}(\lambda) \geq \mathcal{T}(\lambda)$ and $\mathcal{T}^*_{st}(\lambda) \leq \mathcal{T}^*(\lambda)$. Hence $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is finer than $(\mathcal{T}, \mathcal{T}^*)$. Finally, if a stratified IGO $(\mathcal{U}, \mathcal{U}^*)$ is finer than $(\mathcal{T}, \mathcal{T}^*)$, we will show that $\mathcal{T}_{st}(\lambda) \leq \mathcal{U}(\lambda)$ and $\mathcal{T}^*_{st}(\lambda) \geq \mathcal{U}^*(\lambda)$ for all $\lambda \in I^X$.

Suppose there exist $\mu \in I^X$ and $r \in I_0$, $s \in I_1$ such that

$$\mathcal{T}_{st}(\mu) > r > \mathcal{U}(\mu) \text{ and } \mathcal{T}^*_{st}(\mu) < s < \mathcal{U}^*(\mu).$$

Since $\mathcal{T}_{st}(\mu) > r$ and $\mathcal{T}^*_{st}(\mu) < s$, there exists a family $\{\mu_j \mid j \in J\}$ with $\mu = \bigvee_{j\in J}(\mu_j \wedge \overline{\alpha_j})$ such that

$$\mathcal{T}_{st}(\mu) \geq \bigwedge_{j\in J} \mathcal{T}(\mu_j) > r \text{ and } \mathcal{T}^*_{st}(\mu) \leq \bigvee_{j\in J} \mathcal{T}^*(\mu_j) < s.$$

On the other hand, since $\mathcal{U}(\mu_j) \geq \mathcal{T}(\mu_j)$ and $\mathcal{U}^*(\mu_j) \leq \mathcal{T}^*(\mu_j)$ for each $j \in J$, we have

$$\begin{aligned}\mathcal{U}(\mu) = \mathcal{U}(\bigvee_{j\in J}(\mu_j \wedge \overline{\alpha_j})) &\geq \bigwedge_{j\in J}\mathcal{U}(\mu_j \wedge \overline{\alpha_j})\\ &\geq \bigwedge_{j\in J}(\mathcal{U}(\mu_j) \wedge \mathcal{U}(\overline{\alpha_j})) = \bigwedge_{j\in J}\mathcal{U}(\mu_j)\\ &\geq \bigwedge_{j\in J}\mathcal{T}(\mu_j) > r,\end{aligned}$$

$$\begin{aligned}\mathcal{U}^*(\mu) = \mathcal{U}^*(\bigvee_{j\in J}(\mu_j \wedge \overline{\alpha_j})) &\leq \bigvee_{j\in J}\mathcal{U}^*(\mu_j \wedge \overline{\alpha_j})\\ &\leq \bigvee_{j\in J}(\mathcal{U}^*(\mu_j) \vee \mathcal{U}^*(\overline{\alpha_j})) = \bigvee_{j\in J}\mathcal{U}^*(\mu_j)\\ &\leq \bigvee_{j\in J}\mathcal{T}^*(\mu_j) < s.\end{aligned}$$

It is a contradiction. □

DEFINITION 3.2. In the above theorem $(\mathcal{T}_{st}, \mathcal{T}^*_{st})$ is called the *stratification* of an IGO $(\mathcal{T}, \mathcal{T}^*)$ on X.

EXAMPLE 3.3. Let $X = \{a, b\}$ be a set. Let $\mu, \rho \in I^X$ as follows:

$$\mu(x) = 0.5,\ \mu(y) = 0.5 \text{ and } \rho(x) = 0.4,\ \rho(y) = 0.6.$$

We define IGO on X as follows: for each $\lambda \in I^X$

$$\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = \overline{1}, \overline{0} \\ \frac{1}{3} & \text{if } \lambda = \mu \\ \frac{1}{2} & \text{if } \lambda = \rho \\ \frac{3}{4} & \text{if } \lambda = \mu \vee \rho \\ \frac{2}{3} & \text{if } \lambda = \mu \wedge \rho \\ 0 & \text{otherwise,} \end{cases} \qquad \mathcal{T}^*(\lambda) = \begin{cases} 0, & \text{if } \lambda = \overline{1}, \overline{0} \\ \frac{2}{3} & \text{if } \lambda = \mu \\ \frac{1}{2} & \text{if } \lambda = \rho \\ \frac{1}{4} & \text{if } \lambda = \mu \vee \rho \\ \frac{1}{3} & \text{if } \lambda = \mu \wedge \rho \\ 1 & \text{otherwise.} \end{cases}$$

If $\lambda(x) = \alpha$ for $0.5 < \alpha < 0.6$ and $\lambda(y) = 0.6$, for each $\beta \geq 0.6$, since

$$\begin{aligned}\lambda &= (\overline{\alpha} \wedge \overline{1}) \vee (\overline{\beta} \wedge (\mu \vee \rho))\\ &= (\overline{\alpha} \wedge \overline{1}) \vee (\overline{\beta} \wedge \rho)\end{aligned}$$

we have $\mathcal{T}_{st}(\lambda) = [\mathcal{T}(\overline{1}) \wedge \mathcal{T}(\mu \vee \rho)] \vee [\mathcal{T}(\overline{1}) \wedge \mathcal{T}(\rho)] = \frac{3}{4}$, and $\mathcal{T}^*_{st}(\lambda) = [\mathcal{T}^*(\overline{1}) \vee \mathcal{T}^*(\mu \vee \rho)] \wedge [\mathcal{T}^*(\overline{1}) \vee \mathcal{T}^*(\rho)] = \frac{1}{4}$.

If $\lambda(x) = \alpha$ for $0.5 < \alpha < 0.6$ and $\lambda(y) = \beta$ for $0.5 < \alpha,\ \beta < 0.6$ and $\alpha < \beta$, we have $\mathcal{T}_{st}(\lambda) = \frac{3}{4}$ and $\mathcal{T}^*_{st}(\lambda) = \frac{1}{4}$.

If $\lambda(x) = 0.5$ and $\lambda(y) = 0.6$, since for $\alpha \geq 0.5$ and $\beta \geq 0.6$,

$$\lambda = \overline{\beta} \wedge (\mu \vee \rho) = (\overline{\alpha} \wedge \mu) \vee (\overline{\beta} \wedge \rho),$$

we have $\mathcal{T}_{st}(\lambda) = \frac{3}{4}$ and $\mathcal{T}^*_{st}(\lambda) = \frac{1}{4}$.

If $\lambda(x) = 0.5$ and $\lambda(y) = \beta$ for $0.5 < \beta < 0.6$, since

$$\lambda = \overline{\beta} \wedge (\mu \vee \rho) = (\overline{\beta} \wedge \mu) \vee (\overline{\beta} \wedge \rho),$$

we have $\mathcal{T}_{st}(\lambda) = \frac{3}{4}$ and $\mathcal{T}^*_{st}(\lambda) = \frac{1}{4}$.

If $\lambda(x) = \alpha$ and $\lambda(y) = \beta$ for $0.4 < \alpha,\ \beta < 0.5$ and $\alpha < \beta$, since, for $\lambda_1 \in \{\overline{1}, \mu, \mu \vee \rho\}$, $\lambda_2 = \{\rho, \mu \wedge \rho\}$,

$$\lambda = (\overline{\alpha} \wedge \lambda_1) \vee (\overline{\beta} \wedge \lambda_2),$$

we have $\mathcal{T}^*_{st}(\lambda) = \frac{2}{3}$ and $\mathcal{T}^*_{st}(\lambda) = \frac{1}{3}$. By a similar method as the above cases, we can obtain the following:

$$\mathcal{T}_{st}(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{\alpha},\ \forall \alpha \in I, \\ \frac{3}{4} & \text{if } \lambda(x) = \alpha \text{ and } \lambda(y) = \beta \text{ for } 0.5 \leq \alpha, \beta \leq 0.6, \alpha < \beta, \\ \frac{1}{2} & \text{if } \lambda(x) = \alpha \text{ for } 0.4 \leq \alpha < 0.5, \lambda(y) = \beta \text{ for } 0.5 < \beta \leq 0.6, \\ \frac{2}{3} & \text{if } \lambda(x) = \alpha, \lambda(y) = \beta \text{ for } 0.4 \leq \alpha, \beta \leq 0.5, \alpha < \beta, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{T}^*_{st}(\lambda) = \begin{cases} 0 & \text{if } \lambda = \overline{\alpha},\ \forall \alpha \in I, \\ \frac{1}{4} & \text{if } \lambda(x) = \alpha \text{ and } \lambda(y) = \beta \text{ for } 0.5 \leq \alpha, \beta \leq 0.6, \alpha < \beta, \\ \frac{1}{2} & \text{if } \lambda(x) = \alpha \text{ for } 0.4 \leq \alpha < 0.5, \lambda(y) = \beta \text{ for } 0.5 < \beta \leq 0.6, \\ \frac{1}{3} & \text{if } \lambda(x) = \alpha, \lambda(y) = \beta \text{ for } 0.4 \leq \alpha, \beta \leq 0.5, \alpha < \beta, \\ 1 & \text{otherwise.} \end{cases}$$

THEOREM 3.4. Let $(X,\mathcal{T},\mathcal{T}^*)$ and $(X,\mathcal{U},\mathcal{U}^*)$ be ifts's. Let $(\mathcal{T}_{st},\mathcal{T}^*_{st})$ and $(\mathcal{U}_{st},\mathcal{U}^*_{st})$ be stratification for $(\mathcal{T},\mathcal{T}^*)$ and $(\mathcal{U},\mathcal{U}^*)$ respectively. If $f:(X,\mathcal{T},\mathcal{T}^*)\to(Y,\mathcal{U},\mathcal{U}^*)$ is intuitionistic continuous, then $f:(X,\mathcal{T}_{st},\mathcal{T}^*_{st})\to(Y,\mathcal{U}_{st},\mathcal{U}^*_{st})$ is intuitionistic continuous.

PROOF. Suppose there exist $\nu\in I^Y$ and $r\in I_0$, $s\in I_1$ such that

$$\mathcal{U}_{st}(\nu)>r>\mathcal{T}_{st}(f^{-1}(\nu)),\ \ \mathcal{U}^*_{st}(\nu)<s<\mathcal{T}^*_{st}(f^{-1}(\nu)).$$

Since $\mathcal{U}_{st}(\nu)>r$ and $\mathcal{U}^*_{st}(\nu)<s$, by the definition of $(\mathcal{U}_{st},\mathcal{U}^*_{st})$, there exists a family $\{\nu_j\mid j\in J\}$ with $\nu=\bigvee_{j\in J}(\nu_j\wedge\overline{\alpha_j})$ such that

$$\mathcal{U}_{st}(\nu)\geq\bigwedge_{j\in J}\mathcal{U}(\nu_j)>r,\ \ \mathcal{U}^*_{st}(\nu)\leq\bigvee_{j\in J}\mathcal{U}^*(\nu_j)<s.$$

On the other hand, since

$$f^{-1}(\nu)=f^{-1}(\bigvee_{j\in J}(\nu_j\wedge\overline{\alpha_j}))=\bigvee_{j\in J}f^{-1}(\nu_j)\wedge\overline{\alpha_j},$$

by the definition of $(\mathcal{T}_{st},\mathcal{T}^*_{st})$, we have

$$\mathcal{T}_{st}(f^{-1}(\nu))\geq\bigwedge_{j\in J}\mathcal{T}(f^{-1}(\nu_j)),\ \ \mathcal{T}^*_{st}(f^{-1}(\nu))\leq\bigvee_{j\in J}\mathcal{T}^*(f^{-1}(\nu_j)).$$

Since $f:(X,\mathcal{T},\mathcal{T}^*)\to(Y,\mathcal{U},\mathcal{U}^*)$ is intuitionistic continuous, that is, $\mathcal{T}(f^{-1}(\nu_j))\geq\mathcal{U}(\nu_j)$ and $\mathcal{T}^*(f^{-1}(\nu_j))\leq\mathcal{U}^*(\nu_j)$ for each $j\in J$,

$$\mathcal{T}_{st}(f^{-1}(\nu))\geq\bigwedge_{j\in J}\mathcal{T}(f^{-1}(\nu_j))\geq\bigwedge_{j\in J}\mathcal{U}(\nu_j)>r,$$

$$\mathcal{T}^*_{st}(f^{-1}(\nu))\leq\bigvee_{j\in J}\mathcal{T}^*(f^{-1}(\nu_j))\leq\bigvee_{j\in J}\mathcal{U}^*(\nu_j)<s.$$

It is a contradiction. Hence $f:(X,\mathcal{T}_{st},\mathcal{T}^*_{st})\to(Y,\mathcal{U}_{st},\mathcal{U}^*_{st})$ is intuitionistic continuous. □

The converse of the previous theorem is not true from the following example.

EXAMPLE 3.5. Let X be a nonempty set. Define IGO's $(\mathcal{T},\mathcal{T}^*)$ and $(\mathcal{U},\mathcal{U}^*)$ on X as follows: for each $\lambda \in I^X$,

$$\mathcal{T}(\lambda)=\begin{cases} 1 & \text{if } \lambda=\overline{1},\overline{0}, \\ 0 & \text{otherewise}, \end{cases} \qquad \mathcal{T}^*(\lambda)=\begin{cases} 0 & \text{if } \lambda=\overline{1},\overline{0}, \\ 1 & \text{otherewise}, \end{cases}$$

$$\mathcal{U}(\lambda)=\begin{cases} 1 & \text{if } \lambda=\overline{1},\overline{0}, \\ \frac{1}{3} & \text{if } \lambda=\overline{0.5}, \\ 0 & \text{otherewise}, \end{cases} \qquad \mathcal{U}^*(\lambda)=\begin{cases} 0 & \text{if } \lambda=\overline{1},\overline{0} \\ \frac{2}{3} & \text{if } \lambda=\overline{0.5}, \\ 1 & \text{otherewise}. \end{cases}$$

Since $0 = \mathcal{T}_{st}(\overline{0.5}) < \mathcal{U}(\overline{0.5}) = \frac{1}{3}$ and $1 = \mathcal{T}^*(\overline{0.5}) > \mathcal{U}^*(\overline{0.5}) = \frac{2}{3}$, then the idintity function $id_X : (X,\mathcal{T},\mathcal{T}^*) \to (X,\mathcal{U},\mathcal{U}^*)$ is not an intuitionistic continuous. On the other hand, for a family $\{\overline{1}\}$ with $\overline{0.5} = \overline{0.5} \wedge \overline{1}$, we have $\mathcal{U}_{st}(\overline{0.5}) \geq \mathcal{U}(\overline{1}) = 1$ and $\mathcal{U}^*_{st}(\overline{0.5}) \leq \mathcal{U}^*(\overline{1}) = 0$. Hence $\mathcal{U}_{st}(\overline{0.5}) = 1$ and $\mathcal{U}^*_{st}(\overline{0.5}) = 0$. Thus

$$\mathcal{T}_{st}(\lambda)=\mathcal{U}_{st}(\lambda)=\begin{cases} 1 & \text{if } \lambda=\overline{\alpha}, \forall\alpha\in L \\ 0, & \text{otherewise}, \end{cases}$$

$$\mathcal{T}^*_{st}(\lambda)=\mathcal{U}^*_{st}(\lambda)=\begin{cases} 0 & \text{if } \lambda=\overline{\alpha}, \forall\alpha\in I \\ 1, & \text{otherewise}. \end{cases}$$

Therefore, $id_X : (X,\mathcal{T}_{st},\mathcal{T}^*_{st}) \to (X,\mathcal{U}_{st},\mathcal{U}^*_{st})$ is intuitionistic continuous.

THEOREM 3.6. *Let $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$ be a stratification of an ifts $(X,\mathcal{T},\mathcal{T}^*)$. A fuzzy set λ is (r,s)-component in $(X,\mathcal{T},\mathcal{T}^*)$ iff λ is (r,s)-component in $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$.*

PROOF. (1) Let λ (r,s)-component in $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$. Suppose that λ is not (r,s)-connected in $(X,\mathcal{T},\mathcal{T}^*)$. Then $\mu \neq \overline{0}$ and $\rho \neq \overline{0}$ are (r,s)-separated in $(X,\mathcal{T},\mathcal{T}^*)$ such that $\lambda = \mu \vee \rho$. Since $\mathcal{T} \leq \mathcal{T}_{st}$ and $\mathcal{T}^* \geq \mathcal{T}^*_{st}$ from Theorem 3.1,

$$C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\mu,r,s) \leq C_{\mathcal{T},\mathcal{T}^*}(\mu,r,s),\ C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\rho,r,s) \leq C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s).$$

Hence μ and ρ are (r,s)-separated in $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$. Thus, λ is not (r,s)-component in $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$. It is a contradiction.

(2) We will show that if λ is (r,s)-component in $(X,\mathcal{T},\mathcal{T}^*)$, then λ is (r,s)-connected in $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$. Let λ be a (r,s)-component in $(X,\mathcal{T},\mathcal{T}^*)$. Then $C_{\mathcal{T},\mathcal{T}^*}(\lambda,r,s) = \lambda$ from Theorem 2.12(1). Suppose that λ is not

(r,s)-connected in $(X, \mathcal{T}_{st}, \mathcal{T}^*_{st})$, then $\mu \neq \overline{0}$ and $\rho \neq \overline{0}$ are (r,s)-separated in $(X, \mathcal{T}_{st}, \mathcal{T}^*_{st})$ such that $\lambda = \mu \vee \rho$. Since $\mathcal{T} \leq \mathcal{T}_{st}$ and $\mathcal{T}^* \geq \mathcal{T}^*_{st}$, $C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\lambda, r, s) \leq C_{\mathcal{T},\mathcal{T}^*}(\lambda, r, s) = \lambda$. So, $C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\lambda, r, s) = \lambda$. Since $\mu \leq \lambda$, we have $C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\mu, r, s) \leq \lambda$. It implies $\lambda = C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\mu, r, s) \vee \rho$. Put $C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\mu, r, s) = \omega$. If $x \in supp(\omega)$, then $x \in supp(\lambda)$. Since λ is (r,s)-component in $(X, \mathcal{T}, \mathcal{T}^*)$, by Theorem 2.12(4), $x_1 \in \lambda = \omega \vee \rho$, that is, $\omega(x) \vee \rho(x) = 1$. Since $\omega \wedge \rho = \overline{0}$, then $\rho(x) = 0$. So, $\omega(x) = 1$. Hence ω is a crisp set. Since $\mathcal{T}_{st}(\overline{1} - \omega) \geq r$ and $\mathcal{T}^*_{st}(\overline{1} - \omega) \leq s$ from Theorem 1.3, and Theorem 1.4, for any family $\{\overline{\alpha_i} \wedge \eta_i \mid \overline{1} - \omega = \bigvee_{i \in J} \overline{\alpha_i} \wedge \eta_i\}$,

$$\mathcal{T}_{st}(\overline{1} - \omega) = \bigvee \{ \bigwedge_{i \in J} \mathcal{T}(\eta_i) \} \geq r, \ \mathcal{T}^*_{st}(\overline{1} - \omega) = \bigwedge \{ \bigvee_{i \in J} \mathcal{T}^*(\eta_i) \} \leq s.$$

Without loss of generality, we assume that $\overline{\alpha_i} \neq \overline{0}$. Since $\omega(x) = 1$ for $x \in supp(\omega)$,

$$\begin{aligned} (\overline{1} - \omega)(x) &= \bigvee_{i \in J} (\overline{\alpha_i} \wedge \eta_i)(x) \\ \Rightarrow 1 = \omega(x) &= \bigwedge_{i \in J} (\overline{1} - \overline{\alpha_i})(x) \vee (\overline{1} - \eta_i)(x). \end{aligned}$$

Thus, $(\overline{1} - \omega)(x) = \bigvee_{i \in J} \eta_i(x)$ for $x \in supp(\omega)$. If $y \notin supp(\omega)$, then $1 = (\overline{1} - \omega)(y) = (\overline{\alpha_i} \wedge \eta_i)(y) \leq \bigvee_{i \in J} \eta_i(y)$. Hence, for any family $\{\overline{\alpha_i} \wedge \eta_i \mid \overline{1} - \omega = \bigvee_{i \in J} \overline{\alpha_i} \wedge \eta_i\}$, we have

$$\overline{1} - \omega = \bigvee_{i \in J} \eta_i.$$

It implies

$$\mathcal{T}_{st}(\overline{1} - \omega) = \mathcal{T}(\overline{1} - \omega) = \bigwedge_{i \in J} \mathcal{T}(\eta_i) \geq r$$

$$\mathcal{T}^*_{st}(\overline{1} - \omega) = \mathcal{T}^*(\overline{1} - \omega) = \bigwedge_{i \in J} \mathcal{T}^*(\eta_i) \leq s.$$

So, $C_{\mathcal{T},\mathcal{T}^*}(\omega, r, s) = \omega$. It implies

$$C_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\mu, r, s), r, s) = C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\mu, r, s).$$

Similarly, we have $C_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\rho,r,s),r,s) = C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\rho,r,s)$. So, μ and ρ are (r,s)-separated in $(X,\mathcal{T},\mathcal{T}^*)$ from

$$\begin{aligned}
&C_{\mathcal{T},\mathcal{T}^*}(\mu,r,s)\wedge\rho\\
&\le (C_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\mu,r,s),r,s)\wedge\rho) = (C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\mu,r,s)\wedge\rho) = \overline{0},\\
&\mu\wedge C_{\mathcal{T},\mathcal{T}^*}(\rho,r,s)\\
&\le (\mu\wedge C_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\rho,r,s),r,s)) = (\mu\wedge C_{\mathcal{T}_{st},\mathcal{T}^*_{st}}(\rho,r,s)) = \overline{0}.
\end{aligned}$$

Thus, λ is not (r,s)-component in $(X,\mathcal{T},\mathcal{T}^*)$. It is a contradiction.

(3) Let λ be a (r,s)-component in $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$. From (1), λ is (r,s)-connected in $(X,\mathcal{T},\mathcal{T}^*)$. There exists a (r,s)-component μ in $(X,\mathcal{T},\mathcal{T}^*)$ containing λ. From (2), μ is (r,s)-connected in $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$. Thus $\lambda=\mu$.

Let ρ be a (r,s)-component in $(X,\mathcal{T},\mathcal{T}^*)$. Similarly, ρ is a (r,s)-component in $(X,\mathcal{T}_{st},\mathcal{T}^*_{st})$. □

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