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ALMOST SURE CONVERGENCE FOR LINEAR PROCESS GENERATED BY ASYMPTOTICALLY LINEAR NEGATIVE QUADRANT DEPENDENCE PROCESSES

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ABSTRACT. In this paper, we obtain strong law of large numbers for linear process generated by asymptotically linear negative quadrant dependence processes.

1. Introduction

In 2000, Zhang gave the concept of asymptotically linear negative quadrant dependence. The concept of asymptotically linear negative quadrant dependence see the following definition.

DEFINITION 1 A random variables sequence $\{X_k, k \in N\}$ is said to be asymptotically linear negative quadrant dependence (ALNQD), if

$$\rho^-(r) = \sup\{\rho^-(S,T);\ \mathrm{dist(S,T)} \geq \mathrm{r},\ \mathrm{S},\ \mathrm{T} \subset \mathrm{N}\ \text{ are finite}\} \to 0$$

as $r \to \infty$, where

$$\rho^-(S,T) = 0 \vee \sup\{\frac{\mathrm{Cov(f(X),g(Y))}}{(\mathrm{Varf(X)})^{1/2}(\mathrm{Varg(Y)})^{1/2}}; X \in F(S), Y \in F(T)\}.$$

In this paper, we assume that $\{Y_i,\ 0 \leq i < \infty\}$ be an asymptotically linear negative quadrant dependence sequence. Let X_t be a linear process generated by Y_t, that is

$$X_t = \sum_{i=0}^{\infty} a_i Y_{t-i} \tag{1}$$

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where

$$\sum_{i=0}^{\infty} |a_i| < \infty. \tag{2}$$

For the linear process, Ho and Hsing (1997), Phillips and Solo (1992) and Wang et al (2002) got central limit theorems for linear process under independent assumptions. Kim and Baek (2001) got a central limit theorem for strongly stationary linear process under linear positive quadrant dependent assumptions. Salvadori (2003) have obtained Linear combinations of order statistics to estimate the quantiles of generalized pareto and extreme values distributions. Kim et al (2004) got a strong law of large numbers for linear process generated by linear positive quadrant dependence or associated.

In this paper, we obtain a strong law of large numbers for linear process generated by asymptotically linear negative quadrant dependence processes.

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n \ll b_n$ will mean $a_n \leq Cb_n$.

2. Proof of the main theorem

In order to proof our theorems, we need the following lemmas.

LEMMA 1.(Zhang, 2000) *Let $\{Y_i,\ i \geq 1\}$ be a sequence of centered asymptotically linear negative quadrant dependence (ALNQD) random variables and $E|Y_i|^p < \infty$ for some $p > 2$ and every $i \geq 1$. Then there exists $C = C(p, \rho^-(n))$, such that*

$$E \max_{1\leq k\leq n} |\sum_{i=1}^{k} Y_i|^p \leq Cn^{p/2} \max_{1\leq k\leq n} E|Y_i|^p$$

LEMMA 2. *Let $\{Y_i,\ i \geq 1\}$ be a sequence of centered asymptotically linear negative quadrant dependence (ALNQD) random variables and $E|Y_i|^p < \infty$ for some $p > 2$ and every $i \geq 1$. Let Then as $n \to \infty$, we have $\frac{\sum_{i=1}^n Y_i}{n} \to 0$ a.s.*

Proof of Lemma 2. $\forall \varepsilon > 0$ and for some $p > 2$. Using Lemma 1 and

Markov inequality, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} Y_i| > \varepsilon n) \\
&\leq \sum_{n=1}^{\infty} \frac{E|\sum_{i=1}^{n} Y_i|^p}{(\varepsilon n)^p} \\
&\leq \sum_{n=1}^{\infty} \frac{E \max_{1\leq k\leq n} |\sum_{i=1}^{k} Y_i|^p}{(\varepsilon n)^p} \\
&\leq \sum_{n=1}^{\infty} C n^{p/2} \max_{1\leq k\leq n} E|Y_i|^p (\varepsilon n)^{-p} \\
&\ll \sum_{n=1}^{\infty} C n^{-p/2} < \infty. \qquad (3)
\end{aligned}$$

By Borel-Cantelli Lemma, when $n \to \infty$, we have $\frac{\sum_{i=1}^{n} Y_i}{n} \to 0 \quad a.s.$ □

The following theorem is the strong law of large numbers for linear process generated by asymptotically linear negative quadrant dependence processes.

THEOREM 1. *Let $\{Y_n,\ n \geq 0\}$ be an asymptotically linear negative quadrant dependence sequence of indentically distributed random variables with $EY_i = 0,\ E|Y_i|^p < \infty$, for some $p > 2$, and let $\{X_t,\ t \geq 0\}$ be a linear process defined by (1). Suppose that (2) holds, $S_n = \sum_{i=1}^{n} X_i$, then as $n \to \infty$, we have $\frac{S_n}{n} \to 0 \quad a.s.$*

PROOF OF THEOREM 1. Let $\widetilde{X}_t = (\sum_{i=0}^{\infty} a_i) Y_t$, $\widetilde{S}_n = \sum_{i=1}^{n} \widetilde{X}_t$ It is clear that

$$\begin{aligned}
\widetilde{S}_k &= \sum_{t=1}^{k} \widetilde{X}_t \\
&= \sum_{t=1}^{k} (\sum_{i=0}^{k-t} a_i) Y_t + \sum_{t=1}^{k} (\sum_{i=k-t+1}^{\infty} a_i) Y_t \\
&= \sum_{t=1}^{k} (\sum_{i=0}^{t-1} a_i Y_{t-i}) + \sum_{t=1}^{k} (\sum_{i=k-t+1}^{\infty} a_i) Y_t.
\end{aligned}$$

(4)

Then

$$(5)\quad \widetilde{S}_k - S_k = -\sum_{t=1}^{k}(\sum_{i=t}^{\infty} a_i Y_{t-i}) + \sum_{t=1}^{k}(\sum_{i=k-t+1}^{\infty} a_i)Y_t =: A + B.$$

First we proof

$$(6)\qquad n^{-1}\max_{1\le k\le n}|\widetilde{S}_k - S_k| \xrightarrow{P} 0.$$

In order to proof (6), we need only to show

$$(7)\qquad n^{-1}\max_{1\le k\le n}|A| \xrightarrow{P} 0.$$

and

$$(8)\qquad n^{-1}\max_{1\le k\le n}|B| \xrightarrow{P} 0.$$

Using the Minkowsky inequality, Lemma 1 with $r > 2$ and the dominated convergence theorem, then

$$\begin{aligned}
& n^{-r}E\max_{1\le k\le n}|\sum_{t=1}^{k}\sum_{i=t}^{\infty} a_i Y_{t-i})|^r \\
&= n^{-r}E\max_{1\le k\le n}|\sum_{i=1}^{\infty}\sum_{t=1}^{i\wedge k} a_i Y_{t-i})|^r \\
&\le n^{-r}E(\sum_{i=1}^{\infty}|a_i|\max_{1\le k\le n}|\sum_{t=1}^{i\wedge k} Y_{t-i}|)^r \\
&\le n^{-r}(\sum_{i=1}^{\infty}|a_i|(E\max_{1\le k\le n}|\sum_{t=1}^{i\wedge k} Y_{t-i}|^r)^{1/r})^r \\
&\le n^{-r}(\sum_{i=1}^{\infty}|a_i|C(i\wedge n)^{1/2})^r \\
(9)\qquad &\le C(\sum_{i=1}^{\infty}|a_i|(i\wedge n)^{1/2}n^{-1})^r = o(1).
\end{aligned}$$

By (9), we have (7).
Because

$$
\begin{aligned}
B &= \sum_{t=1}^{k}\Big(\sum_{i=k-t+1}^{\infty} a_i\Big)Y_t \\
&= \sum_{i=1}^{k} a_i \sum_{t=k-i+1}^{k} Y_t + \sum_{i=k+1}^{\infty} a_i \sum_{t=1}^{k} Y_t =: B_1 + B_2. \qquad (10)
\end{aligned}
$$

Let $\{p_n\}$ be a positive integers $\{p_n\}$ such that $p_n \to \infty$ and $p_n/n \to 0$, we have

$$
\begin{aligned}
& n^{-1} \max_{1\le k\le n} |B_2| \\
& \le \Big(\sum_{i=0}^{\infty} |a_i|\Big) n^{-1} \max_{1\le k\le p_n} \Big|\sum_{i=1}^{k} Y_i\Big| + \Big(\sum_{i=p_n+1}^{\infty} |a_i|\Big) n^{-1} \max_{1\le k\le n} \Big|\sum_{i=1}^{k} Y_i\Big| \\
& =: B_{21} + B_{22}. \qquad (11)
\end{aligned}
$$

Using Lemma 1 with $r > 2$, we have

$$
\begin{aligned}
E|B_{21}|^r &= \Big(\sum_{i=0}^{\infty} |a_i|\Big)^r n^{-r} E \max_{1\le k\le p_n} \Big|\sum_{i=1}^{k} Y_i\Big|^r \\
&\le \Big(\sum_{i=0}^{\infty} |a_i|\Big)^r n^{-r} C (p_n)^{r/2} E|Y_1|^r \\
&\le C\Big(\sum_{i=0}^{\infty} |a_i|\Big)^r (p_n/n)^{r/2} n^{-r/2} = o(1). \qquad (12)
\end{aligned}
$$

Using Lemma 1 with $r > 2$, we have

$$
\begin{aligned}
E|B_{22}|^r &= \Big(\sum_{i=p_n+1}^{\infty} |a_i|\Big)^r n^{-r} E \max_{1\le k\le n} \Big|\sum_{i=1}^{k} Y_i\Big|^r \\
&\le \Big(\sum_{i=p_n+1}^{\infty} |a_i|\Big)^r n^{-r} C n^{r/2} E|Y_1|^r \\
&\le C\Big(\sum_{i=p_n+1}^{\infty} |a_i|\Big)^r n^{-r/2} = o(1). \qquad (13)
\end{aligned}
$$

By (11), (12) and (13), we have

$$
n^{-1} \max_{1<k<n} |B_2| \xrightarrow{P} 0. \qquad (14)
$$

Next, we want to proof

$$L_n = n^{-1} \max_{1\le k\le n} |B_1| \xrightarrow{P} 0. \tag{15}$$

For each $m \ge 1$, let

$$B_{1,m} = \sum_{i=1}^{k} b_i \sum_{t=k-i+1}^{k} Y_t,$$

where $b_i = a_i I(i \le m)$. Let

$$L_{n,m} = n^{-1} \max_{1\le k\le n} |B_{1,m}|,$$

for each $m \ge 1$, then

$$L_{n,m} \le (|a_1| + \cdots + |a_m|) n^{-1} (|Y_1| + \cdots + |Y_m|) \xrightarrow{P} 0. \tag{16}$$

$\forall \varepsilon > 0$, by Lemma 1, we have

$$\begin{aligned}
& P(|L_n - L_{n,m}| > \varepsilon) \\
& \le \varepsilon^{-r} (L_n - L_{n,m})^r \\
& \le \varepsilon^{-r} n^{-r} E \max_{m\le k\le n} |\sum_{i=1}^{k} (a_i - b_i)(Y_k + \cdots + Y_{k-i+1})|^r \\
& \le \varepsilon^{-r} n^{-r} E \max_{m\le k\le n} (\sum_{i=m+1}^{k} |a_i| (\sum_{i=1}^{k} Y_i - \sum_{i=1}^{k-i} Y_i))^r \\
& \le 2^r \varepsilon^{-r} (\sum_{i=m+1}^{\infty} |a_i|)^r n^{-r} E \max_{1\le k\le n} |\sum_{i=1}^{k} Y_i|^r \\
& \le 2^r \varepsilon^{-r} (\sum_{i=m+1}^{\infty} |a_i|)^r n^{-r} C n^{r/2} E Y_1^r \\
& \le C (\sum_{i=m+1}^{\infty} |a_i|)^r n^{-r/2} \to 0, \qquad (17)
\end{aligned}$$

when $n \to \infty$. By (17), we have

$$|L_n - L_{n,m}| \xrightarrow{P} 0. \tag{18}$$

Using (16) and (18), we have (15). By (14), (15) and (10), we have (8) Therefore we have (6). By

$$E\widetilde{X}_t = (\sum_{i=0}^{\infty} a_i) E Y_t = 0, E|\widetilde{X}_t|^p = (\sum_{i=0}^{\infty} a_i)^p E|Y_t|^p < \infty,$$

By Lemma 2, we have

$$\frac{\widetilde{S}_n - E\widetilde{S}_n}{n} \to 0 a.s. \tag{19}$$

By

$$E\widetilde{S}_n = \sum_{i=1}^{n} E\widetilde{X}_t = 0,$$

and

$$EX_t = \sum_{i=0}^{\infty} a_i EY_{t-i} = 0, ES_n = \sum_{i=1}^{n} EX_i = 0,$$

thus by (19) and (6), we have $\frac{S_n}{n} \to 0$ $a.s.$ Now we complete the proof of Theorem 1. □

Because asymptotically linear negative quadrant dependence (ALNQD) sequence is more general than linear negative quadrant dependence (LNQD) sequence or ρ^*-mixing sequence. So we have the following two Corollaries.

Corollary 1. *Let $\{Y_n,\ n \geq 0\}$ be a LNQD sequence of indentically distributed random variables with $EY_i = 0$, $E|Y_i|^p < \infty$, for some $p > 2$, and let $\{X_t,\ t \geq 0\}$ be a linear process defined by (1). Suppose that (2) holds, $S_n = \sum_{i=1}^{n} X_i$, then as $n \to \infty$, we have $\frac{S_n}{n} \to 0$ $a.s.$*

Corollary 2. *Let $\{Y_n,\ n \geq 0\}$ be a ρ^*-mixing sequence of indentically distributed random variables with $EY_i = 0$, $E|Y_i|^p < \infty$, for some $p > 2$, and let $\{X_t,\ t \geq 0\}$ be a linear process defined by (1). Suppose that (2) holds, $S_n = \sum_{i=1}^{n} X_i$, then as $n \to \infty$, we have $\frac{S_n}{n} \to 0$ $a.s.$*

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