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CONFORMAL TRANSFORMATIONS IN A TWISTED PRODUCT SPACE

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Abstract. The conharmonic transformation is a conformal transformation which satisfies a specified differential equation. Such a transformation was defined by Y. Ishii and we have generalized his results. Twisted product space is a generalized warped product space with a warping function defined on a whole space. In this paper, we partially classified the twisted product space and obtain a sufficient condition for a twisted product space to be locally Riemannian products.

1. Introduction

It is well known that conformal transformation on the Riemannian manifold does not change the angle between two vectors at a point. But, in general, the harmonicity of functions, vectors and forms are not preserved by the conformal transformation. Related this fact, Y. Ishii [4] have studied conharmonic transformation and calculated conharmonic curvature tensor which is a geometric invariants under conharmonic transformation. On the other hand generalizing the idea of warped products, B. Y. Chen [3] defined the twisted products to construct a family of totally umbilical submanifolds with various properties. The projectable metric on fibred Riemannian space and a bundle-like metric defined on foliation are kind of twisted metric. Machida and Sato [7], Ponge and

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Reckziegel [9] have studied twisted products and twistor spaces in the case of pseudo-Riemannian manifolds. In this point of view, we are to classify the twisted product space by use of Tashiro's theorem [10] for a complete manifolds admitting a special concircular scalar field and study the sufficient condition for a conharmonically flat twisted product space to be locally a Riemannian product.

2. A concircular scalar field

Let M be an m-dimensional Riemannian manifold with metric tensor G. We call a non-constant scalar field f in M a concircular scalar field if it satisfies the equation

$$\nabla_X \nabla_Y f = \phi G(X,Y) \tag{2.1}$$

where ∇ indicates covariant differentiation with respect to G and ϕ is a scalar field called the characteristic function of f. If ϕ is of the form $\phi = -\alpha f + \beta$ with constant coefficients α and β, then f is called a special concircular scalar field. The term "concircular" comes from the concircular transformation introduced by K. Yano [11]. A concircular transformation is by definition a conformal transformation preserving geodesic circles. We denote the number of isolated stationary points of a concircular scalar field f in M by $n(f)$. Y. Tashiro [10] have proved $n(f) \leq 2$ and classified the complete manifolds admitting concircular or special concircular scalar fields.

THEOREM 2.1. [10] *Let M be a complete Riemannian manifold of dimension $m \geq 2$ and suppose that it admits a special concircular scalar field f satisfying the equation*

$$\nabla_X \nabla_Y f = (-\alpha f + \beta) G(X,Y) \tag{2.2}$$

Then M is one of the following manifolds :

(I.A) if $\alpha = \beta = 0$, the direct product $V \times I$ of an (m-1)-dimensional complete Riemannian manifold V with a straight line I,

(I.B) if $\alpha = 0$ but $\beta \neq 0$, a Euclidean space,

(II,A) if $\alpha = -c^2$ and $n(f) = 0$, a pseudo-hyperbolic space of zero or negative type,

(II,B) if $\alpha = -c^2 < 0$ and $n(f) = 1$, hyperbolic space of curvature $-c^2$ and

(III) if $\alpha = c^2 > 0$, a spherical space of curvature c^2, where c is a positive constant.

3. Twisted product space

Let (B, g) and $(F, \bar{g})$ be n-dimensional and p-dimensional Riemannain manifolds and f a positive smooth function on $B \times F$. Consider the product manifold $B \times F$ with projections

$$\pi : B \times F \to B \quad \textit{and} \quad \sigma : B \times F \to F.$$

The twisted product manifold $M = B \times_f F$ is by definition the manofolds $B \times F$ with the Riemannian structure given by

$$(3.1) \qquad \|X\|^2 = \|\pi_* X\|^2 + (f(b,p))^2 \|\sigma_* X\|^2$$

for any vector X tangent to M at (b, p) ([1], [6], [7], [9]). If f depends on B only, this is the warped product of B and F. Let G be the Riemannian metric on M and $\dim M = m = n + p$.

For a local coordinate system (u^a) of B, the metric tensor g has the components (g_{ab}). Similarily, for a local coordinate system (u^x) of F, $\bar{g}$ has the components $(\bar{g}_{xy})$. Then with respect to the local coordinate system (u^a, u^x) of M, G has the components

$$(3.2) \qquad (G_{ji}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & f^2 \bar{g}_{xy} \end{pmatrix}.$$

Throughout this paper, the ranges of indices are as follows :

$$i, j, k, \cdots : 1, 2, \cdots, n + p = m$$
$$a, b, c, \cdots : 1, 2, \cdots, n$$
$$x, y, z, \cdots : n + 1, \cdots, n + p$$

unless otherwise stated.

Let ∇_b(resp. ∇_x) be the components of the covariant derivative with respect to g(resp. $\bar{g}$) and $\{{}^{a}_{bc}\}$(resp. $\{{}^{\bar{x}}_{yz}\}$) the Christoffel symbol of g(resp. $\bar{g}$). Then the Christoffel symbol $\{{}^{\tilde{k}}_{ji}\}$ of G on M are given as follows :

$$(3.3) \qquad \{{}^{\tilde{a}}_{bc}\} = \{{}^{a}_{bc}\},$$

$$\widetilde{\left\{ {x \atop yz} \right\}} = \overline{\left\{ {x \atop yz} \right\}} + \frac{1}{f}(f_y\delta_z^x + f_z\delta_y^x - f^x\bar{g}_{yz}), \tag{3.4}$$

$$\widetilde{\left\{ {x \atop ya} \right\}} = \frac{1}{f}f_a\delta_y^x, \tag{3.5}$$

$$\widetilde{\left\{ {a \atop xy} \right\}} = -ff^a\bar{g}_{xy} \tag{3.6}$$

and the others are zero, where $f_a = \frac{\partial f}{\partial u^a}$ and $f_y = \frac{\partial f}{\partial u^y}$.

Let $\tilde{R}$, R and $\bar{R}$ be the curvature tensors of M, B and F respectively, then we have

$$\tilde{R}_{dcb}{}^a = R_{dcb}{}^a, \tag{3.7}$$

$$\tilde{R}_{dxy}{}^z = \frac{1}{f}(\partial_d f_y)\delta_x^z - \frac{1}{f}(\partial_d f^z)\bar{g}_{xy} - \frac{1}{f^2}f_df_y\delta_x^z + \frac{1}{f^2}f_df^z\bar{g}_{xy}, \tag{3.8}$$

$$\tilde{R}_{dxb}{}^z = \frac{1}{f}(\nabla_d f_b)\delta_x^z, \tag{3.9}$$

$$\tilde{R}_{xyz}{}^a = -f(\partial_x f^a)\bar{g}_{yz} + f(\partial_y f^a)\bar{g}_{xz} - f^af_y\bar{g}_{xz} + f^af_x\bar{g}_{yz}, \tag{3.10}$$

(3.11)

$$\begin{aligned}\tilde{R}_{xyz}{}^w = \bar{R}_{xyz}{}^w &+ \frac{1}{f}[(\nabla_x f_z)\delta_y^w + (\nabla_y f^w)\bar{g}_{xz} - (\nabla_x f^w)\bar{g}_{yz} - (\nabla_y f_z)\delta_x^w] \\ &- \frac{2}{f^2}[f_xf_z\delta_y^w + f_yf^w\bar{g}_{xz} - f_xf^w\bar{g}_{yz} - f_yf_z\delta_x^w] \\ &+ \|f_e\|^2[\bar{g}_{xz}\delta_y^w - \bar{g}_{yz}\delta_x^w] - \frac{\|f_v\|^2}{f^2}[\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w],\end{aligned}$$

and the others are zero.

The components of Ricci tensors are given by

$$\tilde{S}_{ab} = S_{ab} - \frac{p}{f}(\nabla_a f_b), \tag{3.12}$$

$$\tilde{S}_{ax} = -(p-1)(\frac{1}{f}\partial_a f_x - \frac{1}{f^2} f_a f_x), \tag{3.13}$$

$$\begin{aligned} \tilde{S}_{yx} = \bar{S}_{yx} - f(\triangle f)\bar{g}_{yx} - \frac{1}{f}(\bar{\triangle} f)\bar{g}_{yx} \\ - \frac{(p-2)}{f}\nabla_y f_x + \frac{2(p-2)}{f^2} f_y f_x \\ - (p-1)\|f_e\|^2 \bar{g}_{yx} - \frac{(p-3)}{f^2}\|f_w\|^2 \bar{g}_{yx}, \end{aligned} \tag{3.14}$$

where $\triangle f = \nabla_e f^e$, $\bar{\triangle} f = \nabla_x f^x$ and $\tilde{S}$, S and $\bar{S}$ are the Ricci tensors of M, B and F respectively.

Let $\tilde{K}$, K and $\bar{K}$ be the scalar curvatures of M, B and F respectively, then we have

$$\begin{aligned} \tilde{K} = K + \frac{1}{f^2}\bar{K} - \frac{2p}{f}(\triangle f) - \frac{2(p-1)}{f^3}(\bar{\triangle} f) \\ - \frac{p(p-1)}{f^2}\|f_e\|^2 - \frac{(p-1)(p-4)}{f^4}\|f_x\|^2. \end{aligned} \tag{3.15}$$

It is well known that [1] if M is conformally flat and $n > 1, p > 1$, then $\tilde{S}_{ax} = 0$. If $\tilde{S}_{ax} = 0$, then $M = B \times_f F$ is called mixed Ricci-flat[6]. Since

$$\tilde{S}_{ax} = -(p-1)\partial_a \partial_y (log f) \tag{3.16}$$

in (3.13), f is a product of a certain function f^* on B and $\bar{f}$ on F. Let F^* be a fibre with the metric $g^*_{xy} = \bar{f}^2 \bar{g}_{xy}$, then we get the following theorems.

THEOREM 3.1. [1] *If the twisted product manifold $M = B \times_f F$ of B and F is conformally flat and $n > 1, p > 1$, then M is the warped product $B \times_{f^*} F^*$ of B and F^**

COROLLARY 3.2. [6] *Let $M = B \times_f F$ be a twisted product of B and F with $p > 1$. If M is mixed Ricci-flat, then M is the warped product $B \times_{f^*} F^*$ of B and F^*.*

4. Conformal transformation in twisted product spaces

A conformal transformation is called formerly a conformal change of metric and recently a conformal diffeomorphism. It is well known that a conformal transformation between two Riemannian manifolds (M, g) and (M^*, g^*) is a diffeomorphism preserving angles measured by the metric g and g^*([2, 8]). It is characterized by $g^* = e^{2\rho}g$ for a scalar function ρ. The conformal curvature tensor C is defined by

$$\begin{aligned}C(X,Y)Z = R(X,Y)Z - &\frac{1}{m-2}\{S(Y,Z)X - g(X,Z)QY \\ &+ g(Y,Z)X - S(X,Z)Y\} \\ &+ \frac{r}{(m-1)(m-2)}\{g(Y,Z)QX - g(X,Z)Y\},\end{aligned} \tag{4.1}$$

where R, S and r are the Riemannian curvature, Ricci curvature and scalar curvature of the m-dimensional manifold M respectively and $g(QX,Y) = S(X,Y)$. It is well known that C is invariant under any conformal change of the metric and $C = 0$ for $m > 3$ is a necessary and sufficient condition for a Riemannian manifold to be conformally flat([1, 2, 3, 8]).

On the other hand, a harmonic function f is defined as a function whose Laplacian $\triangle f$ vanishes. It is easily seen that ([4, 5])

$$\triangle^* f = e^{-2\rho}\triangle f + (m-2)e^{-2\rho}\rho^h(\partial_h f) \tag{4.2}$$

for a smooth function f, where $\rho_h = \partial_h \rho$. Therefore a harmonic function is not in general transformed into a harmonic function by the conformal transformation except 2-dimensional case. In this point of view, Y. Ishii [4] defined a conharmonic transformation which is the conformal transformation preserving harmonicity of a certain function. The conharmonic curvature tensor T defined by

$$\begin{aligned}&T(X,Y)Z = R(X,Y)Z \\ &\quad - \frac{1}{m-2}\{S(Y,Z)X - g(X,Z)QY + g(Y,Z)QX - S(X,Z)Y\}\end{aligned} \tag{4.3}$$

is invariant under conharmonic transformation. When T vanishes identically on M, we call M is conharmonically flat. It is well known that the conformally flat manifold is conharmonically flat if and only if the scalar curvature vanishes.

Let $M = B \times_f F$ be a conharmonically flat twisted product and $m > 2$. Then, the identities (3.7)–(3.11) and (4.3) imply

$$(4.4)\quad \begin{aligned} R_{dcb}{}^{a} = &\frac{1}{(m-2)}(S_{cb}\delta_d^a - S_{db}\delta_c^a + S_d{}^a g_{cb} - S_c{}^a g_{db}) \\ &- \frac{p}{(m-2)f}(\delta_d^a \nabla_c f_b - \delta_c^a \nabla_d f_b + g_{cb}\nabla_d f^a - g_{db}\nabla_c f^a) \end{aligned}$$

$$(4.5)\quad \begin{aligned} &\frac{n-1}{f}(\partial_d f_y)\delta_x^z - \frac{n-1}{f}(\partial_d f^z)\bar{g}_{xy} - \frac{n-1}{f^2} f_d f_y \delta_x^z \\ &+ \frac{n-1}{f^2} f_d f^z \bar{g}_{xy} = 0 \end{aligned}$$

$$(4.6)\quad \begin{aligned} &\frac{n-2}{f}(\nabla_d f_b)\delta_x^z + S_{bd}\delta_x^z + \frac{1}{f^2}\{\bar{S}_x{}^z - f(\triangle f)\delta_x^z \\ &- \frac{1}{f}(\bar{\triangle} f)\delta_x^z - \frac{p-2}{f}\nabla_x f^z + \frac{2(p-2)}{f^2} f_x f^z \\ &- (p-1)\|f_e\|^2\delta_x^z - \frac{p-3}{f^2}\|f_y\|^2\delta_x^z\} g_{bd} = 0 \end{aligned}$$

$$(4.7)\quad (n-1)\{f(\partial_x f^a)\bar{g}_{yz} - f(\partial_y f^a)\bar{g}_{xz} + f^a f_y \bar{g}_{xz} - f^a f_x \bar{g}_{yz}\} = 0$$

and

$$(4.8)\quad \begin{aligned} &\bar{R}_{xyz}{}^{w} \\ &= \frac{1}{m-2}\{\bar{S}_{zy}\delta_x^w + \bar{S}_x{}^w \bar{g}_{zy} - \bar{S}_{zx}\delta_y^w - \bar{S}_y{}^w \bar{g}_{zx}\} \\ &+ \frac{n}{(m-2)f}\{(\nabla_y f_z)\delta_x^w + (\nabla_x f^w)\bar{g}_{yz} - (\nabla_x f_z)\delta_y^w - (\nabla_y f^w)\bar{g}_{xz}\} \\ &- \frac{2n}{(m-2)f^2}\{f_y f_z \delta_x^w + f_x f^w \bar{g}_{yz} - f_x f_z \delta_y^w - f_y f^w \bar{g}_{xz}\} \\ &+ \frac{n-p}{m-2}\|f_e\|^2\{\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w\} \\ &+ \frac{n-p+4}{(m-2)f^2}\|f_w\|^2\{\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w\} \\ &- \frac{2}{m-2} f(\triangle f)\{\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w\} \\ &- \frac{2}{(m-2)f}(\bar{\triangle} f)\{\bar{g}_{yz}\delta_x^w - \bar{g}_{xz}\delta_y^w\}. \end{aligned}$$

Since the scalar curvature tensor vanishies on the conharmonically flat manifold, we obtain from (3.15).

$$
\begin{aligned}
(4.9)\qquad & K + \frac{1}{f^2}\bar{K} - \frac{2p}{f}(\triangle f) - \frac{2(p-1)}{f^3}(\bar{\triangle} f) \\
& - \frac{p(p-1)}{f^2}\|f_e\|^2 - \frac{(p-1)(p-4)}{f^4}\|f_x\|^2 = 0.
\end{aligned}
$$

On the other hand, we can get from (4.6)

$$
\begin{aligned}
(4.10)\qquad & \frac{p(n-2)}{f}\nabla_d f_b + pS_{bd} \\
& + \frac{1}{f^2}\{\bar{K} - pf\triangle f - \frac{2(p-1)}{f}(\bar{\triangle} f) - \frac{(p-1)(p-4)}{f^2}\|f_x\|^2 \\
& - p(p-1)\|f_e\|^2\}g_{bd} = 0.
\end{aligned}
$$

and so

$$
\begin{aligned}
(4.11)\qquad & pK + \frac{n\bar{K}}{f^2} - \frac{2p}{f}(\triangle f) - \frac{2n(p-1)}{f^3}(\bar{\triangle} f) \\
& - \frac{n(p-1)(p-4)}{f^4}\|f_x\|^2 - \frac{np(p-1)}{f^2}\|f_e\|^2 = 0.
\end{aligned}
$$

Thus we have from(4.9) and (4.11)

$$
(4.12)\qquad (n-p)K = \frac{2p(n-1)}{f}\triangle f.
$$

Equivalently, we get

$$
(2n-m)K = \frac{2p(n-1)}{f}\triangle f
$$

from (4.4). Hence we get $\triangle f = 0$ in the case of $n = p$ or $K = 0$. Using this fact and (4.9), we obtain

$$
f\bar{K} - 6(\bar{\triangle} f) - 12f\|f_e\|^2 = 0
$$

if $p = 4$. Therefore if we integrate upper equation on the compact manifold M, then we have the following theorem due to Hopf theorem.

THEOREM 4.1. *Let $M = B \times_f F$ be the compact conharmonically flat twisted product manifold with $K = 0$ and $\bar{K} < 0$. If $n > 1$ and $p = 4$, then M becomes a Riemannian product of B and F. In this case, M is locally Euclidean.*

If we use (4.4), then we obtain

$$S_{cb} = \frac{K}{p} g_{cb} - \frac{n-2}{f} \nabla_c f_b - \frac{\Delta f}{f} g_{cb} \tag{4.13}$$

for $n > 1$, and that

$$S_{cb} = \frac{m-2}{2p(n-1)} K g_{cb} - \frac{n-2}{f} \nabla_c f_b \tag{4.14}$$

due to (4.12) and (4.13). Since 2-dimensional Einstein space is a space of constant curvature, we can state the followings.

THEOREM 4.2. *Let $M = B \times_f F$ be the conharmonically flat twisted product manifold with $n = 2$. If K is constant, then B is a space of constant curvature.*

COROLLARY 4.3. *Let $M = B \times_f F$ be the conharmonically flat twisted product manifold with $n = 2$. If $K = 0$, then B is locally Euclidean.*

Using (4.14), we see that B is Einstein if and only if f is the concircular scalar field that is ,

$$\nabla_c f_b = -\alpha f g_{cb} \tag{4.15}$$

From the equations (4.14) and (4.15), we have

$$\frac{-(n-p)(n-2)}{2p(n-1)} K = (n-2)\alpha.$$

So $-\alpha$ is reduced to

$$-\alpha = \frac{n-p}{2p(n-1)} K \tag{4.16}$$

if $n \neq 2$. Hence (4.4), (4.15), (4.16) and Theorem 2.1 give

THEOREM 4.4. *Let $M = B \times_f F$ be the conharmonically flat twisted product manifold and B a complete Einstein manifold with $n > 2$. Then B is one of the following manifolds :*

(I) The direct product $V \times I$ of an $(n-1)$-dimensional complete Riemannian manifold V with a straight line I if $K = 0$ or $n = p$,
(II) a pseudo-hyperbolic space of zero or negative type if $n(f) = 0$ and $(n-p)K > 0$,
(III) hyperbolic space of curvature α if $n(f) = 1$ and $(n-p)K > 0$,
(IV) a spherical space of curvature α if $(n-p)K < 0$.

REMARK 4.5. Using (4.15) and (4.16), we can easily see that the followings are equivalent: (1) $\alpha = 0$, (2) $\nabla_c f_b = 0$ (3) $K = 0$ or $n = p$.

Using (4.5), we get

$$(n-1)(p-1)(\frac{1}{f}\partial_a f_x - \frac{1}{f^2} f_a f_x) = (n-1)(p-1)\partial_d \partial_y (\log f) = 0.$$

Hence f is a product of a certain function f^* on B and $\bar{f}$ on F if $n > 1$ and $p > 1$. Let F^* be a fibre with the metric $g^*_{xy} = \bar{f}^2 \bar{g}_{xy}$, then we have

THEOREM 4.6. *Let $M = B \times_f F$ be a conharmonically flat twisted product of B and F. If $n > 1$ and $p > 1$, then M is the warped product $B \times_{f^*} F^*$ of B and F^*.*

From (4.12), we see that $\triangle f = 0$ if $n = p$ (> 1). By virtue of $\triangle f = (\triangle f^*)\bar{f}$ and $\triangle f^* = 0$ on B, we get $f = 0$, which is contradiction. Thus we have

THEOREM 4.7. *The conharmonically flat twisted product manifold with $n = p$ (> 1) does not exist.*

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