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HYPONORMAL TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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ABSTRACT. In this note we consider the hyponormality of Toeplitz operators B_φ on the Bergman space $L^2_a(\mathbb{D})$ with symbol in the class of functions $f+\overline{g}$ with polynomials f and g

1. Introduction

A bounded linear operator A on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive semidefinite. Let $\mathbb{D}$ denote the open unit disk in the complex plane, dA the area measure on the plane. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f,\ g\rangle = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\overline{g(z)}dA(z).$$

The Bergman space $L^2_a(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of functions analytic on $\mathbb{D}$. Let $L^\infty(\mathbb{D})$ be the space of bounded area measureable function on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator M_φ on the Bergman space are defined by $M_\varphi(f) = \varphi \cdot f$, where f is in L^2_a. If P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto the Bergman space L^2_a, the Toeplitz operator B_φ on the Bergman space is defined by $B_\varphi(f) = P(\varphi \cdot f)$, where φ is measurable and f is in L^2_a. It is clear that those operators are bounded if φ is in $L^\infty(\mathbb{D})$. The Hankel operator on the Bergman space is defined by

$$H_\varphi : L^2_a \longrightarrow {L^2_a}^\perp$$
$$H_\varphi(f) = (I-P)(\varphi\cdot f).$$

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Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T} = \partial\mathbb{D}$. Recall that given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator on the Hardy space is the operator T_φ on $H^2(\mathbb{T})$ defined by $T_\varphi f = P_+(\varphi \cdot f)$, where f is in $H^2(\mathbb{T})$ and P_+ denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Basic properties of the Bergman space and the Hardy space can be found in [1], [5] and [6]. The hyponormality of Toeplitz operators on the Hardy space has been studied by C. Cowen [2], [3], P. Fan [7], C. Gu [9], [10], T. Nakazi and K. Takahashi [13], K. Zhu [16], W.Y. Lee [4], [8], [11], [12] and others. In [3], Cowen characterized the hyponormality of Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. The solution is based on a dilation theorem of Sarason [15]. It also exploited the fact that functions in ${H^2}^\perp$ are conjugates of functions in zH^2. For the Bergman space, ${L_a^2}^\perp$ is much larger than the conjugates of functions in zL_a^2, and no dilation theorem(similar to Sarason's theorem) is available. Indeed it is quite difficult to determine the hyponormality of B_φ. In fact the study of hyponormal Toeplitz operators on the Bergman space seems to be scarce from the literature. Very recently, in [14], it was shown that

(i) If $n \geq m$, $B_{z^n+\alpha\bar{z}^m}$ is hyponormal if and only if $|\alpha| \leq \sqrt{\frac{m+1}{n+1}}$.
(ii) If $m \geq n$, $B_{z^n+\alpha\bar{z}^m}$ is hyponormal if and only if $|\alpha| \leq \frac{n}{m}$.

We record here some results on the hyponormality of Toeplitz operators on the Hardy space with polynomial symbols, which have been recently developed in [4], [8], and [16].

PROPOSITION 1.1. *Suppose that φ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where a_{-m} and a_N are nonzero.*

(i) *If T_φ is a hyponormal operator then $m \leq N$ and $|a_{-m}| \leq |a_N|$.*

(ii) *If T_φ is a hyponormal operator then $N-m \leq \text{rank}\,[T_\varphi^*, T_\varphi] \leq N$.*

(iii) *The hyponormality of T_φ is independent of the particular values of the Fourier coefficients $a_0, a_1, \cdots, a_{N-m}$ of φ. Moreover the rank of the self-commutator $[T_\varphi^*, T_\varphi]$ is also independent of those coefficients.*

(iv) *If $|a_{-m}| = |a_N|$, then T_φ is hyponormal if and only if the follow-*

ing equation holds:

$$\overline{a_N}\begin{pmatrix} a_{-1} \\ a_{-2} \\ \vdots \\ \vdots \\ a_{-m} \end{pmatrix} = a_{-m}\begin{pmatrix} \overline{a_{N-m+1}} \\ \overline{a_{N-m+2}} \\ \vdots \\ \vdots \\ \overline{a_N} \end{pmatrix}. \tag{1.1}$$

In this case, the rank of $[T_\varphi^*, T_\varphi]$ *is* $N-m$.

(v) T_φ *is normal if and only if* $m = N$, $|a_{-m}| = |a_N|$, *and (1.1) holds with* $m = N$.

We will now consider the hyponormality of Toeplitz operators on the Bergman space with a symbol in the class of functions $\overline{g}+f$, where f and g are polynomials. Since the hyponormality of operators is translation invariant we may assume that $f(0) = g(0) = 0$. We shall list the well-known properties of Toeplitz operators B_φ on the Bergman space.

If f, g are in $L^\infty(\mathbb{D})$ then we can easily check that

1) $B_{f+g} = B_f + B_g$
2) $B_f{}^* = B_{\overline{f}}$
3) $B_{\overline{f}}B_g = B_{\overline{f}g}$ if f or g is analytic.

These properties enable us establish several consequences of hyponormality.

Proposition 1.2. [14] *Let* f, g *be bounded and analytic. Then the followings are equivalent.*

(i) $B_{\overline{g}+f}$ *is hyponormal.*
(ii) $H_{\overline{g}}^* H_{\overline{g}} \le H_{\overline{f}}^* H_{\overline{f}}$.
(iii) $||(I-P)(\overline{g}k)|| \le ||(I-P)(\overline{f}k)||$ *for any* k *in* L_a^2.
(iv) $||\overline{g}k||^2 - ||P(\overline{g}k)||^2 \le ||\overline{f}k||^2 - ||P(\overline{f}k)||^2$ *for any* k *in* L_a^2.
(v) $H_{\overline{g}} = CH_{\overline{f}}$ *where* C *is of norm less than or equal to one.*

Proposition 1.3. *Let* f, g *be bounded and analytic. If* $B_{\overline{g}+f}$ *is hyponormal then* $||f|| \ge ||g||$.

Proof. Put $k = 1$ in Proposition 1.2 (iii). □

2. An extremal case

In this section we establish a necessary and sufficient condition for the hyponormality of Toeplitz operator B_φ on the Bergman space under certain additional assumption concerning the symbol φ.

A straightforward calculation shows that for any s, t nonnegative integers,

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t. \end{cases} \tag{2.1}$$

For $0 \leq i \leq N-1$, write

$$k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}.$$

The following two lemmas will be used for proving the main result of this section.

LEMMA 2.1. *For $0 \leq m \leq N$, we have*

(i) $||\overline{z}^m k_i(z)||^2 = \sum_{n=0}^{\infty} \frac{1}{Nn+i+m+1} |c_{Nn+i}|^2$;

(ii) $||P(\overline{z}^m k_i(z))||^2 = \begin{cases} \sum_{n=0}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2} |c_{Nn+i}|^2 & \text{if } m \leq i \\ \sum_{n=1}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2} |c_{Nn+i}|^2 & \text{if } m > i. \end{cases}$

Proof. Let $0 \leq m \leq N$. Then we have

$$\begin{aligned} ||\overline{z}^m k_i(z)||^2 &= \left\langle \overline{z}^m k_i(z), \overline{z}^m k_i(z) \right\rangle \\ &= \left\langle z^m k_i(z), z^m k_i(z) \right\rangle \\ &= \left\langle \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i+m}, \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i+m} \right\rangle \\ &= \sum_{n=0}^{\infty} \frac{1}{Nn+i+m+1} |c_{Nn+i}|^2. \end{aligned}$$

This proves the equation (i). For the equation (ii) if $m \leq i$ then by (2.1) we have

$$\begin{aligned}
&||P(\overline{z}^m k_i(z))||^2 \\
&= \left\langle \sum_{n=0}^{\infty} c_{Nn+i} \frac{Nn+i-m+1}{Nn+i+1} z^{Nn+i-m}, \right. \\
&\qquad \left. \sum_{n=0}^{\infty} c_{Nn+i} \frac{Nn+i-m+1}{Nn+i+1} z^{Nn+i-m} \right\rangle \\
&= \sum_{n=0}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2} |c_{Nn+i}|^2.
\end{aligned}$$

If instead $m > i$, again by (2.1) we get

$$\begin{aligned}
&||P(\overline{z}^m k_i(z))||^2 \\
&= \left\langle \sum_{n=1}^{\infty} c_{Nn+i} \frac{Nn+i-m+1}{Nn+i+1} z^{Nn+i-m}, \right. \\
&\qquad \left. \sum_{n=1}^{\infty} c_{Nn+i} \frac{Nn+i-m}{Nn+i+1} z^{Nn+i-m+1} \right\rangle \\
&= \sum_{n=1}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2} |c_{Nn+i}|^2.
\end{aligned}$$

□

LEMMA 2.2. *Let $f(z) = a_m z^m + a_N z^N$ and $g(z) = a_{-m} z^m + a_{-N} z^N$ $(0 < m < N)$. If $a_m \overline{a_N} = a_{-m} \overline{a_{-N}}$, then for $i \neq j$, we have*

$$\left\langle H_{\overline{f}} k_i(z), H_{\overline{f}} k_j(z) \right\rangle = \left\langle H_{\overline{g}} k_i(z), H_{\overline{g}} k_j(z) \right\rangle.$$

Proof. Observe that

$$M_{\overline{f}} k_i(z) = \overline{a_m} \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \overline{z}^m + \overline{a_N} \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \overline{z}^N$$

and for $0 \le i \neq j \le N-1$,

$$\begin{aligned}
&\left\langle \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \overline{z}^m, \sum_{n=0}^{\infty} c_{Nn+j} z^{Nn+j} \overline{z}^m \right\rangle \\
&= \left\langle \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \overline{z}^N, \sum_{n=0}^{\infty} c_{Nn+j} z^{Nn+j} \overline{z}^N \right\rangle \\
&= 0,
\end{aligned}$$

which implies that for $i \neq j$

$$\begin{aligned}
&\Big\langle M_{\overline{f}} k_i(z), M_{\overline{f}} k_j(z)\Big\rangle \\
(2.2) \qquad &= \overline{a_m} a_N \Big\langle \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \overline{z}^m,\ \sum_{n=0}^{\infty} c_{Nn+j} z^{Nn+j} \overline{z}^N \Big\rangle \\
&+ a_m \overline{a_N} \Big\langle \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \overline{z}^N,\ \sum_{n=0}^{\infty} c_{Nn+j} z^{Nn+j} \overline{z}^m \Big\rangle.
\end{aligned}$$

Similarly, for $i \neq j$, we get

$$\begin{aligned}
&\Big\langle M_{\overline{g}} k_i(z), M_{\overline{g}} k_j(z)\Big\rangle \\
(2.3) \qquad &= \overline{a_{-m}} a_{-N} \Big\langle \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \overline{z}^m,\ \sum_{n=0}^{\infty} c_{Nn+j} z^{Nn+j} \overline{z}^N \Big\rangle \\
&+ a_{-m} \overline{a_{-N}} \Big\langle \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \overline{z}^N,\ \sum_{n=0}^{\infty} c_{Nn+j} z^{Nn+j} \overline{z}^m \Big\rangle.
\end{aligned}$$

Combining (2.2), (2.3) and the assumption $a_m \overline{a_N} = a_{-m} \overline{a_{-N}}$, we get

$$(2.4) \qquad \Big\langle M_{\overline{f}} k_i(z), M_{\overline{f}} k_j(z)\Big\rangle = \Big\langle M_{\overline{g}} k_i(z), M_{\overline{g}} k_j(z)\Big\rangle \qquad \text{for } i \neq j.$$

On the other hand, it follows from (2.1) that

$$\Big\langle P(\overline{z}^p k_i(z)), P(\overline{z}^p k_j(z))\Big\rangle = 0 \ \text{ for all } 0 \leq i \neq j \leq N-1,$$

$p = 0, 1, 2, \cdots$, so that for $0 \leq i \neq j \leq N-1$,

$$\begin{aligned}
&\Big\langle B_{\overline{f}} k_i(z), B_{\overline{f}} k_j(z)\Big\rangle \\
&= \Big\langle \overline{a_m} P(\overline{z}^m k_i(z)) + \overline{a_N} P(\overline{z}^N k_i(z)), \\
&\qquad \overline{a_m} P(\overline{z}^m k_j(z)) + \overline{a_N} P(\overline{z}^N k_j(z))\Big\rangle \\
&= \overline{a_m} a_N \Big\langle P(\overline{z}^m k_i(z)),\ P(\overline{z}^N k_j(z))\Big\rangle \\
&\qquad + a_m \overline{a_N} \Big\langle P(\overline{z}^N k_i(z)),\ P(\overline{z}^m k_j(z))\Big\rangle.
\end{aligned}$$

Similarly, we also have that for $0 \le i \ne j \le N-1$

$$\Big\langle B_{\overline{g}}k_i(z), B_{\overline{g}}k_j(z)\Big\rangle = \overline{a_{-m}}a_{-N}\Big\langle P(\overline{z}^m k_i(z)), P(\overline{z}^N k_j(z))\Big\rangle + a_{-m}\overline{a_{-N}}\Big\langle P(\overline{z}^N k_i(z)), P(\overline{z}^m k_j(z))\Big\rangle.$$

Hence, again by assumption $a_m\overline{a_N} = a_{-m}\overline{a_{-N}}$, we get

(2.5)
$$\Big\langle B_{\overline{f}}k_i(z), B_{\overline{f}}k_j(z)\Big\rangle = \Big\langle B_{\overline{g}}k_i(z), B_{\overline{g}}k_j(z)\Big\rangle \qquad \text{for } 0 \le i \ne j \le N-1.$$

Combining (2.4) and (2.5) it follows that for $0 \le i \ne j \le N-1$

$$\begin{aligned}\Big\langle H_{\overline{f}}k_i(z), H_{\overline{f}}k_j(z)\Big\rangle &= \Big\langle M_{\overline{f}}k_i(z), M_{\overline{f}}k_j(z)\Big\rangle - \Big\langle B_{\overline{f}}k_i(z), B_{\overline{f}}k_j(z)\Big\rangle \\ &= \Big\langle H_{\overline{g}}k_i(z), H_{\overline{g}}k_j(z)\Big\rangle.\end{aligned}$$

This completes the proof. □

Our main result now follows:

THEOREM 2.3. *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m}z^m + a_{-N}z^N \quad (0 < m < N).$$

If $a_m\overline{a_N} = a_{-m}\overline{a_{-N}}$, *then* B_φ *is hyponormal*

$$\Longleftrightarrow \begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \ge \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & \text{if } |a_{-N}| \le |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \le m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \le |a_{-N}|. \end{cases}$$

Proof. Put $K_i := \{k_i(z) \in L_a^2 : k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i}z^{Nn+i}\}$ for $i = 0, 1, 2, \cdots, N-1$. By Proposition 1.2 (ii), B_φ is hyponormal if and only if

(2.6)
$$\Big\langle (H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}) \sum_{i=0}^{N-1} k_i(z), \ \sum_{i=0}^{N-1} k_i(z)\Big\rangle \ge 0$$

for all $k_i \in K_i$ $(i = 0, 1, 2, \cdots, N-1)$. Also we have that

(2.7)
$$\begin{aligned}&\Big\langle H_{\overline{f}}^* H_{\overline{f}} \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z)\Big\rangle \\ &= \sum_{i=0}^{N-1}\Big\langle H_{\overline{f}}k_i(z), \ H_{\overline{f}}k_i(z)\Big\rangle + \sum_{i\ne j, \ i,j\ge 0}^{N-1}\Big\langle H_{\overline{f}}k_i(z), H_{\overline{f}}k_j(z)\Big\rangle\end{aligned}$$

and

$$\text{(2.8)}\quad \begin{aligned}&\Big\langle H_{\overline{g}}^{*}H_{\overline{g}}\sum_{i=0}^{N-1}k_i(z),\sum_{i=0}^{N-1}k_i(z)\Big\rangle\\ &=\sum_{i=0}^{N-1}\Big\langle H_{\overline{g}}k_i(z),\ H_{\overline{g}}k_i(z)\Big\rangle+\sum_{i\neq j,\ i,j\geq 0}^{N-1}\Big\langle H_{\overline{g}}k_i(z),H_{\overline{g}}k_j(z)\Big\rangle.\end{aligned}$$

Substituting (2.7) and (2.8) into (2.6), it follows from Lemma 2.2 that

$$\begin{aligned}&B_{\varphi}\text{ is hyponormal}\\ \iff&\sum_{i=0}^{N-1}\Big\langle (H_{\overline{f}}^{*}H_{\overline{f}}-H_{\overline{g}}^{*}H_{\overline{g}})k_i(z),k_i(z)\Big\rangle\geq 0.\\ \iff&\sum_{i=0}^{N-1}\Big(\|\overline{f}k_i\|^2-\|\overline{g}k_i\|^2+\|P(\overline{g}k_i)\|^2-\|P(\overline{f}k_i)\|^2\Big)\geq 0.\end{aligned}$$

Therefore it follows from Lemma 2.1 that B_{φ} is hyponormal if and only if

$$\begin{aligned}&(|a_m|^2-|a_{-m}|^2)\Bigg\{\sum_{i=0}^{m-1}\Big(\frac{1}{m+i+1}|c_i|^2\\ &+\sum_{n=1}^{\infty}\Big(\frac{1}{Nn+i+m+1}-\frac{Nn+i-m+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2\Big)\\ &+\sum_{i=m}^{N-1}\sum_{n=0}^{\infty}\Big(\frac{1}{Nn+i+m+1}-\frac{Nn+i-m+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2\Bigg\}\\ &+(|a_N|^2-|a_{-N}|^2)\sum_{i=0}^{N-1}\Big(\frac{1}{N+i+1}|c_i|^2\\ &+\sum_{n=1}^{\infty}\Big(\frac{1}{N(n+1)+i+1}-\frac{N(n-1)+i+1}{(Nn+i+1)^2}\Big)|c_{Nn+i}|^2\Big)\geq 0,\end{aligned}$$

or equivalently

$$\text{(2.8)}\quad \begin{aligned}&(|a_m|^2-|a_{-m}|^2)\Big(\sum_{n=0}^{m-1}\frac{1}{n+m+1}|c_n|^2\\ &+\sum_{n=m}^{\infty}\Big(\frac{1}{n+m+1}-\frac{n-m+1}{(n+1)^2}\Big)|c_n|^2\Big)\end{aligned}$$

$$+ (|a_N|^2 - |a_{-N}|^2)\Big(\sum_{n=0}^{N-1} \frac{1}{n+N+1}|c_n|^2 +$$

$$\sum_{n=N}^{\infty}\Big(\frac{1}{n+N+1} - \frac{n-N+1}{(n+1)^2}\Big)|c_n|^2\Big) \geq 0.$$

Now if $|a_{-N}| \leq |a_N|$ and hence $|a_m| \leq |a_{-m}|$, define ζ by

$$\zeta(n) := \frac{\frac{1}{n+m+1} - \frac{n-m+1}{(n+1)^2}}{\frac{1}{n+N+1} - \frac{n-N+1}{(n+1)^2}} \qquad (n \geq 1).$$

Then ζ is a strictly decreasing function and

$$\lim_{n\to\infty}\zeta(n) = \frac{m^2}{N^2}. \tag{2.9}$$

Observe that

$$\frac{N+1}{m+1} \geq \frac{n+N+1}{n+m+1} \geq \zeta(N) \qquad \text{for } n = 1, 2, 3, \cdots, N-1. \tag{2.10}$$

Therefore (2.8) and (2.10) give that B_φ is hyponormal if and only if

$$\frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2).$$

If instead $|a_N| \leq |a_{-N}|$, define ξ by

$$\xi(n) := \frac{\frac{1}{n+m+1} - \frac{n-m+1}{(n+1)^2}}{\frac{1}{n+N+1}} \qquad \text{for } n = m, m+1, m+2, \cdots, N-1.$$

Since $\xi(n) \geq \frac{m^2}{N^2}$, it follows from (2.8), (2.9) and (2.10) that B_φ is hyponormal if and only if

$$N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2).$$

This completes the proof. □

If $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, then the hyponormality of the Toeplitz operator T_φ on the Hardy space of the unit circle implies $|a_N| \geq |a_{-m}|$ (cf. Proposition 1.1). But the above theorem shows that it is not the case for the Toeplitz operator B_φ on the Bergman space.

Corollary 2.4. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where*

$$f(z) = a_m z^m + a_N z^N, \ g(z) = a_{-m} z^m + a_{-N} z^N \quad (0 < m < N).$$

If $a_m \overline{a_N} = \alpha a_{-m} \overline{a_{-N}}$ for some $\alpha \geq 1$, then the following statements are sufficient condition for the hyponormality of B_φ.

(i) $\frac{1}{N+1}(|a_N|^2 - \sqrt{\alpha}|a_{-N}|^2) \geq \frac{1}{m+1}(\sqrt{\alpha}|a_{-m}|^2 - |a_m|^2)$ *if* $\sqrt{\alpha}|a_{-N}| \leq |a_N|$.

(ii) $N^2(\sqrt{\alpha}|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - \sqrt{\alpha}|a_{-m}|^2)$ *if* $|a_N| \leq \sqrt{\alpha}|a_{-N}|$.

Proof. If $\varphi_\alpha(z) = \overline{\sqrt{\alpha} g(z)} + f(z)$ then $\varphi_\alpha(z)$ satisfies the condition of Theorem 2.3. Hence (i) and (ii) are the necessary and sufficient condition for the hyponormality of B_{φ_α}. Note that $\alpha \geq 1$ and apply Proposition 1.2 (ii) to get the result. □

Corollary 2.5. [14]

(i) *If $n \geq m$, $B_{z^n + \alpha \overline{z}^m}$ is hyponormal if and only if $|\alpha| \leq \sqrt{\frac{m+1}{n+1}}$.*

(ii) *If $m \geq n$, $B_{z^n + \alpha \overline{z}^m}$ is hyponormal if and only if $|\alpha| \leq \frac{n}{m}$.*

Proof. Immediate from Theorem 2.3. □

If $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, then the hyponormality of T_φ on the Hardy space implies $m \leq N$ (cf. Proposition 1.1), but it is not the case for the hyponormality of B_φ on the Bergman space. For example, if $\varphi(z) = \frac{1}{2}\overline{z}^2 + z$ then by Corollary 2.5 we can see that B_φ is hyponormal.

3. Necessary conditions for hyponormality

Let $\varphi(z) = \overline{g(z)} + f(z)$, where

$$f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.$$

Then for $m, n = 1, 2, \cdots, N$, define

$$A_{m,n} := \det \begin{pmatrix} a_m & a_{-m} \\ \overline{a_{-n}} & \overline{a_n} \end{pmatrix}$$

and we abbreviate $A_{n,n}$ to A_n. In section 2, we investigated a necessary and sufficient condition for the hyponormality of the Toeplitz operator B_φ on the Bergman space when $A_{m,N} = 0$. In this section, we give a necessary condition which the hyponormality of B_φ gives $A_{m,n} = 0$.

We begin with:

PROPOSITON 3.1. *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.$$

Suppose B_φ *is hyponormal. Then*

(i) *For each* $i = 0, 1, 2 \cdots, N-1$,

$$\sum_{n=1}^{i} \frac{n^2 A_n}{(i+n+1)(i+1)^2} + \sum_{n=i+1}^{N} \frac{A_n}{i+n+1} \geq 0.$$

(ii) *For each* $i \geq N$

$$\sum_{n=1}^{N} \frac{n^2 A_n}{(i+n+1)(i+1)^2} \geq 0.$$

(iii) *If* $|a_1| \leq |a_{-1}|$ *and* $|a_i| \geq |a_{-i}|$ *for* $i \geq 2$, *then* $||f|| \geq ||g||$ *implies* (i) *and* (ii).

Proof. For each $i = 0, 1, 2, \cdots, N-1$, let $k_i(z) = \sum_{k=0}^{\infty} c_{Nk+i} z^{Nk+i}$. If B_φ is hyponormal, then Proposition 1.2 (iv) gives that

$$||\overline{f}k_i||^2 - ||\overline{g}k_i||^2 + ||P(\overline{g}k_i)||^2 - ||P(\overline{f}k_i)||^2 \geq 0 \tag{3.1}$$

$(i = 0, 1, 2, \cdots, N-1)$. Note that

$$\begin{aligned} ||\overline{f(z)}k_i(z)||^2 &= \sum_{n=1}^{N} |a_n|^2 ||\overline{z}^n k_i(z)||^2, \\ ||\overline{g(z)}k_i(z)||^2 &= \sum_{n=1}^{N} |a_{-n}|^2 ||\overline{z}^n k_i(z)||^2 \end{aligned} \tag{3.2}$$

and

$$(3.3)\qquad \begin{aligned} ||P(\overline{f(z)}k_i(z))||^2 &= \sum_{n=1}^{N} |a_n|^2 ||P(\overline{z}^n k_i(z))||^2, \\ ||P(\overline{g(z)}k_i(z))||^2 &= \sum_{n=1}^{N} |a_{-n}|^2 ||P(\overline{z}^n k_i(z))||^2. \end{aligned}$$

Substituting Lemma 2.1 (i) and (ii), respectively, into (3.2) and (3.3) and applying (3.1), we see that if B_φ is hyponormal then we have

$$(3.4)\qquad \begin{aligned} &\sum_{n=1}^{i} A_n \sum_{k=0}^{\infty} \Big(\frac{1}{Nk+i+n+1} - \frac{Nk+i-n+1}{(Nk+i+1)^2} \Big) |c_{Nk+i}|^2 \\ &+ \sum_{n=i+1}^{N} A_n \Big(\frac{1}{i+n+1} |c_i|^2 + \sum_{k=1}^{\infty} \Big(\frac{1}{Nk+i+n+1} \\ &- \frac{Nk+i-n+1}{(Nk+i+1)^2} \Big) |c_{Nk+i}|^2 \Big) \geq 0. \end{aligned}$$

If we let $c_j = 1$ for $0 \leq j \leq N-1$ and the other c_j's be 0 into (3.4), then we have (i). If we also let $c_{Nk+i} = 1$ for $0 \leq i \leq N-1,\ k = 1, 2, 3, \cdots$ and the other c_j's be 0 into (3.4), then we have

$$\sum_{n=1}^{N} \frac{n^2 A_n}{(Nk+i+n+1)(Nk+i+1)^2} \geq 0,$$

or equivalently,

$$\sum_{n=1}^{N} \frac{n^2 A_n}{(j+n+1)(j+1)^2} \geq 0 \qquad \text{for each } j \geq N,$$

which proves (ii).

For $1 \leq n \leq N,\ k \geq 0$ and $0 \leq i \leq N-1$, define D by

$$D(i,k,n) := \frac{1}{Nk+i+n+1} - \frac{Nk+i-n+1}{(Nk+i+1)^2}$$

and define Q by

$$Q(i,k,n) := \frac{D(i,k,n)}{D(i,k,n+1)}.$$

Then for each $1 \le n \le N-1$,

$$Q(i,n,k) < 1. \tag{3.5}$$

Let $|a_1| \le |a_{-1}|$ and $|a_i| \ge |a_{-i}|$ for $i \ge 2$. Note that

$$\frac{\frac{1}{2}}{\frac{1}{n+1}} \ge \frac{\frac{1}{i+2}}{\frac{1}{i+n+1}} \ge \frac{\frac{1}{i+2} - \frac{i}{(i+1)^2}}{\frac{1}{i+n+1}} \qquad \text{for } 1 \le n \le N,\ 1 \le i \le N-1. \tag{3.6}$$

Since $Q(i,n,k)$ is a strictly decreasing function of i and k, it follows from (3.5) and (3.6) that $||f|| \ge ||g||$ implies (i) and (ii). This proves (iii). □

LEMMA 3.2. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where*

$$f(z) = \sum_{n=1}^{N} a_n z^n \quad \textit{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.$$

Suppose that B_φ is hyponormal and

$$\sum_{n=1}^{i_0} \frac{n^2 A_n}{(i_0+n+1)(i_0+1)^2} + \sum_{n=i_0+1}^{N} \frac{A_n}{i_0+n+1} = 0 \tag{3.7}$$

for some $i_0 = 0, 1, 2, \cdots, N-1$. Then

$$\Big\langle \big(H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}\big) z^{i_0}, z^m \Big\rangle = 0 \qquad (0 \le m \le N-1).$$

Proof. Let B_φ be a hyponormal operator and suppose the equality (3.7) holds for some i_0. Then for $0 \le m \ne i_0 \le N-1$ and $c_{i_0}, c_m \in \mathbb{C}$, we have

$$\Big\langle \big(H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}\big)(c_{i_0} z^{i_0} + c_m z^m), c_{i_0} z^{i_0} + c_m z^m \Big\rangle \ge 0,$$

or equivalently,

$$\begin{aligned} &|c_{i_0}|^2 \Big\langle \big(H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}\big) z^{i_0}, z^{i_0} \Big\rangle \\ &+ |c_m|^2 \Big\langle \big(H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}\big) z^m, z^m \Big\rangle \\ &+ 2\mathrm{Re}\Big(c_{i_0}\overline{c_m} \Big\langle \big(H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}\big) z^{i_0}, z^m \Big\rangle \Big) \ge 0. \end{aligned} \tag{3.8}$$

Observe that

$$\begin{aligned}(3.9)\qquad \left\langle M_{\overline{f}}z^{i_0}, M_{\overline{f}}z^{i_0}\right\rangle &= \left\langle M_f z^{i_0}, M_f z^{i_0}\right\rangle \\ &= \left\langle \sum_{n=1}^{N} a_n z^{n+i_0}, \sum_{n=1}^{N} a_n z^{n+i_0}\right\rangle \\ &= \sum_{n=1}^{N} \frac{1}{n+i_0+1}|a_n|^2\end{aligned}$$

and

$$\left\langle B_{\overline{f}}z^{i_0}, B_{\overline{f}}z^{i_0}\right\rangle = \left\langle P\Big(\sum_{n=1}^{N} \overline{a_n}\overline{z^n}z^{i_0}\Big),\ P\Big(\sum_{n=1}^{N} \overline{a_n}\overline{z^n}z^{i_0}\Big)\right\rangle.$$

Therefore it follows from (2.1) that

$$\begin{aligned}&\left\langle B_{\overline{f}}z^{i_0}, B_{\overline{f}}z^{i_0}\right\rangle \\ (3.10)\qquad &= \left\langle \sum_{n=1}^{i_0} \frac{i_0-n+1}{i_0+1}\overline{a_n}z^{i_0-n}, \sum_{n=1}^{i_0} \frac{i_0-n+1}{i_0+1}\overline{a_n}z^{i_0-n}\right\rangle \\ &= \sum_{n=1}^{i_0} \frac{i_0-n+1}{(i_0+1)^2}|a_n|^2.\end{aligned}$$

Combining (3.9) and (3.10) it follows that

$$(3.11)\quad \left\langle H^*_{\overline{f}}H_{\overline{f}}z^{i_0}, z^{i_0}\right\rangle = \sum_{n=1}^{i_0} \frac{n^2|a_n|^2}{(i_0+n+1)(i_0+1)^2} + \sum_{n=i_0+1}^{N} \frac{|a_n|^2}{i_0+n+1}.$$

Similarly, we also have that

$$(3.12)\quad \left\langle H^*_{\overline{g}}H_{\overline{g}}z^{i_0}, z^{i_0}\right\rangle = \sum_{n=1}^{i_0} \frac{n^2|a_{-n}|^2}{(i_0+n+1)(i_0+1)^2} + \sum_{n=i_0+1}^{N} \frac{|a_{-n}|^2}{i_0+n+1}.$$

Combining (3.7), (3.11) and (3.12) we have

$$\begin{aligned}&\left\langle (H^*_{\overline{f}}H_{\overline{f}} - H^*_{\overline{g}}H_{\overline{g}})z^{i_0}, z^{i_0}\right\rangle \\ (3.13)\qquad &= \sum_{n=1}^{i_0} \frac{n^2 A_n}{(i_0+n+1)(i_0+1)^2} + \sum_{n=i_0+1}^{N} \frac{A_n}{i_0+n+1} = 0.\end{aligned}$$

Since c_{i_0} and c_m are arbitrary, it follows from (3.8) and (3.13) that

$$\left\langle \left(H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}\right) z^{i_0}, z^m \right\rangle = 0.$$

This completes the proof. □

We are ready for:

THEOREM 3.3. *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.$$

If B_φ *is hyponormal and* $||f|| = ||g||$, *then we have*

$$\begin{pmatrix} A_{1,1} & A_{2,2} & \dots & \dots & \dots & A_{N,N} \\ 0 & A_{1,2} & A_{2,3} & \dots & \dots & A_{N-1,N} \\ 0 & 0 & A_{1,3} & \dots & \dots & A_{N-2,N} \\ 0 & 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & A_{1,N-1} & A_{2,N} \\ 0 & 0 & \dots & \dots & 0 & A_{1,N} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{N} \\ \frac{1}{N+1} \end{pmatrix} = 0.$$

Proof. The assumption $||f|| = ||g||$ implies that the equality (3.7) holds for $i_0 = 0$. Therefore by Lemma 3.2 we have that

$$\left\langle \left(H_{\overline{f}}^* H_{\overline{f}} - H_{\overline{g}}^* H_{\overline{g}}\right) 1, z^m \right\rangle = 0 \qquad (0 \le m \le N-1). \tag{3.14}$$

Observe that

(3.15)

$$\begin{aligned} \left\langle H_{\overline{f}}^* H_{\overline{f}} 1, z^m \right\rangle &= \left\langle M_{\overline{f}}^* M_{\overline{f}} 1, z^m \right\rangle - \left\langle B_{\overline{f}}^* B_{\overline{f}} 1, z^m \right\rangle \\ &= \left\langle M_{\overline{f}} 1, M_{\overline{f}} z^m \right\rangle \\ &= \left\langle \sum_{n=1}^{N} a_n z^n, \sum_{n=1}^{N} a_n z^{n+m} \right\rangle \\ &= \sum_{n=1}^{N-m} \frac{1}{m+n+1} a_{m+n} \overline{a_n}. \end{aligned}$$

Similarly, we also have that

$$\left\langle H_{\overline{g}}^{*}H_{\overline{g}}1, z^{m}\right\rangle = \sum_{n=1}^{N-m} \frac{1}{m+n+1} a_{-m-n}\overline{a_{-n}}. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) we have

$$\sum_{n=1}^{N-m} \frac{1}{m+n+1} \overline{A_{n,m+n}} = 0 \quad (0 \leq m \leq N-1),$$

which gives the result. □

COROLLARY 3.4. *Let* $\varphi(z) = \overline{g(z)} + f(z)$, *where*

$$f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N \quad (0 < m < N).$$

If B_φ *is hyponormal and* $||f|| = ||g||$, *then* $A_{m,N} = 0$.

Proof. Immediate from Theorem 3.3. □

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