ROTATION SURFACES WITH 1-TYPE GAUSS MAP

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ABSTRACT. In this paper, we study rotation surfaces in a Euclidean space with pointwise 1-type Gauss map and obtain by the use of the concept of pointwise finite type Gauss map, a characterization theorem for rotation surfaces of constant mean curvature.

1. Introduction

Recently in 2000, and in the framework of the theory of finite type submanifolds (see [2], [3]), the authors of [7] raising the following problem: classify all submanifolds in an $m-$Euclidean space $\mathbb{E}^m$ (or in the Minkowski space $\mathbb{E}^m_1$) satisfying the following equation

\[(1.1) \quad \Delta G = fG,\]

where $\Delta$ in the Laplacian of the induced metric and $G$ the Gauss map of the submanifold, and for some function $f$ on the submanifold.

The authors of [7] have studied ruled surfaces in 3-dimensional Minkowski space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map, and obtain a classification theorem for them. Also, submanifolds in pseudo-Euclidean space with finite type Gauss map are studied (cf [1], [5] among others).

In the paper [6], a characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map in 3-dimensional Euclidean space is obtained.

On the other hand, Chen and Piccini [4] made a general study on submanifolds of Euclidean space with finite type Gauss map and classified the compact surfaces of 1-type Gauss map.

In this paper we use the concept of pointwise 1-type Gauss map introduced in [7] to obtain the following theorem.

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THEOREM 1.1. A rotation surface $M$ in 3–Euclidean space $\mathbb{E}^3$ is pointwise 1-type Gauss map if and only if its mean curvature is a constant.

Throughout this paper, we assume that all surfaces are connected and all objects are at least of class $C^3$.

2. Preliminaries

Let $(x(s), y(s)), s \in I$ be any smooth curve parametrised by the arc length and the domain of definition $I$ is any open interval of the set of real numbers.

We define a surface of revolution $M$ in $\mathbb{E}^3$ by an isometric immersion $X$ defined:

$$X(s, \theta) = (x(s), y(s) \cos(\theta), y(s) \sin(\theta)); \ s \in I, \ 0 \leq \theta \leq 2\pi.$$  

We will assume that $y(s) > 0$.

The first and the second fundamental forms of $M$ are $ds^2 + y^2(s)d\theta^2$ and $(x''(s)y'(s) - x'(s)y''(s))ds^2 + x'(s)y(s)d\theta^2$, respectively. Then one can easily get that its mean curvature $H$ is given by

$$2H = (x''(s)y'(s) - x'(s)y''(s)) + \frac{x'(s)}{y(s)}.$$ (2.1)

The Gauss map $G$ of $M$ is given by

$$G = (y(s), -x'(s)\cos(\theta), -x'(s)\sin(\theta)),$$ (2.2)

which is obtained from the classical formula $G = \frac{X_s \times X_\theta}{|X_s \times X_\theta|}$, where $\times$ denotes the cross-product in $\mathbb{E}^3$.

To obtain the Laplacian $\Delta$ of $M$ we apply the following formula:

$$\Delta = -\frac{1}{\sqrt{|\det(g_{ij})|}} \sum \frac{\partial}{\partial x^i}(\sqrt{|\det(g_{ij})|}g^{ij} \frac{\partial}{\partial x^j}).$$ (2.3)

So by using 2.2 and 2.3, and the first fundamental form given above, one gets by an easy computaton the Laplacian $\Delta G$ of the Gauss map $G$ of $M$:

$$\Delta G = \begin{pmatrix} -y''' - \frac{y'}{y}y'' \\ (x'' + \frac{y'}{y}x'' - \frac{1}{y^2}x') \cos \theta \\ (x'' + \frac{y'}{y}x'' - \frac{1}{y^2}x') \sin \theta \end{pmatrix}.$$ (2.4)
3. Proof of the theorem

Now we consider a rotation surface $M$ given as in preliminaries, that satisfies moreover the equation: $\Delta G = fG$, for some function $f$ in $M$. That is $M$ pointwise 1-type Gauss map; and this is also equivalent to the following condition:

$$\Delta G - \langle \Delta G, G \rangle G = 0$$

where $\langle , \rangle$ denotes the inner product of $\mathbb{E}^3$.

**Step 1.** We first compute the function $f := \langle \Delta G, G \rangle$, by using the relations 2.2 and 2.4, and obtain that

$$f = y'\{(-y'' - y' y'') - x'\{(x''' + y' x'' - \frac{1}{y^2} x')\}.$$ 

It is convenient to make the notations:

$$A = y''' + \frac{y'}{y} y'' \text{ and } B = x''' + \frac{y'}{y} x'' - \frac{1}{y^2} x',$$

then the function $f$ becomes $f = -y' A - x' B$. With these notations we have

$$\Delta G = (-A, B \cos \theta, B \sin \theta).$$

Then the condition in 3.1 becomes

$$\begin{cases}
-A + y'(y'A + x'B) = 0, \\
B - x'(y'A + x'B).
\end{cases}$$

Since the curve $(x(s), y(s)), s \in I$ is parametrised by the arc length, then we have

$$x'^2 + y'^2 = 1.$$ 

And by using this equation, the above condition becomes

$$\begin{cases}
-x^2 A + x'y' B = 0 \\
-x'y'A + y'^2 B = 0.
\end{cases}$$

**Step 2.** We assume that neither $x'$ nor $y'$ is the zero function in a subinterval $J$ of the interval $I$. Then the condition given by 3.3 becomes

$$-x' A + y' B = 0.$$

By using the relation 3.2 and its derivative, the condition above is now:

$$\begin{cases}
(y' x''' - x'y''') + (\frac{x'}{y})' = 0.
\end{cases}$$
Now we take the derivative of the formula 2.1 for mean curvature $H$ to obtain that
\[
(y' x''' - x' y''') = 2H' - \left(\frac{x'}{y}\right)'.
\]
Inserting the left member of this relation in 3.4, we see that $H'$ is zero on $M$, and conclude that the mean curvature is a constant.

**STEP 3.** Now we assume either one the functions $x'$ or $y'$ is the zero function in subinterval $J$ of the interval $I$. If we assume $x'$ is the zero function on $J$ then the function $y'$ is a nonzero constant on $J$ and vice versa. We might also assume the interval $J$ to an open interval. Assume $x'$ is the zero function on an open interval $J$. Then one see easily that the open set $U = \{(s, \theta); s \in I, 0 \leq \theta \leq 2\pi\}$ of $M$ is planar and have zero mean curvature.

Since the manifold $M$ is connected, its mean curvature $H$ cannot jump between the zero value and constant nonzero values. And this proof the one part of the theorem.

**STEP 4.** Conversely assume the mean curvature $H$ be constant. We just have to show how to obtain the condtion 3.3 which is equivalent to fact that $M$ is pointwise 1-type Gauss map.

As we seen above we have:
\[
(y' x''' - x' y''') = 2H' - \left(\frac{x'}{y}\right)'.
\]
And therefore we get
\[
(a): \quad (y' x''' - x' y''') = \left(\frac{x'}{y}\right)'.
\]

On the other hand we have
\[
(b): \quad -x'A + y'B = (y' x''' - x' y''') + \left(\frac{x'}{y}\right)'.
\]
From (a) and (b) one we get the following equation
\[
(c): \quad -x'A + y'B = 0.
\]
To get the two equations condition 3.3, we multiply the equation (c) by $x'$ and by $y'$, respectively. This proves the theorem.

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References


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