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An Optimal Ordering policy on Both Way Substitutable Two-Commodity Inventory Control System

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Abstract. There are a lot of raw materials, work-in-processes and finished goods in manufacturing industry. Here, the less stock of materials and work-in-processes manufacturing industry has, the worse the rate of the production is. Inversely, the more manufacturing industry has, the more expensive the cost to support them is. Thus, it is important for us to balance them efficiently. In general, inventory problems are to decide appropriate times to produce goods and to determine appropriate quantities of goods. Therefore, inventory problems require as more useful information as possible. For example, there are demand, lead time, ordering point and so on. In this paper, we deal with an optimal ordering policy on both way substitutable two-commodity inventory control system. That is, there is a problem of how to allocate the produced two kinds of goods in a factory to m areas so as to minimize the total expected inventory cost. The demand of each area is probabilistic, and we adopt the exponential distribution as a probability density function of demand. Moreover, we provide numerical examples of the problem.

Keywords: Both Way Substitutable Two-Commodity Inventory Control System, Optimal Ordering Policy, Lagrange's Multiplier, Convex Function

1. INTRODUCTION

Manufacturing industry has a lot of raw materials, work-in-processes and finished goods. Here, the less stock of materials and work-in-processes manufacturing industry has, the worse the rate of production is. Inversely, the more manufacturing industry has, the more expensive the cost to support them is. Thus, it is important for us to balance them efficiently. Also, it is necessary to make rules such as "When should you produce goods?" or "How many goods should you produce?" Hence, inventory problems are to decide appropriate times to produce goods and to determine appropriate quantities of goods. Therefore, it is very important to obtain as more useful information as possible to solve inventory problems. The demand may be deterministic in some situations, or probabilistic in other situations. In addition to this, we must get as more useful inventory information as possible such as lead time, ordering cost, holding cost, shortage cost and so on.

Hitherto, in previous papers, for example, Pasternack and Drezner (1991), H. Shang Lau and A. Hing-Ling Lau (1991), Parlar and Goyal (1984), Khouja *et al*. (1996), Yoshikawa and Tanaka (1997), and Shimosakon

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et al. (1984) proposed optimal ordering policies on substitutable two-commodity system. These papers are based on the circumstances of suppliers. If one commodity becomes surplus stock, they are reused as a substitution of the other commodity. Consequently, inventory carrying cost is reduced. However, we have to consider the shortage cost. Shimosakon *et al*. (1984), and Tanaka *et al*. (1997) took it into consideration. They explained the model on one way substitutable two-commodity inventory control system. Strictly speaking, when one commodity becomes out of stock, the customer who wants to buy the commodity can purchase another commodity as a substitution for one commodity. But in actual situation, there exists a problem of both way substitutable two-commodity inventory control system. For example, consider the two commodities which are composed of a high quality machine and a low quality machine such as a computer product. A customer who can buy the high quality one can purchase whichever he/she likes. But a customer who can only buy the low quality one can only purchase the low quality one. However, we must consider the case where the value between company A's one and company B's one is the same grade such as A's DVD and B's DVD. If one commodity is out of stock, a customer can purchase another commodity as a substitution for one commodity.

On the other hand, there are problems of how to allocate the resources to many areas so as to minimize the total expected cost or to maximize the total expected profit. These problems are reported by Tanaka *et al*. (1995; 1997), and Yoshikawa and Tanaka (2001). They are called resource allocation problems, and they are applied production planning and so on.

In actual situation, we have to take both way substitutable two-commodity inventory control system into consideration. In this paper, we deal with an optimal ordering policy on both way substitutable two-commodity inventory control system. That is, there is a problem of how to allocate the produced two kinds of goods in a factory to m areas so as to minimize the total expected inventory cost. The demand of each area is probabilistic. In particular, we assume the exponential distribution as a probability density function of demand. Actually, in previous papers, Shang Lau and Hing-Ling Lau (1991), Shimosakon *et al*. (1984), and Yoshikawa and Tanaka (2001) employed the exponential distribution. Moreover, we provide numerical examples to explain the problem.

2. THE SUBSTITUTABLE TWO-COMMODITY INVENTORY MODEL

To develop the stochastic, single-period, both way substitutable two-commodity inventory model, we define the following assumptions and notation.

2.1 Assumptions

In our model, the lead time is zero. During the period, a depletion of the inventory of the two commodities are induced by customer's demand described by the random variable $y_i \geq 0$, $i = 1, 2$, where the notation i defines the commodity i. When commodity 1 becomes out of stock, the customer that wants to buy the commodity purchase the commodity 2 as a substitution for the commodity 1. Inversely, when commodity 2 becomes out of stock, commodity 1 is substitutable for the commodity 2.

2.2 Notation

In our model, we use the following symbols.

i : commodity index, $i = 1, 2$.

a_i : the inventory level before ordering for commodity i.

x_i : the inventory level after ordering for commodity i.

y_i : the demand during the period, y_i is a random variable, $y_i \geq 0$, $i = 1, 2$.

z_i : the ordering quantity for commodity i ($x_i = a_i + z_i$).

$\phi(y_1, y_2)$: the joint probability density function of the demand y_i, $i = 1, 2$.

h_i : the holding cost per unit of the commodity i.

p_i : the shortage cost per unit of the commodity i ($p_i \geq h_i$).

$L(x_1, x_2)$: the expected cost function for given x_1 and x_2 over the period.

α_{i1i2} : the substitution ratio in the case of commodity i_1 becomes out of stock, the other commodity i_2 is substitutable for the commodity i_1.

2.3 The Expected Cost Function

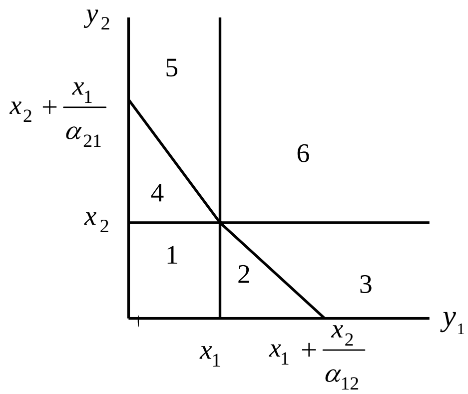

Figure 1. A calculation method of the total of the expected inventory cost for both way substitutable two-commodity inventory control system

We determine an optimal policy so as to minimize $L(x_1, x_2)$ for given x_1 and x_2 over the period.

To analyze $L(x_1, x_2)$, we divide the domain of $L(x_1, x_2)$ into 6 parts as shown in Figure 1. We consider the expected cost function of each part.

[Part 1]

Both commodities 1 and 2 are held in the part 1, then we obtain the expected cost function $L_1(x_1, x_2)$ as follows.

$$L_1(x_1, x_2) = \int_0^{x_1}\int_0^{x_2}\left\{h_1(x_1 - y_1) + h_2(x_2 - y_2)\right\} \times \phi(y_1, y_2)dy_2\,dy_1 \quad (1)$$

[Part 2]

Commodity 1 is sold out while commodity 2 is held in the part 2. From the assumption of the substitution, the α_{12} percent of the shortage quantities of the commodity 1 ($\alpha_{12}(y_1 - x_1)$) are substituted from the commodity 2. Therefore, the expected cost function $L_2(x_1, x_2)$ is given as follows.

$$L_2(x_1, x_2) = \int_0^{x_2}\int_{x_1}^{x_1 + \frac{(x_2 - y_2)}{\alpha_{12}}}\Big[p_1(1 - \alpha_{12})(y_1 - x_1) + h_2\left\{(x_2 - y_2) - \alpha_{12}(y_1 - x_1)\right\}\Big]\phi(y_1, y_2)dy_1\,dy_2 \quad (2)$$

[Part 3]

In the part 3, demands of the commodity 1 over the upper limit that substitutes the commodity 2 for the commodity 1 occurs. All the remaining stock is sold out because of the substitution. Then, the expected cost function $L_3(x_1, x_2)$ is represented as follows.

$$L_3(x_1, x_2) = \int_0^{x_2}\int_{x_1 + \frac{(x_2 - y_2)}{\alpha_{12}}}^{\infty} p_1\left\{(y_1 - x_1) - (x_2 - y_2)\right\} \times \phi(y_1, y_2)dy_1\,dy_2 \quad (3)$$

[Part 4]

Commodity 2 is sold out while commodity 1 is held in the part 4. From the assumption of the substitution, the α_{21} percent of the shortage quantities of the commodity 2 ($\alpha_{21}(y_2 - x_2)$) are substituted from the commodity 1. Therefore, the expected cost function $L_4(x_1, x_2)$ is given as follows.

$$L_4(x_1, x_2) = \int_0^{x_1}\int_{x_2}^{x_2 + \frac{(x_1 - y_1)}{\alpha_{21}}}\Big[p_2(1 - \alpha_{21})(y_2 - x_2) + h_1\left\{(x_1 - y_1) - \alpha_{21}(y_2 - x_2)\right\}\Big]\phi(y_1, y_2)dy_2\,dy_1 \quad (4)$$

[Part 5]

In the part 5, demands of the commodity 2 over the upper limit that substitutes the commodity 1 for the commodity 2 occurs. All the remaining stock is sold out because of the substitution. Then, the expected cost function $L_5(x_1, x_2)$ is represented as follows.

$$L_5(x_1, x_2) = \int_0^{x_1}\int_{x_2 + \frac{(x_1 - y_1)}{\alpha_{21}}}^{\infty} p_2\left\{(y_2 - x_2) - (x_1 - y_1)\right\} \times \phi(y_1, y_2)dy_2\,dy_1 \quad (5)$$

[Part 6]

Both the commodities 1 and 2 are sold out in the part 6. Then, the expected cost function is given as

$$L_6(x_1, x_2) = \int_{x_1}^{\infty}\int_{x_2}^{\infty}\left\{p_1(y_1 - x_1) + p_2(y_2 - x_2)\right\} \times \phi(y_1, y_2)dy_2\,dy_1\,, \quad (6)$$

where

$$\int_0^{\infty}\int_0^{\infty}\phi(y_1, y_2)dy_2\,dy_1 = 1.$$

Therefore, we obtain the total expected cost function $L(x_1, x_2)$ for given x_1 and x_2 is represented as follows.

$$L(x_1, x_2) = \sum_{k=1}^{6} L_k(x_1, x_2) \quad (7)$$

The aim of our research in the inventory policy is to determine the optimal quantities (x_1^*, x_2^*) that minimizes the total expected cost function $L(x_1, x_2)$.

In our model, the substitution is limited only to the case where a commodity becomes out of stock, and to a part of the shortage expressed by the substitution ratio α_{12}, α_{21}. There may be the case where a customer substitute a commodity even if both commodities are not out of stock. However, if we consider such situation, the model will be very complicated one. Then, there exists a case where we can't solve analytically. Therefore, in this case, we confined our model to enable us to solve analytically.

3. A FORMULATION OF AN OPTIMAL RESOURCE ALLOCATION PROBLEM ON BOTH WAY SUBSTITUTABLE TWO-COMMODITY INVENTORY SYSTEM

In this paper, we consider an optimal ordering policy on both way substitutable two-commodity inventory system. That is, we determine the optimal quantities x_1^* and x_2^* that minimizes the total expected cost function $L(x_1, x_2)$.

3.1 Notation and assumptions

Here, we redefine all symbols and assumptions.

j : area index, $j = 1, 2, \cdots, m$.

a_{ij} : the inventory level before ordering for commodity i for area j.

x_{ij} : the inventory level after ordering for commodity i for area j.

y_{ij} : the demand during the period, y_{ij} is a random variable, $y_{ij} \geq 0$, $i = 1, 2$, $j = 1, 2, \cdots, m$.

z_{ij} : the ordering quantity for commodity i for area j ($x_{ij} = a_{ij} + z_{ij}$).

$\phi(y_{1j}, y_{2j})$: the joint probability density function of the demand y_{ij} for area j, $i = 1, 2$.

h_i : the holding cost per unit of the commodity i for area j (This symbol does not depend on area j).

p_{ij} : the shortage cost per unit of the commodity i for area $j (p_{ij} \geq h_i)$.

$L(x_{1j}, x_{2j})$: the expected cost function for given x_{1j} and x_{2j} for area j over the period.

α_{i1i2} : the substitution ratio in the case of commodity i_1 becomes out of stock, the other commodity i_2 is substitutable for the commodity i_1.

$TC_j(x_{1j}, x_{2j})$: the total expected cost function for area j.

$TC_j(x_{1j})$: the total expected cost function of commodity 1 for area j with the non-substitutable case.

$TC_j(x_{2j})$: the total expected cost function of commodity 2 for area j with the non-substitutable case.

We consider the optimal resource allocation problem on both way substitutable two-commodity inventory system as the following assumptions.

1. The production is performed in one factory.
2. A factory produces two kinds of commodities.
3. The total amount of the commodity i ($i = 1, 2$) is less than or equal to X_i ($i = 1, 2$) because of the production ability.
4. The number of areas we are going to allocate is composed of m areas.
5. We must allocate these commodities to m areas so as to minimize the total expected cost function.

In this paper, it is assumed that the production is performed in one factory. Therefore, we can set up h_i for each area as uniform cost. As for h_i, the decision of h_i can be entrusted to manufacturer (Notice that various establishment of h_i for each area is also possible). However, as for p_{ij}, the decision of p_{ij} can't be entrusted for manufacturer. Not a few losses might be yield by giving the dissatisfactions to customer. The degree of dissatisfaction to customer will differ among each area. So it is difficult to treat p_{ij} uniformly, we set up p_{ij} for each area as different cost.

To solve this problem, we use the total expected cost function that we have already discussed in previous section. First of all, we formulate the total expected cost function for each area. Next, we lead to all area's total expected cost function. Let x_{ij} (≥ 0) be a number of products of the commodity $i (i = 1, 2)$ for area j ($j = 1, 2, \cdots, m$). Then, the total expected cost function $TC_j(x_{1j}, x_{2j})$ for area j is given as follows.

$$
\begin{aligned}
TC_j(x_{1j}, x_{2j}) &= \sum_{k=1}^{6} L_k(x_{1j}, x_{2j}) \\
&= \int_0^{x_{1j}} \int_0^{x_{2j}} \left\{ h_1 (x_{1j} - y_{1j}) + h_2 (x_{2j} - y_{2j}) \right\} \\
&\quad \times \phi(y_{1j}, y_{2j}) dy_{2j} dy_{1j} \\
&+ \int_0^{x_{2j}} \int_{x_{1j}}^{x_{1j} + \frac{(x_{2j} - y_{2j})}{\alpha_{12}}} \Big[p_{1j} (1 - \alpha_{12})(y_{1j} - x_{1j}) \\
&+ h_2 \left\{ (x_{2j} - y_{2j}) - \alpha_{12} (y_{1j} - x_{1j}) \right\} \Big] \phi(y_{1j}, y_{2j}) dy_{1j} dy_{2j} \\
&+ \int_0^{x_{2j}} \int_{x_{1j} + \frac{(x_{2j} - y_{2j})}{\alpha_{12}}}^{\infty} p_{1j} \left\{ (y_{1j} - x_{1j}) - (x_{2j} - y_{2j}) \right\} \\
&\quad \times \phi(y_{1j}, y_{2j}) dy_{1j} dy_{2j} \\
&+ \int_0^{x_{1j}} \int_{x_{2j}}^{x_{2j} + \frac{(x_{1j} - y_{1j})}{\alpha_{21}}} \Big[p_{2j} (1 - \alpha_{21})(y_{2j} - x_{2j}) \\
&+ h_1 \left\{ (x_{1j} - y_{1j}) - \alpha_{21} (y_{2j} - x_{2j}) \right\} \Big] \phi(y_{1j}, y_{2j}) dy_{2j} dy_{1j} \\
&+ \int_0^{x_{1j}} \int_{x_{2j} + \frac{(x_{1j} - y_{1j})}{\alpha_{21}}}^{\infty} p_{2j} \left\{ (y_{2j} - x_{2j}) - (x_{1j} - y_{1j}) \right\} \\
&\quad \times \phi(y_{1j}, y_{2j}) dy_{2j} dy_{1j} \\
&+ \int_{x_{1j}}^{\infty} \int_{x_{2j}}^{\infty} \left\{ p_{1j} (y_{1j} - x_{1j}) + p_{2j} (y_{2j} - x_{2j}) \right\} \\
&\quad \times \phi(y_{1j}, y_{2j}) dy_{2j} dy_{1j}, \qquad (8)
\end{aligned}
$$

where p_{ij} is the shortage cost per unit of the commodity i for area j. Therefore, an optimal resource allocation problem on both way substitutable two-commodity inventory system we propose here is represented as follows.

$$
\begin{aligned}
&\textit{Minimize} \qquad TCV = \sum_{j=1}^{m} TC_j(x_{1j}, x_{2j}) \\
&\textit{subject to} \qquad \sum_{j=1}^{m} x_{1j} \leq X_1, \quad \sum_{j=1}^{m} x_{2j} \leq X_2 \qquad (9) \\
&(x_{1j} \geq 0, x_{2j} \geq 0, \quad j = 1, 2, \cdots, m)
\end{aligned}
$$

Here, TCV is the objective function.

3.2 Characteristics of the proposed model

In this paper, our purpose is to allocate the commodities for area j so as to minimize the total expected cost function. However, it is hard to solve this problem analytically. First, we examine the characteristics of the proposed model. The $TC_j(x_{1j}, x_{2j})$ in the non-substitution case is expressed as a sum of the function of x_{1j} and x_{2j} only, namely, $TC_j(x_{1j}, x_{2j})$ is separable as $TC_j \; (x_{1j}, x_{2j}) = TC_j(x_{1j}) + TC_j(x_{2j})$. Because $TC_j(x_{1j}, x_{2j})$ is a strictly convex function, we can divide $TC_j(x_{1j}, x_{2j})$ into $TC_j(x_{1j})$ and $TC_j(x_{2j})$, where $TC_j(x_{1j})$ is the cost function for given x_{1j} for commodity 1 for area j over the period as shown in Sivazlian and Stanfel (1975). Therefore, we can calculate the following subtraction easily.

$$\begin{aligned}
\Delta &= TC_j\,(x_{1j}) + TC_j\,(x_{2j}) - TC_j\,(x_{1j}, x_{2j}) \\
&= \int_0^{x_{2j}} \int_{x_{1j}}^{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}}} \left\{ \alpha_{12} h_2\,(y_{1j} - x_{1j}) + \alpha_{12} p_{1j}(y_{1j} - x_{1j}) \right\} \\
&\qquad \times \phi(y_{1j}, y_{2j}) dy_{1j} dy_{2j} \\
&\quad + \int_0^{x_{2j}} \int_{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}}}^{\infty} \left\{ h_2\,(y_{1j} - x_{1j}) + p_{1j}(y_{2j} - x_{2j}) \right\} \\
&\qquad \times \phi(y_{1j}, y_{2j}) dy_{1j} dy_{2j} \\
&\quad + \int_0^{x_{1j}} \int_{x_{2j}}^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}}} \left\{ \alpha_{21} h_1\,(y_{1j} - x_{1j}) + \alpha_{21} p_{2j}(y_{2j} - x_{2j}) \right\} \\
&\qquad \times \phi(y_{1j}, y_{2j}) dy_{2j} dy_{1j} \\
&\quad + \int_0^{x_{1j}} \int_{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}}}^{\infty} \left\{ h_1\,(y_{2j} - x_{2j}) + p_{2j}(y_{1j} - x_{1j}) \right\} \\
&\qquad \times \phi(y_{1j}, y_{2j}) dy_{2j} dy_{1j}
\end{aligned} \tag{10}$$

Here, we can derive **Proposition 1** as follows.

Proposition 1. *If a two-commodity inventory system involves the both way substitutable property, then the total expected cost decreases in comparison with the non-substitutable case*.

Proof. From equation (10), $\Delta > 0$. Therefore,

$$TC_j\,(x_{1j}) + TC_j\,(x_{2j}) - TC_j\,(x_{1j}, x_{2j}) > 0\,. \qquad \text{[Q.E.D.]}$$

In order to investigate the effect of the substitution ratio α_{12} and α_{21} for this system, we differentiate $TC_j(x_{1j}, x_{2j})$ partially with respect to α_{12} and α_{21}, respectively. The following equation and **Proposition 2** are obtained.

$$\begin{aligned}
\frac{\partial TC_j\,(x_{1j}, x_{2j})}{\partial \alpha_{12}} &= -(p_{1j} + h_2) \\
&\times \int_0^{x_{2j}} \int_{x_{1j}}^{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}}} (y_{1j} - x_{1j}) \phi(y_{1j}, y_{2j}) dy_{1j} dy_{2j} \\
&- \alpha_{12}(p_{1j} + h_2) \int_0^{x_{2j}} \left(\frac{x_{2j} - y_{2j}}{\alpha_{12}} \right) \left(-\frac{x_{2j} - y_{2j}}{\alpha_{12}^2} \right) \\
&\times \phi(x_{1j} + \frac{(x_{2j} - y_{2j})}{\alpha_{12}}, y_{2j}) dy_{2j} \\
&+ \int_0^{x_{2j}} \left[p_{1j} \left(\frac{x_{2j} - y_{2j}}{\alpha_{12}} \right) + h_2\,(x_{2j} - y_{2j}) \right] \\
&\times \left(-\frac{x_{2j} - y_{2j}}{\alpha_{12}^2} \right) \phi(x_{1j} + \frac{(x_{2j} - y_{2j})}{\alpha_{12}}, y_{2j}) dy_{2j} \\
&- \int_0^{x_{2j}} p_{1j} \left[\left(\frac{x_{2j} - y_{2j}}{\alpha_{12}} \right) - (x_{2j} - y_{2j}) \right] \\
&\times \left(-\frac{x_{2j} - y_{2j}}{\alpha_{12}^2} \right) \phi(x_{1j} + \frac{(x_{2j} - y_{2j})}{\alpha_{12}}, y_{2j}) dy_{2j} \\
&= -(p_{1j} + h_2) \\
&\times \int_0^{x_{2j}} \int_{x_{1j}}^{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}}} (y_{1j} - x_{1j}) \phi(y_{1j}, y_{2j}) dy_{1j} dy_{2j}
\end{aligned} \tag{11}$$

$$\begin{aligned}
\frac{\partial TC_j\,(x_{1j}, x_{2j})}{\partial \alpha_{21}} &= -(p_{2j} + h_1) \\
&\times \int_0^{x_{1j}} \int_{x_{2j}}^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}}} (y_{2j} - x_{2j}) \phi(y_{1j}, y_{2j}) dy_{2j} dy_{1j}
\end{aligned} \tag{12}$$

Proposition 2. *$TC_j(x_{1j}, x_{2j})$ is a monotonically decreasing function of α_{12} and α_{21}.*

Proof. From equations (11) and (12), we obtain

$$\frac{\partial TC_j\,(x_{1j}, x_{2j})}{\partial \alpha_{12}} < 0, \quad \frac{\partial TC_j\,(x_{1j}, x_{2j})}{\partial \alpha_{21}} < 0\,,$$

respectively. [Q.E.D.]

Also, in order to show the effect of this inventory system for each commodity, we calculate first order and second order partial differentiation of $TC_j(x_{1j}, x_{2j})$ with respect to x_{1j} and x_{2j}, respectively.

$$\begin{aligned}
\frac{\partial TC_j\,(x_{1j}, x_{2j})}{\partial x_{1j}} &= h_1 \int_0^{x_{1j}} \int_0^{x_{2j}} \phi(y_{1j}, y_{2j}) dy_{2j} dy_{1j} \\
&+ h_2 \int_0^{x_{2j}} (x_{2j} - y_{2j}) \phi(x_{1j}, y_{2j}) dy_{2j} \\
&+ \left\{ -p_{1j}(1 - \alpha_{12}) + \alpha_{12} h_2 \right\} \\
&\quad \times \int_0^{x_{2j}} \int_{x_{1j}}^{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}}} \phi(y_{1j}, y_{2j}) dy_{1j} dy_{2j} \\
&- \left(\frac{\alpha_{12} - 1}{\alpha_{12}} \right) p_{1j} \int_0^{x_{2j}} (x_{2j} - y_{2j}) \phi(x_{1j} + \frac{(x_{2j} - y_{2j})}{\alpha_{12}}, y_{2j}) dy_{2j} \\
&- h_2 \int_0^{x_{2j}} (x_{2j} - y_{2j}) \phi(x_{1j}, y_{2j}) dy_{2j} \\
&- p_{1j} \int_0^{x_{2j}} \int_{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha 12}}^{\infty} \phi(y_{1j}, y_{2j}) dy_{1j} dy_{2j}
\end{aligned}$$

$$+\left(\frac{\alpha_{12}-1}{\alpha_{12}}\right)p_{1j}\int_0^{x_{2j}}(x_{2j}-y_{2j})\phi(x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}},y_{2j})dy_{2j}$$

$$+h_1\int_0^{x_{1j}}\int_{x_{2j}}^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha 21}}\phi(y_{1j},y_{2j})dy_{2j}dy_{1j}$$

$$+\int_{x_{2j}}^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha 21}}\left\{p_{2j}(y_{2j}-x_{2j})-\alpha_{21}(p_{2j}+h_1)(y_{2j}-x_{2j})\right\}$$

$$\times\phi(x_{1j},y_{2j})dy_{2j}$$

$$+\left(\frac{1-\alpha_{21}}{\alpha_{21}^2}\right)p_{2j}\int_0^{x_{1j}}(x_{1j}-y_{1j})\phi(y_{1j},x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}})dy_{1j}$$

$$-p_{2j}\int_0^{x_{1j}}\int_{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha 21}}^{\infty}\phi(y_{1j},y_{2j})dy_{2j}dy_{1j}$$

$$-\left(\frac{1-\alpha_{21}}{\alpha_{21}^2}\right)p_{2j}\int_0^{x_{1j}}x_{1j}\phi(y_{1j},x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}})dy_{1j}$$

$$+\left(\frac{1-\alpha_{21}}{\alpha_{21}^2}\right)p_{2j}\int_0^{x_{1j}}y_{1j}\phi(y_{1j},x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}})dy_{1j}$$

$$+p_{2j}\int_{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha 21}}^{\infty}(y_{2j}-x_{2j})\phi(x_{1j},y_{2j})dy_{2j}$$

$$-p_{1j}\int_{x_{1j}}^{\infty}\int_{x_{2j}}^{\infty}\phi(y_{1j},y_{2j})dy_{2j}dy_{1j}$$

$$-p_{2j}\int_{x_{2j}}^{\infty}(y_{2j}-x_{2j})\phi(x_{1j},y_{2j})dy_{2j}$$

$$=h_1\int_0^{x_{1j}}\int_0^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha 21}}\phi(y_{1j},y_{2j})dy_{2j}dy_{1j}$$

$$-p_{1j}\int_0^{\infty}\int_{x_{1j}}^{\infty}\phi(y_{1j},y_{2j})dy_{1j}dy_{2j}$$

$$-p_{2j}\int_0^{x_{1j}}\int_{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha 21}}^{\infty}\phi(y_{1j},y_{2j})dy_{2j}dy_{1j}$$

$$+\alpha_{12}(p_{1j}+h_2)\int_0^{x_{2j}}\int_{x_{1j}}^{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}}}\phi(y_{1j},y_{2j})dy_{1j}dy_{2j}$$

$$-\alpha_{21}(p_{2j}+h_1)\int_{x_{2j}}^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}}}(y_{2j}-x_{2j})\phi(x_{1j},y_{2j})dy_{2j} \quad (13)$$

$$\frac{\partial^2 TC_j(x_{1j},x_{2j})}{\partial x_{1j}^2}=h_1\int_0^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha 21}}\phi(x_{1j},y_{2j})dy_{2j}$$

$$+p_{1j}\int_0^{\infty}\phi(x_{1j},y_{2j})dy_{2j}$$

$$+\alpha_{12}(p_{1j}+h_2)\int_0^{x_{2j}}\phi(x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}},y_{2j})dy_{2j}$$

$$-\alpha_{12}(p_{1j}+h_2)\int_0^{x_{2j}}\phi(x_{1j},y_{2j})dy_{2j}$$

$$-p_{2j}\int_{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha 21}}^{\infty}\phi(x_{1j},y_{2j})dy_{2j}$$

$$+\left(\frac{h_1+p_{2j}}{\alpha_{21}}\right)\int_0^{x_{1j}}\phi(y_{1j},x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}})dy_{1j} \quad (14)$$

$$\frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{2j}}=h_2\int_0^{x_{2j}}\int_0^{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha 12}}\phi(y_{1j},y_{2j})dy_{1j}dy_{2j}$$

$$-p_{2j}\int_0^{\infty}\int_{x_{2j}}^{\infty}\phi(y_{1j},y_{2j})dy_{2j}dy_{1j}$$

$$-p_{1j}\int_0^{x_{2j}}\int_{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha 12}}^{\infty}\phi(y_{1j},y_{2j})dy_{1j}dy_{2j}$$

$$+\alpha_{21}(p_{2j}+h_1)\int_0^{x_{1j}}\int_{x_{2j}}^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}}}\phi(y_{1j},y_{2j})dy_{2j}dy_{1j}$$

$$-\alpha_{12}(p_{1j}+h_2)$$

$$\times\int_{x_{1j}}^{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}}}(y_{1j}-x_{1j})\phi(y_{1j},x_{2j})dy_{1j} \quad (15)$$

$$\frac{\partial^2 TC_j(x_{1j},x_{2j})}{\partial x_{2j}^2}=h_2\int_0^{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha 12}}\phi(y_{1j},x_{2j})dy_{1j}$$

$$+p_{2j}\int_0^{\infty}\phi(y_{1j},x_{2j})dy_{1j}$$

$$+\alpha_{21}(p_{2j}+h_1)\int_0^{x_{1j}}\phi(y_{1j},x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}})dy_{1j}$$

$$-\alpha_{21}(p_{2j}+h_1)\int_0^{x_{1j}}\phi(y_{1j},x_{2j})dy_{1j}$$

$$-p_{1j}\int_{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha 12}}^{\infty}\phi(y_{1j},x_{2j})dy_{1j}$$

$$+\left(\frac{h_2+p_{1j}}{\alpha_{12}}\right)\int_0^{x_{2j}}\phi(x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}},y_{2j})dy_{2j} \quad (16)$$

As for equations (14), (15), (16), these equations can be calculated by similar process used in equation (13).

From the above differentiations, we can obtain the following equations.

$$\lim_{x_{1j}\to\infty}\frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{1j}}=h_1 \quad (17)$$

$$\lim_{x_{2j}\to\infty}\frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{2j}}=h_2 \quad (18)$$

While, we can derive **Proposition 3** using the initial stock level (a_{1j},a_{2j}) and

$$\frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{1j}}=0,\quad \frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{2j}}=0.$$

Proposition 3. *We have the following optimal ordering policy. If*

$$\frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{1j}} < 0 \text{ and } \frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{2j}} < 0,$$

then we should take an order. If

$$\frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{1j}} \geq 0 \text{ and } \frac{\partial TC_j(x_{1j},x_{2j})}{\partial x_{2j}} \geq 0,$$

then we should not take an order. Notice that $x_{1j} = a_{1j}$, $x_{2j} = a_{2j}$ *(On account of an initial inventory level, we set* $z_{1j} = 0$ *and* $z_{2j} = 0$*, respectively.*).

Proof. If both first order partial differentiations of $TC_j(x_{1j},x_{2j})$ with respect to x_{1j} and x_{2j} are negative, then the object function $TC_j(x_{1j},x_{2j})$ decreases. Therefore, when the inventory level is less than (a_{1j},a_{2j}), we need to order. If both first order partial differentiations of $TC_j(x_{1j},x_{2j})$ with respect to x_{1j} and x_{2j} are nonnegative, then the object function $TC_j(x_{1j},x_{2j})$ increases. Therefore, when the inventory level is larger than or equal to (a_{1j},a_{2j}), we need not to order. [Q.E.D.]

3.3 A case of exponential distribution

In this section, we formulate the total expected cost function $TC_j(x_{1j},x_{2j})$, and the characteristics of this system. Therefore, the optimal inventory policy will be calculated when the combination of (x_{1j},x_{2j}) that minimizes the total expected cost function is obtained. Now, we assume that the demand distribution is expressed as the exponential distribution. This assumption is done by Shang Lau and Hing-Ling Lau (1991), and so on. Namely, the joint probability density function of demand, $\phi(y_{1j},y_{2j})$, is expressed as

$$\phi(y_{1j},y_{2j}) = \lambda_1 \lambda_2 \exp\left[-\lambda_1 y_{1j} - \lambda_2 y_{2j}\right], \qquad (19)$$

where λ_{ij} ($i = 1, 2$) is a shape parameter of exponential distribution of the demand of commodity i for area j. The method that can analytically solve the proposed model is only the case where the density function is the exponential distribution. We confined the problem and emphasized the derivation of efficient solution algorithm. In general, if we use nonlinear theory, we can solve the problem even if we assume the other distributions. But, in that case, it is very hard to solve analytically. If we force to solve the problem, it will be solved by numerical calculation.

From the equations (8) and (19), total expected cost function $TC_j(x_{1j},x_{2j})$ ($j = 1, 2, \cdots, m$) when the commodity 1 and 2 are allocated x_{1j} and x_{2j} respectively, for area j, is expressed as

$$\begin{aligned} TC_j(x_{1j},x_{2j}) &= h_1 x_{1j} + h_2 x_{2j} - \frac{h_1}{\lambda_{1j}} - \frac{h_2}{\lambda_{2j}} \\ &+ \frac{\{h_1 + p_{1j}(1-\alpha_{12}) - h_2\alpha_{12}\}}{\lambda_{1j}} \exp\left[-\lambda_{1j}x_{1j}\right] \\ &+ \frac{\{h_2 + p_{2j}(1-\alpha_{21}) - h_1\alpha_{21}\}}{\lambda_{2j}} \exp\left[-\lambda_{2j}x_{2j}\right] \\ &+ \frac{(h_2 + p_{1j})\alpha_{12}^2\lambda_{2j}}{\lambda_{1j}(\lambda_{1j} - \alpha_{12}\lambda_{2j})} \exp\left[-\lambda_{1j}x_{1j} - \frac{\lambda_{1j}x_{2j}}{\alpha_{12}}\right] \\ &+ \frac{(h_1 + p_{2j})\alpha_{21}^2\lambda_{1j}}{\lambda_{2j}(\lambda_{2j} - \alpha_{21}\lambda_{1j})} \exp\left[-\frac{\lambda_{2j}x_{1j}}{\alpha_{21}} - \lambda_{2j}x_{2j}\right] \\ &+ \left(\frac{(h_2 + p_{1j})\alpha_{12}}{(\lambda_{1j} - \alpha_{12}\lambda_{2j})} + \frac{(h_1 + p_{2j})\alpha_{21}}{(\lambda_{2j} - \alpha_{21}\lambda_{1j})}\right) \exp\left[-\lambda_{1j}x_{1j} - \lambda_{2j}x_{2j}\right]. \end{aligned}$$

(See **Appendix**) Thus, all area's total expected cost function TCV is expressed as

$$\begin{aligned} TCV &= \sum_{j=1}^{m} TC_j(x_{1j},x_{2j}) \\ &= \sum_{j=1}^{m} \Bigg\{ h_1 x_{1j} + h_2 x_{2j} - \frac{h_1}{\lambda_{1j}} - \frac{h_2}{\lambda_{2j}} \\ &+ \frac{\{h_1 + p_{1j}(1-\alpha_{12}) - h_2\alpha_{12}\}}{\lambda_{1j}} \exp\left[-\lambda_{1j}x_{1j}\right] \\ &+ \frac{\{h_2 + p_{2j}(1-\alpha_{21}) - h_1\alpha_{21}\}}{\lambda_{2j}} \exp\left[-\lambda_{2j}x_{2j}\right] \\ &+ \frac{(h_2 + p_{1j})\alpha_{12}^2\lambda_{2j}}{\lambda_{1j}(\lambda_{1j} - \alpha_{12}\lambda_{2j})} \exp\left[-\lambda_{1j}x_{1j} - \frac{\lambda_{1j}x_{2j}}{\alpha_{12}}\right] \\ &+ \frac{(h_1 + p_{2j})\alpha_{21}^2\lambda_{1j}}{\lambda_{2j}(\lambda_{2j} - \alpha_{21}\lambda_{1j})} \exp\left[-\frac{\lambda_{2j}x_{1j}}{\alpha_{21}} - \lambda_{2j}x_{2j}\right] \\ &+ \left(\frac{(h_2 + p_{1j})\alpha_{12}}{(\lambda_{1j} - \alpha_{12}\lambda_{2j})} + \frac{(h_1 + p_{2j})\alpha_{21}}{(\lambda_{2j} - \alpha_{21}\lambda_{1j})}\right) \exp\left[-\lambda_{1j}x_{1j} - \lambda_{2j}x_{2j}\right] \Bigg\}. \end{aligned}$$

Now, it is not always guaranteed that the above TCV has limit value with respect to x_{1j} and x_{2j}. In general, if TCV has a convex function, we can solve the above problem using nonlinear theory. But we emphasize that we can construct an efficient solution algorithm. Therefore, we only deal with the case that TCV is a convex function with respect to x_{1j} and x_{2j}. Then, we must obtain the condition that Hessian matrix is greater than or equal to 0. From Hessian matrix, we

easily derive the following conditions. That is,
$\lambda_{1j} > \alpha_{12}\lambda_{2j}$, $\lambda_{2j} > \alpha_{21}\lambda_{1j}$ and $h_1 \ge \alpha_{12}h_2$, $h_2 \ge \alpha_{21}h_1$.

To solve this problem, we introduce Lagrange's multiplier μ_1, μ_2. Let $L(\mathbf{x}_1, \mathbf{x}_2; \mu_1, \mu_2)$ be Lagrangean function. This function is represented as

$$L(\mathbf{x}_1, \mathbf{x}_2; \mu_1, \mu_2) = \sum_{j=1}^{m} TC_j(x_{1j}, x_{2j}) + \mu_1\left(\sum_{j=1}^{m} x_{1j} - X_1\right) + \mu_2\left(\sum_{j=1}^{m} x_{2j} - X_2\right), \quad (20)$$

where

$$\mathbf{x}_1 = \left(x_{11}, x_{12}, \cdots, x_{1m}\right),$$
$$\mathbf{x}_2 = \left(x_{21}, x_{22}, \cdots, x_{2m}\right).$$

From equation (9), $x_{1j} \ge 0$, $x_{2j} \ge 0$ ($j = 1, 2, \cdots, m$). The necessary and sufficient conditions to solve the problem expressed as (9) are described as follows.

$$\begin{aligned}\frac{\partial L}{\partial x_{1j}} &= h_1 - \left\{-\alpha_{12}(h_2 + p_{1j}) + (h_1 + p_{1j})\right\}\exp\left[-\lambda_{1j}x_{1j}\right] \\ &- \frac{\alpha_{12}^2\lambda_{2j}(h_2 + p_{1j})}{\lambda_{1j} - \alpha_{12}\lambda_{2j}}\exp\left[-\lambda_{1j}x_{1j} - \frac{\lambda_{1j}x_{2j}}{\alpha_{12}}\right] \\ &- \frac{\alpha_{21}\lambda_{1j}(h_1 + p_{2j})}{\lambda_{2j} - \alpha_{21}\lambda_{1j}}\exp\left[-\frac{\lambda_{2j}x_{1j}}{\alpha_{21}} - \lambda_{2j}x_{2j}\right] \\ &- \lambda_{1j}\left(\frac{\alpha_{12}(h_2 + p_{1j})}{\lambda_{1j} - \alpha_{12}\lambda_{2j}} + \frac{\alpha_{21}(h_1 + p_{2j})}{\lambda_{2j} - \alpha_{21}\lambda_{1j}}\right) \\ &\quad \times \exp\left[-\lambda_{1j}x_{1j} - \lambda_{2j}x_{2j}\right] \\ &+ \mu_1 \ge 0, \quad (j = 1, 2, \cdots, m),\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial x_{2j}} &= h_2 - \left\{-\alpha_{21}(h_1 + p_{2j}) + (h_2 + p_{2j})\right\}\exp\left[-\lambda_{2j}x_{2j}\right] \\ &- \frac{\alpha_{21}^2\lambda_{1j}(h_1 + p_{2j})}{\lambda_{2j} - \alpha_{21}\lambda_{1j}}\exp\left[-\frac{\lambda_{2j}x_{1j}}{\alpha_{21}} - \lambda_{2j}x_{2j}\right] \\ &- \frac{\alpha_{12}\lambda_{2j}(h_2 + p_{1j})}{\lambda_{1j} - \alpha_{12}\lambda_{2j}}\exp\left[-\lambda_{1j}x_{1j} - \frac{\lambda_{1j}x_{2j}}{\alpha_{12}}\right] \\ &- \lambda_{2j}\left(\frac{\alpha_{12}(h_2 + p_{1j})}{\lambda_{1j} - \alpha_{12}\lambda_{2j}} + \frac{\alpha_{21}(h_1 + p_{2j})}{\lambda_{2j} - \alpha_{21}\lambda_{1j}}\right) \\ &\quad \times \exp\left[-\lambda_{1j}x_{1j} - \lambda_{2j}x_{2j}\right] \\ &+ \mu_2 \ge 0, \quad (j = 1, 2, \cdots, m),\end{aligned}$$

$$\mu_1\left(\sum_{j=1}^{m} x_{1j} - X_1\right) = 0, \quad \mu_2\left(\sum_{j=1}^{m} x_{2j} - X_2\right) = 0,$$
$$x_{1j}\cdot\frac{\partial L}{\partial x_{1j}} = 0, \quad x_{2j}\cdot\frac{\partial L}{\partial x_{2j}} = 0 \quad (j = 1, 2, \cdots, m),$$
$$\mu_1 \ge 0, \quad \mu_2 \ge 0 \quad (21)$$

Here, L represents the Lagrangean function. A mathematical solution method we consider here is Karush-Kuhn-Tucker condition and complementariness. Applying these conditions, the inequalities $\partial L/\partial x_{1j} \ge 0$ and $\partial L/\partial x_{2j} \ge 0$ are derived easily. Also, using complementariness, $x_{1j}\cdot\partial L/\partial x_{1j} = 0$ and $x_{2j}\cdot\partial L/\partial x_{2j} = 0$ are obtained easily. Strictly speaking, if $\partial L/\partial x_{1j} = 0$, then $x_{1j}^* > 0$. Or if $\partial L/\partial x_{1j_*} > 0$, then $x_{1j}^* = 0$. Similarly, if $\partial L/\partial x_{2j} = 0$, then $x_{2j}^* > 0$. Or if $\partial L/\partial x_{2j} > 0$, then $x_{2j}^* = 0$.

Now, for the optimal solutions of the problem (9), that is to say, x_{1j}^*, x_{2j}^*,
[I] the necessary and sufficient condition that satisfies $x_{1j}^* = 0$ ($j = 1, 2, \cdots, m$) is expressed as follows:

$$\begin{aligned}\mu_1 &\ge -h_1 + \left\{-\alpha_{12}(h_2 + p_{1j}) + h_1 + p_{1j}\right\} \\ &+ \frac{\alpha_{12}^2\lambda_{2j}(h_2 + p_{1j})}{\lambda_{1j} - \alpha_{12}\lambda_{2j}}\exp\left[-\frac{\lambda_{1j}x_{2j}}{\alpha_{12}}\right] \\ &+ \frac{2\alpha_{21}\lambda_{1j}(h_1 + p_{2j})}{\lambda_{2j} - \alpha_{21}\lambda_{1j}}\exp\left[-\lambda_{2j}x_{2j}\right] \\ &+ \frac{\alpha_{12}\lambda_{1j}(h_2 + p_{1j})}{\lambda_{1j} - \alpha_{12}\lambda_{2j}}\exp\left[-\lambda_{2j}x_{2j}\right] \\ &= -\frac{\partial TCV(0, x_{2j})}{\partial x_{1j}} \\ &= A_{1j}, \end{aligned} \quad (22)$$

[II] the necessary and sufficient condition that satisfies $x_{2j}^* = 0$ ($j = 1, 2, \cdots, m$) is expressed as follows:

$$\begin{aligned}\mu_2 &\ge -h_2 + \left\{-\alpha_{21}(h_1 + p_{2j}) + h_2 + p_{2j}\right\} \\ &+ \frac{\alpha_{21}^2\lambda_{1j}(h_1 + p_{2j})}{\lambda_{2j} - \alpha_{21}\lambda_{1j}}\exp\left[-\frac{\lambda_{2j}x_{1j}}{\alpha_{21}}\right] \\ &+ \frac{2\alpha_{12}\lambda_{2j}(h_2 + p_{1j})}{\lambda_{1j} - \alpha_{12}\lambda_{2j}}\exp\left[-\lambda_{1j}x_{1j}\right] \\ &+ \frac{\alpha_{21}\lambda_{2j}(h_1 + p_{2j})}{\lambda_{2j} - \alpha_{21}\lambda_{1j}}\exp\left[-\lambda_{1j}x_{1j}\right]\end{aligned}$$

$$= -\frac{\partial TCV(x_{1j}, 0)}{\partial x_{2j}}$$

$$= A_{2j}. \quad (23)$$

We have to discuss whether allocate x_{1j} for each area or not. Then, we substitute $x_{1j}^* = 0$ for $\partial L / \partial x_{1j}$ in the equation (21). We can derive the equation (22) easily. Similarly, we can also derive the equation (23).

Now, we consider the following equations.

$$-\frac{\partial TCV(x_{1j}, x_{2j})}{\partial x_{1j}} - \mu_1 = 0, \quad -\frac{\partial TCV(x_{1j}, x_{2j})}{\partial x_{2j}} - \mu_2 = 0$$

We check the difference between $-\partial TCV(x_{1j}, x_{2j}) / \partial x_{1j}$ and μ_1. If $-\partial TCV(x_{1j}, x_{2j}) / \partial x_{1j} - \mu_1$ is greater than 0, we allocate $x_{1j}^* (> 0)$. If it is less than or equal to 0, we don't allocate ($x_{1j}^* = 0$). Similarly, same rule is applicable to commodity 2. Let $g_{1j}(x_{1j}, x_{2j})$, $g_{2j}(x_{1j}, x_{2j})$ are the inverse functions of $-\partial TCV / \partial x_{1j}$, $-\partial TCV / \partial x_{2j}$, respectively. The solutions of the above equations are obtained as follows.

$$x_{1j} = g_{1j}(\mu_1, x_{2j}), \quad x_{2j} = g_{2j}(x_{1j}, \mu_2)$$

We deal with the case where the objective function (TCV) consists of a convex function. Now, we must pay attention to the left side of objective function. Then, the left side, the more allocation of x_{ij} is, the less the value of objective function is. Inversely, the right side of objective function, the more allocation of x_{ij} is, the larger the value of objective function is. Because the objective function is a convex function. So the function $-\partial TCV / \partial x_{ij}$ is a monotone decreasing function on the left side (Take notice that $\partial TCV / \partial x_{ij}$ is a monotone increasing function on the left side). Therefore, the inverse function $g_{ij}(x_{1j}, x_{2j})$ also becomes a monotone decreasing function.

Hence, using equations (21) and (22), the optimal solution of $x_{1j} (\geq 0)$ is represented as

$$x_{1j}^* = \begin{cases} g_{1j}(\mu_1, x_{2j}), & 0 \leq \mu_1 < -\dfrac{\partial TCV(0, x_{2j})}{\partial x_{1j}} = A_{1j}, \\ 0, & \mu_1 \geq -\dfrac{\partial TCV(0, x_{2j})}{\partial x_{1j}} = A_{1j}. \end{cases}$$

Similarly, using equations (21) and (23), the optimal solution of $x_{2j} (\geq 0)$ is represented as

$$x_{2j}^* = \begin{cases} g_{2j}(x_{1j}, \mu_2), & 0 \leq \mu_2 < -\dfrac{\partial TCV(x_{1j}, 0)}{\partial x_{2j}} = A_{2j}, \\ 0, & \mu_2 \geq -\dfrac{\partial TCV(x_{1j}, 0)}{\partial x_{2j}} = A_{2j}. \end{cases}$$

As both $g_{1j}(x_{1j}, x_{2j})$ and $g_{2j}(x_{1j}, x_{2j})$ are monotone decreasing functions, so if μ_1^* and μ_2^* satisfy the following conditions, they will the optimal solutions of Lagrange's multiplier μ_1 and μ_2, respectively.

$$\sum_{j=1}^{m} \max\left\{0, \quad g_{1j}(\mu_1^*, x_{2j})\right\} = X_1 ,$$

$$\sum_{j=1}^{m} \max\left\{0, \quad g_{2j}(x_{1j}, \mu_2^*)\right\} = X_2$$

Now, we arrange the right side of inequalities (22) and (23) in large orders. And we renumber the indexes from the left side. Namely, 11, 12, $\cdots$, $1m$ and 21, 22, $\cdots$, $2m$, respectively. Those are represented as follows.

$$A_{11} \geq A_{12} \geq \cdots \geq A_{1m}, \quad A_{21} \geq A_{22} \geq \cdots \geq A_{2m}.$$

We investigate from the right side, namely, A_{1m} and A_{2m}. For μ_1^* and μ_2^*, we assume the following rearrangement.

$$A_{11} \geq A_{12} \geq \cdots \geq A_{1k} \geq \mu_1^* \geq A_{1_{k+1}} \geq \cdots \geq A_{1m},$$

$$A_{21} \geq A_{22} \geq \cdots \geq A_{2l} \geq \mu_2^* \geq A_{2_{l+1}} \geq \cdots \geq A_{2m}.$$

Now, for $A_{1_j} \geq \mu_1^* \geq A_{1_{j+1}}$ ($j = 1, 2, \cdots, m$), we have to repeat until we get

$$\sum_{j=1}^{k} g_{1j}(\mu_1^*, x_{2j}) = X_1 . \quad (24)$$

Similarly, for $A_{2_j} \geq \mu_2^* \geq A_{2_{j+1}}$, we have to repeat until we get

$$\sum_{j=1}^{l} g_{2j}(x_{1j}, \mu_2^*) = X_2 . \quad (25)$$

We obtain $x_{1j} = 0$ for $j \geq k+1$ and $x_{2j} = 0$ for $j \geq l+1$, respectively, so μ_1^* and μ_2^* satisfy the equations (24) and (25). The equation (24) means that if $m > k$,then we don't allocate m areas for commodity 1. But if $m = k$, then we allocate m areas for commodity 1. Similarly, the equation (25) means that if $m > l$,then we don't allocate m areas for commodity 2. But if $m = l$, then we allocate m areas for commodity 2. We take a notice that we can't solve these equations analytically. But we have a unique solution because of the convex function. However, if we use Newton method, solutions are obtained easily. Moreover, all area's total expected cost TCV can be calculated.

Here, we specify an algorithm to search for the optimal solutions x_{1j}^*, x_{2j}^* ($j = 1, 2, \cdots, m$) and the total expected cost TCV.

[SOLVING ALGORITHM]

Step 1. For commodity 1 and 2, now we set up $k = m$ and $l = m$ respectively. We ask for μ_i ($i = 1, 2$) that satisfies the following equations:

$$\sum_{j=1}^{k} g_{1j}(\mu_1, x_{2j}) = X_1 \ , \ \sum_{j=1}^{l} g_{2j}(x_{1j}, \mu_2) = X_2$$

Step 2. We ask for x_{1j}, x_{2j} ($j = 1, 2, \cdots, m$) using the following equations:

$$-\frac{\partial TCV(x_{1j}, x_{2j})}{\partial x_{1j}} - \mu_1 = 0 \, , -\frac{\partial TCV(x_{1j}, x_{2j})}{\partial x_{2j}} - \mu_2 = 0 \, ,$$

($j = 1, 2, \cdots, m$).

Step 3. For commodity 1, we ask for A_{1j} ($j = 1, 2, \cdots, m$) using equation (22). Similarly, for commodity 2, we ask for A_{2j} ($j = 1, 2, \cdots, m$) using equation (23).

Step 4. For commodity 1 and 2, we rearrange A_{1j} and A_{2j} in large order.

$$A_{11} \geq A_{12} \geq \cdots \geq A_{1m} \, , \ A_{21} \geq A_{22} \geq \cdots \geq A_{2m} \, .$$

Step 5. Now, if $A_{1m} \geq \mu_1 \geq A_{1_{m+1}}$ and $A_{2m} \geq \mu_2 \geq A_{2_{m+1}}$, then the optimal Lagrange's multipliers $\mu_1^* = \mu_1$ and $\mu_2^* = \mu_2$ are found (Notice that $A_{1_{m+1}} = A_{2_{m+1}} = 0$, $A_{10} = A_{20} = \infty$). Therefore, the optimal solutions x_{1j}^*, x_{2j}^* ($j = 1, 2, \cdots, m$) for commodity 1 and 2 are given. Also, the total expected cost TCV is obtained. **Stop**. If μ_1 and μ_2 are not found in the above ranges, we set up $m = m - 1$ and **Go To Step1**.

4. NUMERICAL EXAMPLES

This section gives examples in order to illustrate the results of 3.2. Each parameters of our problem is represented in Table 1. Here, we try to allocate two commodities for 10 areas so as to minimize TCV. Now, let $X_1 = 200$ and $X_2 = 200$, respectively. Consequently, the results of our problem are shown in Table1, the optimal solution x_{1j}^* and x_{2j}^* ($j = 1, 2, \cdots, m$) and the expected cost function TCV are obtained. In this case, value of the TCV is 2,594.7. From Table 1, the total values of the optimal quantities for commodity 1 and 2 are 96.19 and 147.97, respectively(The minimum value of this model). Note that the investment quantities over the optimal order quantities are inappropriate, because TCV is the convex function (TCV increases).

Second, we explain the Table 2. Each parameter of our model in Table 2 is the same situation in Table 1 except $X_1 = 10$ and $X_2 = 10$. The results of our problem are shown in Table 2, the optimal solution x_{1j}^* and x_{2j}^* ($j = 1, 2, \cdots, 10$) and the value of TCV are obtained. In this case, the value of TCV is $7,018.2$.

First of all, we examine commodity 1. We compare μ_1^* with A_{1j} ($j = 1, 2, \cdots, 10$). The arrangement in large order is

$A_{1_{10}} = 188.99 > A_{1_3} = 170.89 > A_{1_5} = 140.02 > A_{1_4} = 129.97 > A_{1_8} = 125.84 > A_{1_9} = 104.5 > \mu_1^* (= 94.16) > A_{1_2} = 85.85 > A_{1_7} = 84.97 > A_{1_6} = 82.83 > A_{1_1} = 76.05$.

As A_{1_1}, A_{1_2}, A_{1_6}, A_{1_7} are less than μ_1^*, we don't allocate these areas at all. Therefore, we determine to reallocate the remaining areas 3, 4, 5, 8, 9, 10, so minimizing TCV is achieved. Similarly, for commodity 2, the arrangement in large order is

$A_{2_1} = 79.57 > A_{2_6} = 75.45 > A_{2_7} = 73.01 > A_{2_2} = 63.75 > A_{2_4} = 49.91 > \mu_2^* (= 48.87) > A_{2_8} = 48.68 > A_{2_9} = 48.41 > A_{2_3} = 46.20 > A_{2_{10}} = 43.33 > A_{2_5} = 40.31$.

As A_{2_3}, A_{2_5}, A_{2_8}, A_{2_9}, $A_{2_{10}}$ are less than μ_2^*, we don't allocate these areas at all. Therefore, we determine to reallocate the remaining areas 1, 2, 4, 6, 7.

Table 1. The optimal allocation quantity and the total of the expected inventory cost under the whole parameters of all areas

Area	λ_{1j}	λ_{2j}	p_{1j}	p_{2j}	x_{1j}^*	x_{2j}^*	TC_j
1	0.11	0.085	50	20	11.88	14.17	283.2
2	0.15	0.092	49	18	8.90	11.67	220.6
3	0.17	0.058	55	23	9.53	19.91	305.1
4	0.15	0.065	60	19	10.37	15.94	281.0
5	0.21	0.074	53	15	7.72	11.97	204.3
6	0.13	0.083	48	27	9.78	16.02	277.1
7	0.12	0.077	51	23	11.07	15.94	291.8
8	0.14	0.063	62	17	11.26	15.43	287.0
9	0.18	0.082	46	16	7.72	11.61	201.8
10	0.22	0.071	63	21	7.97	15.30	242.8
				total	96.19	147.97	2,594.7
$X_1 = 200$, $X_2 = 200$, $h_1 = 15$, $h_2 = 10$, $\alpha_{12} = 0.5$,							
$\mu_1^* = 0.0$, $\mu_2^* = 0.0$							

Table 2. The optimal allocation quantity and the total of the expected inventory cost under the whole parameters of all areas

Area	λ_{1j}	λ_{2j}	p_{1j}	p_{2j}	x_{1j}^*	x_{2j}^*	TC_j
1	0.11	0.085	50	20	0.00	3.10	743.9
2	0.15	0.092	49	18	0.00	1.35	619.2
3	0.17	0.058	55	23	2.65	0.00	798.3

4	0.15	0.065	60	19	1.56	0.01	775.8
5	0.21	0.074	53	15	1.42	0.00	543.3
6	0.13	0.083	48	27	0.00	2.84	737.3
7	0.12	0.077	51	23	0.00	2.70	793.0
8	0.14	0.063	62	17	1.51	0.00	800.4
9	0.18	0.082	46	16	0.40	0.00	584.6
10	0.22	0.071	63	21	2.46	0.00	616.8
				total	10.00	10.00	7,018.2
$X_1 = 10$, $X_2 = 10$, $h_1 = 15$, $h_2 = 10$, $\alpha_{12} = 0.5$, $\alpha_{21} = 0.2$							
$\mu_1^* = 94.16$, $\mu_2^* = 48.87$							

In the numerical examples, we assumed the number of areas as 10. An important idea we consider here is to construct an efficient algorithm. That is, the efficiency of the computation. If we apply our proposed algorithm, we can solve the mathematical problem at most 20 times. Generally speaking, if we have m areas, the complexity of problem is reduced to $2m$ times. Our solution algorithm will very efficient.

5. CONCLUDING REMARKS

In this paper, we have dealt with an optimal ordering policy on both way substitutable two-commodity inventory system. That is, there is a problem of how to allocate the produced two kinds of goods in a factory to m areas so as to minimize the total expected inventory cost. First of all, we have formulated the model. In particular, we have assumed the exponential distribution as a probability density function of demand and have formulated the solution algorithm to carry out the optimal ordering policy. Moreover, we have provided numerical examples to explain the problem. In example, we attempted the ordering policy for 10 areas. One example is the production ability is larger than the total of each area's optimal order quantity. This example means the investment quantities over the optimal order quantities are inappropriate, because TCV is the convex function. The other one is the production ability is smaller than the total of each area's optimal order quantity. In this case, we used up the production ability. Take notice that commodity 1 and commodity 2 are not allocated on some areas. This consequence is corresponding to the solution algorithm mentioned in 3.2.

Essentially, inventory control activity must be lasted permanently. In this paper, we dealt both way substitutable two commodity inventory system on single-period, but in future, we have to investigate this system on multi-period, for example, using Dynamic Programming. Moreover, both way substitutable three commodity or more, should be researched. Also, we need to examine the estimation method of parameters used in our model.

APPENDIX

Derivation process of $TC_j(x_{1j}, x_{2j})$ using equation (8) and (19)

Each part of $L_k(x_{1j}, x_{2j})$ is calculated as follows.

[Part 1]

$$L_1(x_{1j}, x_{2j}) = \int_0^{x_{1j}} \int_0^{x_{2j}} \left\{ h_1 (x_{1j} - y_{1j}) + h_2 (x_{2j} - y_{2j}) \right\} \times \lambda_{1j} \lambda_{2j} \exp\left[-\lambda_{1j} y_{1j} - \lambda_{2j} y_{2j} \right] \, dy_2 \, dy_1$$

$$= h_1 x_{1j} + h_2 x_{2j} - \frac{h_1}{\lambda_{1j}} - \frac{h_2}{\lambda_{2j}} + \left(\frac{h_1}{\lambda_{1j}} + \frac{h_2}{\lambda_{2j}} - h_2 x_{2j} \right) \exp\left(-\lambda_{1j} x_{1j} \right) + \left(\frac{h_1}{\lambda_{1j}} + \frac{h_2}{\lambda_{2j}} - h_1 x_{1j} \right) \exp\left(-\lambda_{2j} x_{2j} \right) - \left(\frac{h_1}{\lambda_{1j}} + \frac{h_2}{\lambda_{2j}} \right) \exp\left(-\lambda_{1j} x_{1j} - \lambda_{2j} x_{2j} \right)$$

[Part 2]

$$L_2(x_{1j}, x_{2j}) = \int_0^{x_{2j}} \int_{x_{1j}}^{x_{1j} + \frac{(x_{2j} - y_{2j})}{\alpha_{12}}} \left[p_{1j}(1 - \alpha_{12})(y_{1j} - x_{1j}) + h_2 \left\{ (x_{2j} - y_{2j}) - \alpha_{12}(y_{1j} - x_{1j}) \right\} \right] \times \lambda_{1j} \lambda_{2j} \exp\left[-\lambda_{1j} y_{1j} - \lambda_{2j} y_{2j} \right] \, dy_{1j} dy_{2j}$$

$$= \left[-\frac{h_2}{\lambda_{2j}} + h_2 x_{2j} + \frac{1}{\lambda_{1j}} \left\{ p_{1j}(1 - \alpha_{12}) - \alpha_{12} h_2 \right\} \right] \times \exp\left(-\lambda_{1j} x_{1j} \right)$$

$$+ \left[\frac{h_2}{\lambda_{2j}} - \frac{1}{\lambda_{1j}} \left\{ p_{1j}(1 - \alpha_{12}) - \alpha_{12} h_2 \right\} - \frac{\lambda_{2j} \alpha_{12}}{\lambda_{1j} (\lambda_{1j} - \alpha_{12} \lambda_{2j})} \left\{ p_{1j}(1 - \alpha_{12}) - \alpha_{12} h_2 \right\} - \frac{\alpha_{12} p_{1j} \lambda_{2j} (1 - \alpha_{12})}{(\lambda_{1j} - \alpha_{12} \lambda_{2j})^2} \right] \exp\left(-\lambda_{1j} x_{1j} - \lambda_{2j} x_{2j} \right)$$

$$+ \left\{ \frac{p_{1j} \lambda_{2j} x_{2j} (1 - \alpha_{12})}{\lambda_{1j} - \alpha_{12} \lambda_{2j}} + \frac{p_{1j} \lambda_{2j} \alpha_{12} (1 - \alpha_{12})}{(\lambda_{1j} - \alpha_{12} \lambda_{2j})^2} \right.$$

$$-\frac{\lambda_{2j}\alpha_{12}}{\lambda_{1j}(\lambda_{1j}-\alpha_{12}\lambda_{2j})}\left\{p_{1j}(1-\alpha_{12})-\alpha_{12}h_2\right\}\Bigg\}$$
$$\times\exp\left(-\lambda_{1j}x_{1j}-\frac{\lambda_{1j}}{\alpha_{12}}x_{2j}\right)$$

[Part 3]

$$L_3(x_{1j},x_{2j})$$
$$=\int_0^{x_{2j}}\int_{x_{1j}+\frac{(x_{2j}-y_{2j})}{\alpha_{12}}}^{\infty} p_{1j}\left\{(y_{1j}-x_{1j})-(x_{2j}-y_{2j})\right\}$$
$$\times\lambda_{1j}\lambda_{2j}\exp\left[-\lambda_{1j}y_{1j}-\lambda_{2j}y_{2j}\right]\ dy_{1j}dy_{2j}$$
$$=\left\{\frac{p_{1j}\lambda_{2j}x_{2j}(\alpha_{12}-1)}{\lambda_{1j}-\alpha_{12}\lambda_{2j}}+\frac{p_{1j}\lambda_{2j}\alpha_{12}(\alpha_{12}-1)}{(\lambda_{1j}-\alpha_{12}\lambda_{2j})^2}\right.$$
$$\left.+\frac{p_{1j}\lambda_{2j}\alpha_{12}}{\lambda_{1j}(\lambda_{1j}-\alpha_{12}\lambda_{2j})}\right\}\exp\left(-\lambda_{1j}x_{1j}-\frac{\lambda_{1j}}{\alpha_{12}}x_{2j}\right)$$
$$+\left\{\frac{p_{1j}\lambda_{2j}\alpha_{12}}{\lambda_{1j}(\lambda_{1j}-\alpha_{12}\lambda_{2j})}-\frac{p_{1j}\lambda_{2j}\alpha_{12}(\alpha_{12}-1)}{(\lambda_{1j}-\alpha_{12}\lambda_{2j})^2}\right\}$$
$$\times\exp\left(-\lambda_{1j}x_{1j}-\lambda_{2j}x_{2j}\right)$$

[Part 4]

$$L_4(x_{1j},x_{2j})=\int_0^{x_{1j}}\int_{x_{2j}}^{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}}}\left[p_{2j}(1-\alpha_{21})(y_{2j}-x_{2j})\right.$$
$$\left.+h_1\left\{(x_{1j}-y_{1j})-\alpha_{21}(y_{2j}-x_{2j})\right\}\right]$$
$$\times\lambda_{1j}\lambda_{2j}\exp\left[-\lambda_{1j}y_{1j}-\lambda_{2j}y_{2j}\right]\ dy_{2j}dy_{1j}$$
$$=\left[-\frac{h_1}{\lambda_{1j}}+h_1x_{1j}+\frac{1}{\lambda_{2j}}\left\{p_{2j}(1-\alpha_{21})-\alpha_{21}h_1\right\}\right]$$
$$\times\exp\left(-\lambda_{2j}x_{2j}\right)$$
$$+\left[\frac{h_1}{\lambda_{1j}}-\frac{1}{\lambda_{2j}}\left\{p_{2j}(1-\alpha_{21})-\alpha_{21}h_1\right\}\right.$$
$$-\frac{\lambda_{1j}\alpha_{21}}{\lambda_{2j}(\lambda_{2j}-\alpha_{21}\lambda_{1j})}\left\{p_{2j}(1-\alpha_{21})-\alpha_{21}h_1\right\}$$
$$\left.-\frac{\alpha_{21}p_{2j}\lambda_{1j}(1-\alpha_{21})}{(\lambda_{2j}-\alpha_{21}\lambda_{1j})^2}\right]\exp\left(-\lambda_{1j}x_{1j}-\lambda_{2j}x_{2j}\right)$$
$$+\left\{\frac{p_{2j}\lambda_{1j}x_{1j}(1-\alpha_{21})}{\lambda_{2j}-\alpha_{21}\lambda_{1j}}+\frac{p_{2j}\lambda_{1j}\alpha_{21}(1-\alpha_{21})}{(\lambda_{2j}-\alpha_{21}\lambda_{1j})^2}\right.$$

$$\left.-\frac{\lambda_{1j}\alpha_{21}}{\lambda_{2j}(\lambda_{2j}-\alpha_{21}\lambda_{1j})}\left\{p_{2j}(1-\alpha_{21})-\alpha_{21}h_1\right\}\right\}$$
$$\times\exp\left(-\frac{\lambda_{2j}}{\alpha_{21}}x_{1j}-\lambda_{2j}x_{2j}\right)$$

[Part 5]

$$L_5(x_{1j},x_{2j})$$
$$=\int_0^{x_{1j}}\int_{x_{2j}+\frac{(x_{1j}-y_{1j})}{\alpha_{21}}}^{\infty} p_{2j}\left\{(y_{2j}-x_{2j})-(x_{1j}-y_{1j})\right\}$$
$$\times\lambda_{1j}\lambda_{2j}\exp\left[-\lambda_{1j}y_{1j}-\lambda_{2j}y_{2j}\right]\ dy_{2j}dy_{1j}$$
$$=\left\{\frac{p_{2j}\lambda_{1j}x_{1j}(\alpha_{21}-1)}{\lambda_{2j}-\alpha_{21}\lambda_{1j}}+\frac{p_{2j}\lambda_{1j}\alpha_{21}(\alpha_{21}-1)}{(\lambda_{2j}-\alpha_{21}\lambda_{1j})^2}\right.$$
$$\left.+\frac{p_{2j}\lambda_{1j}\alpha_{21}}{\lambda_{2j}(\lambda_{2j}-\alpha_{21}\lambda_{1j})}\right\}\exp\left(-\frac{\lambda_{2j}}{\alpha_{21}}x_{1j}-\lambda_{2j}x_{2j}\right)$$
$$+\left\{\frac{p_{2j}\lambda_{1j}\alpha_{21}}{\lambda_{2j}(\lambda_{2j}-\alpha_{21}\lambda_{1j})}-\frac{p_{2j}\lambda_{1j}\alpha_{21}(\alpha_{21}-1)}{(\lambda_{2j}-\alpha_{21}\lambda_{1j})^2}\right\}$$
$$\times\exp\left(-\lambda_{1j}x_{1j}-\lambda_{2j}x_{2j}\right)$$

[Part 6]

$$L_6(x_{1j},x_{2j})=\int_{x_{1j}}^{\infty}\int_{x_{2j}}^{\infty}\left\{p_{1j}(y_{1j}-x_{1j})+p_{2j}(y_{2j}-x_{2j})\right\}$$
$$\times\lambda_{1j}\lambda_{2j}\exp\left[-\lambda_{1j}y_{1j}-\lambda_{2j}y_{2j}\right]\ dy_{2j}dy_{1j}$$
$$=\left(\frac{p_{1j}}{\lambda_{1j}}+\frac{p_{2j}}{\lambda_{2j}}\right)\exp\left(-\lambda_{1j}x_{1j}-\lambda_{2j}x_{2j}\right)$$

Therefore, the summation

$$\sum_{k=1}^{6}L_k(x_{1j},x_{2j})$$

becomes the right side of $TC_j(x_{1j},x_{2j})$.

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