HYponormal TOEPLITZ OPERATORS
ON THE BERGMAN SPACE

IN SUNG HWANG

ABSTRACT. In this note we consider the hyponormality of Toeplitz operators $B_\varphi$ on the Bergman space $L^2_a(\mathbb{D})$ with symbol in the class of functions $f + \overline{g}$ with polynomials $f$ and $g$

1. Introduction

A bounded linear operator $A$ on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive semidefinite. Let $\mathbb{D}$ denote the open unit disk in the complex plane, $dA$ the area measure on the plane. The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_\mathbb{D} f(z) \overline{g(z)} dA(z).$$

The Bergman space $L^2_a(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of functions analytic on $\mathbb{D}$. Let $L^\infty(\mathbb{D})$ be the space of bounded area measurable function on $\mathbb{D}$. For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator $M_\varphi$ on the Bergman space are defined by $M_\varphi(f) = \varphi \cdot f$, where $f$ is in $L^2_a$. If $P$ denotes the orthogonal projection of $L^2(\mathbb{D})$ onto the Bergman space $L^2_a$, the Toeplitz operator $B_\varphi$ on the Bergman space is defined by $B_\varphi(f) = P(\varphi \cdot f)$, where $\varphi$ is measurable and $f$ is in $L^2_a$. It is clear that those operators are bounded if $\varphi$ is in $L^\infty(\mathbb{D})$. The Hankel operator on the Bergman space is defined by

$$H_\varphi : L^2_a \longrightarrow L^2_a \perp$$

$$H_\varphi(f) = (I - P)(\varphi \cdot f).$$

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Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T} = \partial \mathbb{D}$. Recall that given $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator on the Hardy space is the operator $T_\varphi$ on $H^2(\mathbb{T})$ defined by $T_\varphi f = P_+(\varphi \cdot f)$, where $f$ is in $H^2(\mathbb{T})$ and $P_+$ denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Basic properties of the Bergman space and the Hardy space can be found in [1], [5] and [6]. The hyponormality of Toeplitz operators on the Hardy space has been studied by C. Cowen [2], [3], P. Fan [7], C. Gu [9], [10], T. Nakazi and K. Takahashi [13], K. Zhu [16], W.Y. Lee [4], [8], [11], [12] and others. In [3], Cowen characterized the hyponormality of Toeplitz operator $T_\varphi$ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. The solution is based on a dilation theorem of Sarason [15]. It also exploited the fact that functions in $H^2_\perp$ are conjugates of functions in $z H^2$. For the Bergman space, $L^2_\alpha \perp$ is much larger than the conjugates of functions in $z L^2_\alpha$, and no dilation theorem (similar to Sarason's theorem) is available. Indeed it is quite difficult to determine the hyponormality of $B_\varphi$. In fact the study of hyponormal Toeplitz operators on the Bergman space seems to be scarce from the literature. Very recently, in [14], it was shown that

(i) If $n \geq m$, $B_{z^n + \alpha \bar{z}^m}$ is hyponormal if and only if $|\alpha| \leq \sqrt{\frac{m+1}{n+1}}$.

(ii) If $m \geq n$, $B_{z^n + \alpha \bar{z}^m}$ is hyponormal if and only if $|\alpha| \leq \frac{n}{m}$.

We record here some results on the hyponormality of Toeplitz operators on the Hardy space with polynomial symbols, which have been recently developed in [4], [8], and [16].

**Proposition 1.1.** Suppose that $\varphi$ is a trigonometric polynomial of the form $\varphi(z) = \sum_{n=-m}^{N} a_n z^n$, where $a_{-m}$ and $a_N$ are nonzero.

(i) If $T_\varphi$ is a hyponormal operator then $m \leq N$ and $|a_{-m}| \leq |a_N|$.

(ii) If $T_\varphi$ is a hyponormal operator then $N - m \leq \text{rank}[T_\varphi^*, T_\varphi] \leq N$.

(iii) The hyponormality of $T_\varphi$ is independent of the particular values of the Fourier coefficients $a_0, a_1, \ldots, a_{N-m}$ of $\varphi$. Moreover the rank of the self-commutator $[T_\varphi^*, T_\varphi]$ is also independent of those coefficients.

(iv) If $|a_{-m}| = |a_N|$, then $T_\varphi$ is hyponormal if and only if the follow-
ing equation holds:

\[
\begin{pmatrix}
  a_{-1} \\
  a_{-2} \\
  \vdots \\
  a_{-m}
\end{pmatrix}
\begin{pmatrix}
  a_{N-m+1} \\
  a_{N-m+2} \\
  \vdots \\
  a_N
\end{pmatrix}
= \begin{pmatrix}
  a_{-m} \\
  \vdots \\
  \vdots \\
  \vdots \\
  a_N
\end{pmatrix}.
\]

(1.1)

In this case, the rank of \([T_\varphi^*, T_\varphi]\) is \(N - m\).

(v) \(T_\varphi\) is normal if and only if \(m = N\), \(|a_{-m}| = |a_N|\), and (1.1) holds with \(m = N\).

We will now consider the hyponormality of Toeplitz operators on the Bergman space with a symbol in the class of functions \(\overline{\varphi} + f\), where \(f\) and \(g\) are polynomials. Since the hyponormality of operators is translation invariant we may assume that \(f(0) = g(0) = 0\). We shall list the well-known properties of Toeplitz operators \(B_\varphi\) on the Bergman space.

If \(f, g\) are in \(L^\infty(\mathbb{D})\) then we can easily check that

1) \(B_{f+g} = B_f + B_g\)
2) \(B_f^* = B_{\overline{f}}\)
3) \(B_f B_g = B_{fg}\) if \(f\) or \(g\) is analytic.

These properties enable us establish several consequences of hyponormality.

**Proposition 1.2.** [14] Let \(f, g\) be bounded and analytic. Then the followings are equivalent.

(i) \(B_{g+f}\) is hyponormal.

(ii) \(H_g^* H_g \leq H_f^* H_f\).

(iii) \(||(I - P)(\overline{g} k)|| \leq ||(I - P)(\overline{f} k)||\) for any \(k\) in \(L^2_a\).

(iv) \(||\overline{g} k||^2 - ||P(\overline{g} k)||^2 \leq ||\overline{f} k||^2 - ||P(\overline{f} k)||^2\) for any \(k\) in \(L^2_a\).

(v) \(H_g = CH_{\overline{f}}\) where \(C\) is of norm less than or equal to one.

**Proposition 1.3.** Let \(f, g\) be bounded and analytic. If \(B_{g+f}\) is hyponormal then \(||f|| \geq ||g||\).

*Proof.* Put \(k = 1\) in Proposition 1.2 (iii). □
2. An extremal case

In this section we establish a necessary and sufficient condition for the hyponormality of Toeplitz operator $B_\varphi$ on the Bergman space under certain additional assumption concerning the symbol $\varphi$.

A straightforward calculation shows that for any $s, t$ nonnegative integers,

\[(2.1) \quad P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t. \end{cases} \]

For $0 \leq i \leq N - 1$, write

\[k_i(z) := \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i}.\]

The following two lemmas will be used for proving the main result of this section.

**Lemma 2.1.** For $0 \leq m \leq N$, we have

(i) $||z^m k_i(z)||^2 = \sum_{n=0}^{\infty} \frac{1}{Nn+i+m+1} |c_{Nn+i}|^2$;

(ii) $||P(\overline{z}^m k_i(z))||^2 = \begin{cases} \sum_{n=0}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2} |c_{Nn+i}|^2 & \text{if } m \leq i \\ \sum_{n=1}^{\infty} \frac{Nn+i-m+1}{(Nn+i+1)^2} |c_{Nn+i}|^2 & \text{if } m > i. \end{cases}$

**Proof.** Let $0 \leq m \leq N$. Then we have

\[||z^m k_i(z)||^2 = \left< z^m k_i(z), \overline{z}^m k_i(z) \right> = \left< z^m k_i(z), z^m k_i(z) \right> = \left< \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i+m}, \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i+m} \right> = \sum_{n=0}^{\infty} \frac{1}{Nn+i+m+1} |c_{Nn+i}|^2.\]

This proves the equation (i). For the equation (ii) if $m \leq i$ then by (2.1)
we have
\[
||P(z^m k_i(z))||^2 = \left( \sum_{n=0}^{\infty} c_{Nn+i} \frac{Nn + i - m + 1}{Nn + i + 1} z^{Nn+i-m}, \sum_{n=1}^{\infty} c_{Nn+i+m} \frac{Nn + i - m + 1}{Nn + i + 1} z^{Nn+i-m+1} \right) \\
= \sum_{n=0}^{\infty} \frac{Nn + i - m + 1}{(Nn + i + 1)^2} |c_{Nn+i}|^2.
\]
If instead \( m > i \), again by (2.1) we get
\[
||P(z^m k_i(z))||^2 = \left( \sum_{n=1}^{\infty} c_{Nn+i} \frac{Nn + i - m + 1}{Nn + i + 1} z^{Nn+i-m}, \sum_{n=1}^{\infty} c_{Nn+i+m} \frac{Nn + i - m + 1}{Nn + i + 1} z^{Nn+i-m+1} \right) \\
= \sum_{n=1}^{\infty} \frac{Nn + i - m + 1}{(Nn + i + 1)^2} |c_{Nn+i}|^2.
\]

\[\square\]

**Lemma 2.2.** Let \( f(z) = a_m z^m + a_N z^N \) and \( g(z) = a_{-m} z^m + a_{-N} z^N \) (0 < \( m \) < \( N \)). If \( a_m a_N = a_{-m} a_{-N} \), then for \( i \neq j \), we have
\[
\left< H_{f} k_i(z), H_{f} k_j(z) \right> = \left< H_{g} k_i(z), H_{g} k_j(z) \right>.
\]

**Proof.** Observe that
\[
M_{f} k_i(z) = \overline{a_m} \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} + \overline{a_N} \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} z^N
\]
and for \( 0 \leq i \neq j \leq N - 1 \),
\[
\left< \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} z^m, \sum_{n=0}^{\infty} c_{Nn+j} z^{Nn+j} z^m \right> \\
= \left< \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} z^N, \sum_{n=0}^{\infty} c_{Nn+j} z^{Nn+j} z^N \right> \\
= 0,
\]
which implies that for $i \neq j$

\[
\langle M_j k_i(z), M_j k_j(z) \rangle
\]

\[
= a_m a_N \left( \sum_{n=0}^{\infty} c_{N_n+i} z^{N_n+i} \bar{z}^n, \sum_{n=0}^{\infty} c_{N_n+j} z^{N_n+j} \bar{z}^n \right)
\]

\[
+ a_m \bar{a}_N \left( \sum_{n=0}^{\infty} c_{N_n+i} z^{N_n+i} \bar{z}^n, \sum_{n=0}^{\infty} c_{N_n+j} z^{N_n+j} \bar{z}^n \right).
\]

(2.2)

Similarly, for $i \neq j$, we get

\[
\langle M_g k_i(z), M_g k_j(z) \rangle
\]

\[
= a_m a_{-N} \left( \sum_{n=0}^{\infty} c_{N_n+i} z^{N_n+i} \bar{z}^n, \sum_{n=0}^{\infty} c_{N_n+j} z^{N_n+j} \bar{z}^n \right)
\]

\[
+ a_{-m} \bar{a}_{-N} \left( \sum_{n=0}^{\infty} c_{N_n+i} z^{N_n+i} \bar{z}^n, \sum_{n=0}^{\infty} c_{N_n+j} z^{N_n+j} \bar{z}^n \right).
\]

(2.3)

Combining (2.2), (2.3) and the assumption $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$, we get

\[
\langle M_j k_i(z), M_j k_j(z) \rangle = \langle M_g k_i(z), M_g k_j(z) \rangle \quad \text{for } i \neq j.
\]

(2.4)

On the other hand, it follows from (2.1) that

\[
\langle P(\bar{z}^p k_i(z)), P(\bar{z}^p k_j(z)) \rangle = 0 \quad \text{for all } 0 \leq i \neq j \leq N - 1,
\]

$p = 0, 1, 2, \cdots$, so that for $0 \leq i \neq j \leq N - 1$,

\[
\langle B_j k_i(z), B_j k_j(z) \rangle
\]

\[
= \langle a_m P(\bar{z}^m k_i(z)) + a_N P(\bar{z}^N k_i(z)),
\]

\[
= a_m \bar{a}_N \left( \sum_{n=0}^{\infty} c_{N_n+i} z^{N_n+i} \bar{z}^n, \sum_{n=0}^{\infty} c_{N_n+j} z^{N_n+j} \bar{z}^n \right),
\]

\[
+ a_{-m} a_{-N} \left( \sum_{n=0}^{\infty} c_{N_n+i} z^{N_n+i} \bar{z}^n, \sum_{n=0}^{\infty} c_{N_n+j} z^{N_n+j} \bar{z}^n \right).
\]
Similarly, we also have that for \(0 \leq i \neq j \leq N - 1\)
\[
\left\langle B\bar{g}k_i(z), B_g k_j(z) \right\rangle = a_{-m}a_{-N} \left\langle P(z^m k_i(z)), P(z^N k_j(z)) \right\rangle + a_{-m}a_{-N} \left\langle P(z^N k_i(z)), P(z^m k_j(z)) \right\rangle.
\]
Hence, again by assumption \(a_{m\bar{a}_N} = a_{-m\bar{a}_N}\), we get
\[
(2.5) \quad \left\langle B_f k_i(z), B_f k_j(z) \right\rangle = \left\langle B_g k_i(z), B_g k_j(z) \right\rangle \quad \text{for } 0 \leq i \neq j \leq N - 1.
\]
Combining (2.4) and (2.5) it follows that for \(0 \leq i \neq j \leq N - 1\)
\[
\left\langle H_f k_i(z), H_f k_j(z) \right\rangle = \left\langle M_f k_i(z), M_f k_j(z) \right\rangle - \left\langle B_f k_i(z), B_f k_j(z) \right\rangle
\]
\[
= \left\langle H_g k_i(z), H_g k_j(z) \right\rangle.
\]
This completes the proof. \(\square\)

Our main result now follows:

**Theorem 2.3.** Let \(\varphi(z) = \overline{g(z)} + f(z)\), where
\[
f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N \quad (0 < m < N).
\]
If \(a_m a_{N} = a_{-m} a_{-N}\), then \(B_\varphi\) is hyponormal
\[
\iff \left\{ \begin{array}{l}
\frac{1}{N+1} (|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1} (|a_{-m}|^2 - |a_m|^2) \quad \text{if } |a_{-N}| \leq |a_N| \\
N^2 (|a_{-N}|^2 - |a_N|^2) \leq m^2 (|a_{-m}|^2 - |a_m|^2) \quad \text{if } |a_N| \leq |a_{-N}|.
\end{array} \right.
\]

**Proof.** Put \(K_i := \{ k_i(z) \in L^2_a : k_i(z) = \sum_{n=0}^{\infty} c_{Nn+i} z^{Nn+i} \} \) for \(i = 0, 1, 2, \ldots, N - 1\). By Proposition 1.2 (ii), \(B_\varphi\) is hyponormal if and only if
\[
(2.6) \quad \left\langle (H_f \overline{H_f} - H_g \overline{H_g}) \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \right\rangle \geq 0
\]
for all \(k_i \in K_i \quad (i = 0, 1, 2, \ldots, N - 1)\). Also we have that
\[
(2.7) \quad \left\langle H_f \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \right\rangle
\]
\[
= \sum_{i=0}^{N-1} \left\langle H_f k_i(z), H_f k_i(z) \right\rangle + \sum_{i \neq j, i, j \geq 0} \left\langle H_f k_i(z), H_f k_j(z) \right\rangle.
\]
and

$$\langle H_g^* H g \sum_{i=0}^{N-1} k_i(z), \sum_{i=0}^{N-1} k_i(z) \rangle$$

(2.8)

$$= \sum_{i=0}^{N-1} \langle H_g k_i(z), H_g k_i(z) \rangle + \sum_{i \neq j, i,j \geq 0} \langle H_g k_i(z), H_g k_j(z) \rangle.$$ 

Substituting (2.7) and (2.8) into (2.6), it follows from Lemma 2.2 that

$$B_\varphi$$ is hyponormal

$$\iff \sum_{i=0}^{N-1} \langle (H_f^* H_f - H_g^* H_g) k_i(z), k_i(z) \rangle \geq 0.$$ 

$$\iff \sum_{i=0}^{N-1} (||f k_i||^2 - ||g k_i||^2 + ||P(g k_i)||^2 - ||P(f k_i)||^2) \geq 0.$$ 

Therefore it follows from Lemma 2.1 that $$B_\varphi$$ is hyponormal if and only if

$$\left( |a_m|^2 - |a_m| \right) \left\{ \sum_{i=0}^{m-1} \left( \frac{1}{m+i+1} |c_i|^2 \right) 
+ \sum_{n=1}^{\infty} \left( \frac{1}{Nn+i+m+1} - \frac{Nn+i-m+1}{(Nn+i+1)^2} \right) |c_{Nn+i}|^2 \right\}$$ 

$$+ \sum_{i=m}^{N-1} \sum_{n=0}^{\infty} \left( \frac{1}{Nn+i+m+1} - \frac{Nn+i-m+1}{(Nn+i+1)^2} \right) |c_{Nn+i}|^2 \right\}$$

$$+ \sum_{n=1}^{\infty} \left( \frac{1}{N(n+1)+i+1} - \frac{N(n-1)+i+1}{(Nn+i+1)^2} \right) |c_{Nn+i}|^2 \right\} \geq 0,$$

or equivalently

$$\left( |a_m|^2 - |a_m| \right) \left\{ \sum_{n=0}^{m-1} \left( \frac{1}{n+m+1} |c_n|^2 \right) 
+ \sum_{n=m}^{\infty} \left( \frac{1}{n+m+1} - \frac{n-m+1}{(n+1)^2} \right) |c_n|^2 \right\} \geq 0.$$ 

(2.8)
\[ + (|a_N|^2 - |a_{-N}|^2) \left( \sum_{n=0}^{N-1} \frac{1}{n + N + 1} |c_n|^2 \right) + \sum_{n=N}^{\infty} \left( \frac{1}{n + N + 1} - \frac{n - N + 1}{(n + 1)^2} \right) |c_n|^2 \geq 0. \]

Now if \( |a_{-N}| \leq |a_N| \) and hence \( |a_m| \leq |a_{-m}| \), define \( \zeta \) by

\[ \zeta(n) := \frac{1}{n + m + 1} - \frac{n - m + 1}{(n+1)^2} \quad (n \geq 1). \]

Then \( \zeta \) is a strictly decreasing function and

\[ \lim_{n \to \infty} \zeta(n) = \frac{m^2}{N^2}. \]

Observe that

\[ \frac{N + 1}{m + 1} \geq \frac{n + N + 1}{n + m + 1} \geq \zeta(N) \quad \text{for } n = 1, 2, 3, \ldots, N - 1. \]

Therefore (2.8) and (2.10) give that \( B_\varphi \) is hyponormal if and only if

\[ \frac{1}{N + 1} (|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m + 1} (|a_{-m}|^2 - |a_m|^2). \]

If instead \( |a_N| \leq |a_{-N}| \), define \( \xi \) by

\[ \xi(n) := \frac{1}{n + m + 1} - \frac{n - m + 1}{(n+1)^2} \quad \text{for } n = m, m + 1, m + 2, \ldots, N - 1. \]

Since \( \xi(n) \geq \frac{m^2}{N^2} \), it follows from (2.8), (2.9) and (2.10) that \( B_\varphi \) is hyponormal if and only if

\[ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2). \]

This completes the proof.

If \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n \), then the hyponormality of the Toeplitz operator \( T_\varphi \) on the Hardy space of the unit circle implies \( |a_N| \geq |a_{-m}| \) (cf. Proposition 1.1). But the above theorem shows that it is not the case for the Toeplitz operator \( B_\varphi \) on the Bergman space.
Corollary 2.4. Let $\varphi(z) = \overline{g(z)} + f(z)$, where
\[ f(z) = a_m z^m + a_N z^N, \quad g(z) = a_{-m} z^m + a_{-N} z^N \quad (0 < m < N). \]

If $a_m \overline{a_N} = \alpha a_{-m} \overline{a_{-N}}$ for some $\alpha \geq 1$, then the following statements are sufficient condition for the hyponormality of $B_{\varphi}$.

(i) $\frac{1}{N+1}(|a_N|^2 - \sqrt{\alpha}|a_{-N}|^2) \geq \frac{1}{m+1} (\sqrt{\alpha}|a_{-m}|^2 - |a_m|^2)$
if $\sqrt{\alpha}|a_{-N}| \leq |a_N|$.

(ii) $N^2 (\sqrt{\alpha}|a_{-N}|^2 - |a_N|^2) \leq m^2 (|a_m|^2 - \sqrt{\alpha}|a_{-m}|^2)$
if $|a_N| \leq \sqrt{\alpha}|a_{-N}|$.

Proof. If $\varphi_\alpha(z) = \sqrt{\alpha g(z)} + f(z)$ then $\varphi_\alpha(z)$ satisfies the condition of Theorem 2.3. Hence (i) and (ii) are the necessary and sufficient condition for the hyponormality of $B_{\varphi_\alpha}$. Note that $\alpha \geq 1$ and apply Proposition 1.2 (ii) to get the result. \qed

Corollary 2.5. [14]

(i) If $n \geq m$, $B_{z^n + \alpha z^m}$ is hyponormal if and only if $|\alpha| \leq \sqrt{\frac{m+1}{n+1}}$.

(ii) If $m \geq n$, $B_{z^n + \alpha z^m}$ is hyponormal if and only if $|\alpha| \leq \frac{n}{m}$.

Proof. Immediate from Theorem 2.3. \qed

If $\varphi(z) = \sum_{n=-m}^N a_n z^n$, then the hyponormality of $T_\varphi$ on the Hardy space implies $m \leq N$ (cf. Proposition 1.1), but it is not the case for the hyponormality of $B_{\varphi}$ on the Bergman space. For example, if $\varphi(z) = \frac{1}{2} z^2 + z$ then by Corollary 2.5 we can see that $B_{\varphi}$ is hyponormal.

3. Necessary conditions for hyponormality

Let $\varphi(z) = \overline{g(z)} + f(z)$, where
\[ f(z) = \sum_{n=1}^N a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^N a_{-n} z^n. \]

Then for $m, n = 1, 2, \ldots, N$, define
\[ A_{m,n} := \det \begin{pmatrix} \frac{a_m}{a_n} & \frac{a_{-m}}{a_n} \\ \frac{a_{-m}}{a_{-n}} & \frac{a_m}{a_{-n}} \end{pmatrix} \]
and we abbreviate $A_{n,n}$ to $A_n$. In section 2, we investigated a necessary and sufficient condition for the hyponormality of the Toeplitz operator $B_\varphi$ on the Bergman space when $A_{m,N} = 0$. In this section, we give a necessary condition which the hyponormality of $B_\varphi$ gives $A_{m,n} = 0$.

We begin with:

**Proposition 3.1.** Let $\varphi(z) = \overline{g(z)} + f(z)$, where

$$f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.$$

Suppose $B_\varphi$ is hyponormal. Then

(i) For each $i = 0,1,2,\ldots,N-1$,

$$\sum_{n=1}^{i} \frac{n^2 A_n}{(i+n+1)(i+1)^2} + \sum_{n=i+1}^{N} \frac{A_n}{i+n+1} \geq 0.$$

(ii) For each $i \geq N$

$$\sum_{n=1}^{N} \frac{n^2 A_n}{(i+n+1)(i+1)^2} \geq 0.$$

(iii) If $|a_1| \leq |a_{-1}|$ and $|a_i| \geq |a_{-i}|$ for $i \geq 2$, then $||f|| \geq ||g||$ implies (i) and (ii).

**Proof.** For each $i = 0,1,2,\ldots,N-1$, let $k_i(z) = \sum_{k=0}^{\infty} c_N k^{k+i}$. If $B_\varphi$ is hyponormal, then Proposition 1.2 (iv) gives that

\[ ||f k_i||^2 - ||g k_i||^2 + ||P(g k_i)||^2 - ||P(f k_i)||^2 \geq 0 \quad (3.1) \]

(i = 0,1,2,\ldots,N-1). Note that

\[ ||f(z)k_i(z)||^2 = \sum_{n=1}^{N} |a_n|^2 ||z^n k_i(z)||^2, \quad \text{(3.2)} \]

\[ ||g(z)k_i(z)||^2 = \sum_{n=1}^{N} |a_{-n}|^2 ||\overline{z^n} k_i(z)||^2 \]
and

\[ \|P(f(z)k_i(z))\|^2 = \sum_{n=1}^{N} |a_n|^2 \|P(z^n k_i(z))\|^2, \]

(3.3)

\[ \|P(g(z)k_i(z))\|^2 = \sum_{n=1}^{N} |a_{-n}|^2 \|P(z^n k_i(z))\|^2. \]

Substituting Lemma 2.1 (i) and (ii), respectively, into (3.2) and (3.3) and applying (3.1), we see that if \( B_\varphi \) is hyponormal then we have

\[
\sum_{n=1}^{i} A_n \sum_{k=0}^{\infty} \left( \frac{1}{Nk + i + n + 1} - \frac{Nk + i - n + 1}{(Nk + i + 1)^2} \right) |c_{Nk+i}|^2

+ \sum_{n=i+1}^{N} A_n \left( \frac{1}{i + n + 1} |c_i|^2 + \sum_{k=1}^{\infty} \frac{1}{Nk + i + n + 1} - \frac{Nk + i - n + 1}{(Nk + i + 1)^2} \right) |c_{Nk+i}|^2 \geq 0.
\]

(3.4)

If we let \( c_j = 1 \) for \( 0 \leq j \leq N - 1 \) and the other \( c_j \)'s be 0 into (3.4), then we have (i). If we also let \( c_{Nk+i} = 1 \) for \( 0 \leq i \leq N - 1 \), \( k = 1, 2, 3, \ldots \) and the other \( c_j \)'s be 0 into (3.4), then we have

\[
\sum_{n=1}^{N} \frac{n^2 A_n}{(Nk + i + n + 1)(Nk + i + 1)^2} \geq 0,
\]

or equivalently,

\[
\sum_{n=1}^{N} \frac{n^2 A_n}{(j + n + 1)(j + 1)^2} \geq 0 \quad \text{for each} \ j \geq N,
\]

which proves (ii).

For \( 1 \leq n \leq N \), \( k \geq 0 \) and \( 0 \leq i \leq N - 1 \), define \( D \) by

\[
D(i, k, n) := \frac{1}{Nk + i + n + 1} - \frac{Nk + i - n + 1}{(Nk + i + 1)^2}
\]

and define \( Q \) by

\[
Q(i, k, n) := \frac{D(i, k, n)}{D(i, k, n + 1)}.
\]
Then for each \(1 \leq n \leq N - 1\),

\[
Q(i, n, k) < 1.
\]

(3.5)

Let \(|a_1| \leq |a_{-1}|\) and \(|a_i| \geq |a_{-i}|\) for \(i \geq 2\). Note that

\[
\frac{1}{n+1} \geq \frac{1}{i+2} \geq \frac{i+2 - (i+1)^2}{i(i+1)} \quad \text{for } 1 \leq n \leq N, \ 1 \leq i \leq N - 1.
\]

(3.6)

Since \(Q(i, n, k)\) is a strictly decreasing function of \(i\) and \(k\), it follows from (3.5) and (3.6) that \(||f|| \geq ||g||\) implies (i) and (ii). This proves (iii). \(\square\)

**Lemma 3.2.** Let \(\varphi(z) = \overline{g(z)} + f(z)\), where

\[
f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.
\]

Suppose that \(B_\varphi\) is hyponormal and

\[
\sum_{n=1}^{i_0} \frac{n^2 A_n}{(i_0 + n + 1)(i_0 + 1)^2} + \sum_{n=i_0+1}^{N} \frac{A_n}{i_0 + n + 1} = 0
\]

for some \(i_0 = 0, 1, 2, \ldots, N - 1\). Then

\[
\left\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}) z^{i_0}, z^m \right\rangle = 0 \quad (0 \leq m \leq N - 1).
\]

**Proof.** Let \(B_\varphi\) be a hyponormal operator and suppose the equality (3.7) holds for some \(i_0\). Then for \(0 \leq m \neq i_0 \leq N - 1\) and \(c_{i_0}, c_m \in \mathbb{C}\), we have

\[
\left\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}) (c_{i_0} z^{i_0} + c_m z^m), c_{i_0} z^{i_0} + c_m z^m \right\rangle \geq 0,
\]

or equivalently,

\[
|c_{i_0}|^2 \left\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}) z^{i_0}, z^{i_0} \right\rangle
\]

\[
+ |c_m|^2 \left\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}) z^m, z^m \right\rangle
\]

\[
+ 2 \text{Re} \left( c_{i_0} \overline{c_m} \left\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}}) z^{i_0}, z^m \right\rangle \right) \geq 0.
\]

(3.8)
Observe that

\[
\left\langle M_f z^{i_0}, M_f z^{i_0} \right\rangle = \left\langle M_f z^{i_0}, M_f z^{i_0} \right\rangle
\]

\[
= \left\langle \sum_{n=1}^{N} a_n z^{n+i_0}, \sum_{n=1}^{N} a_n z^{n+i_0} \right\rangle
\]

\[
= \sum_{n=1}^{N} \frac{1}{n+i_0 + 1} |a_n|^2
\]

and

\[
\left\langle B_f z^{i_0}, B_f z^{i_0} \right\rangle = \left\langle P\left(\sum_{n=1}^{N} \frac{z^n z^{i_0}}{a_n}\right), P\left(\sum_{n=1}^{N} \frac{z^n z^{i_0}}{a_n}\right) \right\rangle.
\]

Therefore it follows from (2.1) that

\[
\left\langle B_f z^{i_0}, B_f z^{i_0} \right\rangle
\]

\[
= \left\langle \sum_{n=1}^{i_0} \frac{i_0 - n + 1}{i_0 + 1} a_n z^{i_0 - n}, \sum_{n=1}^{i_0} \frac{i_0 - n + 1}{i_0 + 1} a_n z^{i_0 - n} \right\rangle
\]

\[
= \sum_{n=1}^{i_0} \frac{i_0 - n + 1}{(i_0 + 1)^2} |a_n|^2.
\]

Combining (3.9) and (3.10) it follows that

\[
\left\langle H_f^* H_f z^{i_0}, z^{i_0} \right\rangle = \sum_{n=1}^{i_0} \frac{n^2|a_n|^2}{(i_0 + n + 1)(i_0 + 1)^2} + \sum_{n=i_0+1}^{N} \frac{|a_n|^2}{i_0 + n + 1}.
\]

Similarly, we also have that

\[
\left\langle H_g^* H_g z^{i_0}, z^{i_0} \right\rangle = \sum_{n=1}^{i_0} \frac{n^2|a_{-n}|^2}{(i_0 + n + 1)(i_0 + 1)^2} + \sum_{n=i_0+1}^{N} \frac{|a_{-n}|^2}{i_0 + n + 1}.
\]

Combining (3.7), (3.11) and (3.12) we have

\[
\left\langle (H_f^* H_f - H_g^* H_g) z^{i_0}, z^{i_0} \right\rangle
\]

\[
= \sum_{n=1}^{i_0} \frac{n^2 A_n}{(i_0 + n + 1)(i_0 + 1)^2} + \sum_{n=i_0+1}^{N} \frac{A_n}{i_0 + n + 1} = 0.
\]
Since $c_{i0}$ and $c_m$ are arbitrary, it follows from (3.8) and (3.13) that

$$\left\langle (H_f^*H_f - H_g^*H_g)z^{i_0}, z^m \right\rangle = 0.$$ 

This completes the proof. \qed

We are ready for:

**Theorem 3.3.** Let $\varphi(z) = \overline{g(z)} + f(z)$, where

$$f(z) = \sum_{n=1}^{N} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{N} a_{-n} z^n.$$ 

If $B_\varphi$ is hyponormal and $||f|| = ||g||$, then we have

$$\begin{pmatrix} A_{1,1} & A_{2,2} & \ldots & \ldots & \ldots & A_{N,N} \\ 0 & A_{1,2} & A_{2,3} & \ldots & \ldots & A_{N-1,N} \\ 0 & 0 & A_{1,3} & \ldots & \ldots & A_{N-2,N} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & 0 & A_{1,N-1} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \\ \vdots \\ \frac{1}{N} \end{pmatrix} = 0.$$ 

**Proof.** The assumption $||f|| = ||g||$ implies that the equality (3.7) holds for $i_0 = 0$. Therefore by Lemma 3.2 we have that

$$\left\langle (H_f^*H_f - H_g^*H_g)1, z^m \right\rangle = 0 \quad (0 \leq m \leq N - 1).$$

Observe that

$$\left\langle H_f^*H_f1, z^m \right\rangle = \left\langle M_f^*M_f1, z^m \right\rangle - \left\langle B_f^*B_f1, z^m \right\rangle$$

$$= \left\langle M_f^*1, M_fz^m \right\rangle$$

$$= \left\langle \sum_{n=1}^{N} a_n z^n, \sum_{n=1}^{N} a_n z^{n+m} \right\rangle$$

$$= \sum_{m-n}^{N-m} \frac{1}{m + n + 1} a_{m+n} \overline{a_n}.$$
Similarly, we also have that

\begin{equation}
\langle H_z^+ H_g^+ \|, z^m \rangle = \sum_{n=1}^{N-m} \frac{1}{m + n + 1} a_{-m-n} a_{-n}.
\end{equation}

Substituting (3.15) and (3.16) into (3.14) we have

\[
\sum_{n=1}^{N-m} \frac{1}{m + n + 1} A_{n,m+n} = 0 \quad (0 \leq m \leq N - 1),
\]

which gives the result.

\begin{corollary}
Let \( \varphi(z) = g(z) + f(z) \), where

\[ f(z) = a_m z^m + a_N z^N \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-N} z^N \quad (0 < m < N). \]

If \( B_{\varphi} \) is hyponormal and \( ||f|| = ||g|| \), then \( A_{m,N} = 0 \).
\end{corollary}

\textbf{Proof.} Immediate from Theorem 3.3.

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\textbf{References}


Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea

*E-mail*: ihwang@skku.edu