Bull. Korean Math. Soc. **42** (2005), No. 1, pp. 53–56

ISOMORPHIC MODULAR GROUP ALGEBRAS OF SEMI-COMPLETE PRIMARY ABELIAN GROUPS

PETER V. DANCHEV

ABSTRACT. Suppose G is a semi-complete abelian p-group and $FG \cong FH$ as commutative unitary F-algebras of characteristic p for any fixed group H. Then, it is shown that, $G \cong H$. This improves a result of the author proved in the Proceedings of the American Math. Society (2002) and also completely solves by an another method a long-standing problem of W. May posed in the same Proceedings (1979).

1. Introduction

In [5], the May's problem for isomorphism of commutative modular group algebras of torsion-complete p-primary abelian groups stated in [12], is completely exhausted, by making use of Direct Factor Theorem showed in [3] (see also cf. [2, 4, 6, 7, 8, 9]). An analogous problem in the sense of Beers-Richman-Walker [1] was also examined in [7] for the case of group algebras of direct sums of torsion-complete abelian p-groups, but over the finite modular p-element field. Our aim here is to study a more restricted class of abelian p-torsion groups, called semi-complete [11], but over an arbitrary field in characteristic p. A p-torsion abelian group is said to be semi-complete if it is the direct product of a torsion-complete group and a direct sum of cyclic groups. The technique that we will use is different to this of [5]. It is based on strong group results for semi-complete p-groups, established by Kolettis [11], and our Direct Factor Theorem pertaining to the group $V(FG)$ of all normalized units in the group algebra FG, obtained in [3]. For the sake of completeness and for the convenience of the reader, we shall formulate them. We

Received June 18, 2003.
2000 Mathematics Subject Classification: Primary 20C07; Secondary 16U60, 16S34, 20K10, 20K15.
Key words and phrases: semi-complete p-groups, direct factors, isomorphisms.

start with a restating of Theorem 4 in [11] in an equivalent form in the following manner:

THEOREM. [11] *Suppose $G = T \times C$ and $H = T' \times C'$, where T, T' are torsion-complete abelian p-groups, and C, C' are direct sums of p-cyclics. Then $G \cong H$ if and only if G and H have equal Ulm-Kaplansky invariants and there is a natural number n such that $T^{p^n} \cong T'^{p^n}$ and $C^{p^n} \cong C'^{p^n}$.*

We continue with Direct Factor Theorem deduced in [3]. Let G be an abelian p-group with countable length and F be a perfect field of characteristic p. Then $V(FG)/G$, the group of all normed p-elements in the group algebra FG modulo G, is totally projective. From this fact it follows directly that the following consequence holds (here $\overline{F}$ is the algebraic closure of a field F in characteristic p).

COROLLARY. *The abelian p-group $V(FG)$ is semi-complete if and only if the abelian p-group G is semi-complete.*

Proof. If $V(FG)$ is semi-complete, then G is separable, and so by what we have claimed in the previous Theorem, $V(\overline{F}G)/G$ is separable totally projective, hence a direct sum of cyclic groups, whence so is its subgroup $V(FG)/G$. Thus $V(FG) \cong G \times V(FG)/G$ and the semi-completeness of G is guaranteed from [10].

Oppositely, as above, G separable implies $V(FG) \cong G \times V(FG)/G$, where $V(FG)/G$ is a direct sum of cyclics, whence the definition for semi-complete p-groups [11] yields the desired claim. The proof is over.□

REMARK. The foregoing formulated Direct Factor Theorem with the restriction on the power of G being $\leq \aleph_1$ was proved by May in [13].

We begin with the statement and argumentation of the central affirmation selected in the following section.

2. Main assertion

Well, we are in position to proceed by proving the main attainment motivating the writing of the present article, namely:

THEOREM.(ISOMORPHISM) *Suppose G is a semi-complete abelian p-group and $RH \cong RG$ as R-algebras for any group H and over a commutative unitary ring R with prime characteristic p. Then $H \cong G$.*

Proof. It is well-known that $RH \cong RG$ forces $FH \cong FG$ for some perfect field F in characteristic p, and thus the Ulm-Kaplansky invariants of G and H can be recaptured from the group algebras, so they are equal (see e.g. [12]). Moreover, with no harm of generality, we may presume that $FH = FG$. Therefore, $V(FH) = V(FG)$. On the other hand, because length $(G) \leq \omega$ and length $(H) =$ length(G), it holds that length $(H) \leq \omega$. Consequently, exploiting the Direct Factor Theorem and its Corollary, we can write $G \times V(FG)/G = H \times V(FH)/H$, where $G = T \times C$ and $H = T' \times C'$ are both semi-complete with standard direct decompositions (see for instance [11]). Furthermore, $T \times C \times V(FG)/G = T' \times C' \times V(FH)/H$. Then T torsion-complete implies its projection into the direct sum of cyclics $C' \times V(FH)/H$ would be bounded. Thus there is a natural number m with the property $T^{p^m} \subseteq T'$, i.e., $T^{p^m} \subseteq T'^{p^m}$. By symmetry, we may assume $T^{p^m} = T'^{p^m}$. Consequently, $FG^{p^m} = FH^{p^m}$ with $FT^{p^m} = FT'^{p^m}$ do imply $FC^{p^m} \cong F(G^{p^m}/T^{p^m}) \cong F(H^{p^m}/T'^{p^m}) \cong FC'^{p^m}$, hence a result due to May ([12]) assures that $C^{p^m} \cong C'^{p^m}$. Now, the above listed Kolettis Theorem leads us to $G \cong H$. The proof is fulfilled. □

The next immediate consequence completely settles an old problem of W. May raised in [12] on 1979. It was argued by us in [5] but utilizing another idea for proof.

COROLLARY. [5] *Let $RG \cong RH$ as R-algebras over a commutative ring R with identity in prime characteristic p such that G is a torsion-complete p-group and H is an arbitrary group. Then $G \cong H$.*

Proof. Each torsion-complete group is semi-complete, hence the central Theorem is applicable to finish the conclusions. □

REMARKS. There are two misprints in [3] which we point out and eliminate. First of all, the sign $=$ on p.9, line $5(-)$ of Theorem 14 in [3] should be replaced by $\subset$. Notice that we have done such a choice since each subgroup of a σ-summable p-group with the same length is σ-summable, too. Secondly, the latter symbol $g_{in}^{\varepsilon_n}$ on p.12, line $5(+)$ of Theorem 17 in [3] must be removed.

ACKNOWLEDGMENTS. The author would like to express his indebtedness to the referee and to the Editor, Professor Goansu Kim, for the professional work.

References

[1] D. Beers, F. Richman, and E. Walker, *Group algebras of abelian groups*, Rend. Sem. Mat. Univ. Padova **69** (1983), 41–50.

[2] P. V. Danchev, *Commutative group algebras of cardinality* $\aleph_1$, Southeast Asian Bull. Math. **25** (2002), no. 4, 589–598.

[3] ———, *Commutative group algebras of direct sums of σ-summable abelian p-groups*, Math. J. Okayama Univ. **45** (2003), 1–15.

[4] ———, *Invariances in commutative and noncommutative group algebras*, Compt. rend. Acad. bulg. Sci. **54** (2001), no. 4, 5–8.

[5] ———, *Isomorphism of commutative group algebras of closed p-groups and p-local algebraically compact groups*, Proc. Amer. Math. Soc. **130** (2002), no. 7, 1937–1941.

[6] ———, *Isomorphism of commutative group algebras of mixed splitting groups*, Compt. rend. Acad. bulg. Sci. **51** (1998), no. 1-2, 13–16.

[7] ———, *Isomorphism of modular group algebras of direct sums of torsion-complete abelian p-groups*, Rend. Sem. Mat. Univ. Padova **101** (1999), 51–58.

[8] ———, *The splitting problem and the direct factor problem in modular abelian group algebras*, Math. Balkanica (N.S.) **15** (2000), no. 3-4, 217–226.

[9] ———, *Units in abelian group rings of prime characteristic*, Compt. rend. Acad. bulg. Sci. **48** (1995), no. 8, 5–8.

[10] J. Irwin, F. Richman, and E. Walker, *Countable direct sums of torsion complete groups*, Proc. Amer. Math. Soc. **17** (1966), no. 4, 763–766.

[11] G. Kolettis, Jr., *Semi-complete primary abelian groups*, Proc. Amer. Math. Soc. **11** (1960), no. 2, 200–205.

[12] W. May, *Modular group algebras of totally projective p-primary groups*, Proc. Amer. Math. Soc. **76** (1979), no. 1, 31–34.

[13] ———, *The direct factor problem for modular abelian group algebras*, Contemp. Math. **93** (1989), 303–308.

13 General Kutuzov Street, block 7, floor 2, flat 4, 4003 Plovdiv, Bulgaria

E-mail: pvdanchev@yahoo.com