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ON THE HYERS-ULAM STABILITY OF A GENERALIZED QUADRATIC AND ADDITIVE FUNCTIONAL EQUATION

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Abstract. In this paper, we obtain the general solution of a generalized quadratic and additive type functional equation

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y)$$

for any integer a with $a \neq -1,0,1$ in the class of functions between real vector spaces and investigate the generalized Hyers-Ulam stability problem for the equation.

1. introduction

In 1940, S. M. Ulam [18] raised a question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is approximately a homomorphism, then there exists a true homomorphism near it with small error.

The case of approximately additive functions was solved by D. H. Hyers [7] and generalized by Th. M. Rassias [17]. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [2, 3, 8, 9, 10, 11, 12, 16]. The terminology generalized Hyers-Ulam stability originates from these historical backgrounds.

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The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is related to a symmetric biadditive function ([1, 15]). It is well known that a function f is a solution of (1.1) if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ for all x (see [1]). The biadditive function B is given by

$$B(x,y)=\frac{1}{4}(f(x+y)-f(x-y)). \tag{1.2}$$

Thus we call the equation (1.1) *quadratic functional equation* and every solution of the quadratic equation (1.1) is said to be a *quadratic function.*

A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors ([5, 14] and references there in). Further, Jun and Lee [13] proved the generalized Hyers-Ulam stability of the pexiderized quadratic equation (1.1).

In this paper, we investigate the following quadratic functional equations;

$$f(x+2y)+2f(x-y)=f(x-2y)+2f(x+y), \tag{1.3}$$

$$f(x+ay)+af(x-y)=f(x-ay)+af(x+y) \tag{1.4}$$

for any fixed integer a with $a \neq -1,0,1$. A function f between real vector spaces, defined by $f(x)=B(x,x)+A(x)+c$ for all $x \in X$, where B is symmetric biadditive, A is additive and c in Y, satisfies the functional equation (1.4), which is the main subject of this paper. In fact, we find out the general solution of the generalized quadratic functional equation (1.4) in the class of functions between real vector spaces in Section 2 and we establish the generalized Hyers-Ulam stability problem for the equation (1.4) in the sense of Hyers, Ulam and Rassias in Section 3.

2. General solution of (1.4)

Let E_1 and E_2 be real vector spaces throughout this section. We here present the general solution of (1.4).

THEOREM 2.1. *(i) A function $f: E_1 \to E_2$ satisfies the functional equation (1.3) if and only if (ii) $f: E_1 \to E_2$ satisfies the functional equation (1.4).*

Furthermore, every solution of functional equations (1.3) and (1.4) has the form $f(x) = B(x,x) + A(x) + f(0)$ *for all* $x \in E_1$*, where* $B : E_1 \times E_1 \to E_2$ *is symmetric biadditive, and* $A : E_1 \to E_2$ *is additive.*

Proof. Let $f : E_1 \to E_2$ satisfy the functional equation (1.3). To use an induction argument we may assume that (1.4) is true for all n with $1 < n \leq N$. Putting x by $x + y$ and a by N in (1.4), we obtain

$$f(x + (N+1)y) + Nf(x) = f(x - (N-1)y) + Nf(x + 2y). \tag{2.1}$$

Replacing y by $-y$, we get

$$f(x - (N+1)y) + Nf(x) = f(x + (N-1)y) + Nf(x - 2y). \tag{2.2}$$

Subtracting (2.2) from (2.1) and using an inductive assumption for $N-1$ together with (1.3), we lead to

$$\begin{aligned} & f(x + (N+1)y) - f(x - (N+1)y) \\ &= (N-1)f(x-y) - (N-1)f(x+y) \\ &\quad + N[f(x+2y) - f(x-2y)] \\ &= (N+1)f(x+y) - (N+1)f(x-y), \end{aligned} \tag{2.3}$$

which proves the validity of (1.4) for $N+1$.

For a negative integer $n < -1$, replacing n by $-n > 1$ one can easily prove the validity of (1.4).

Therefore, (1.3) implies (1.4) for any integer a with $a \neq -1, 0, 1$.

Conversely, let $f : E_1 \to E_2$ satisfy the functional equation (1.4). If we put $g(x) = f(x) - f(0)$, we obtain that g is also a solution of (1.4) and $g(0) = 0$. So we may assume without loss of generality that f is a solution of (1.4) and $f(0) = 0$. Let $f_e(x) = \frac{f(x)+f(-x)}{2}$, $f_o(x) = \frac{f(x)-f(-x)}{2}$ for all $x \in E_1$. Then $f_e(0) = 0 = f_o(0)$ and $f(x) = f_e(x) + f_o(x)$, f_e is even and f_o is odd. Since f is a solution of (1.4), f_e and f_o also satisfy the equation (1.4).

First, we may assume that f is a solution of the functional equation (1.4) and f is odd, $f(0) = 0$. Replacing x by y in (1.4), we have

$$f(ax + y) + f(ax - y) = af(x + y) + af(x - y) \tag{2.4}$$

for all $x, y \in E_1$. Letting $y = ax$, $y = x$ in (2.4), separately, we obtain that

$$\begin{aligned} f(2ax) &= af((a+1)x) - af((a-1)x) \\ f((a+1)x) + f((a-1)x) &= af(2x), \end{aligned} \tag{2.5}$$

for all $x \in E_1$. Putting y by $x + ay$ in (2.4), we obtain

(2.6) $f(a(x+y)+x) + f(a(x-y)-x) = af(2x+ay) + af(-ay)$.

Interchange y and $-y$ in (2.6) to get the relation

(2.7) $f(a(x-y)+x) + f(a(x+y)-x) = af(2x-ay) + af(ay)$.

Adding (2.6) to (2.7), by use of (1.4) we lead to

$$f(2x+y) + f(2x-y) = f(2x+ay) + f(2x-ay) \tag{2.8}$$

for all $x, y \in E_1$. Substitution $\frac{ax}{2}$ for x in (2.8) together with (2.4) gives the relation

$$af(x+y) + af(x-y) = f(ax+ay) + f(ax-ay). \tag{2.9}$$

Taking y as 0 in (2.9) yields $af(x) = f(ax)$ and whence by (2.5)

(2.10) $(a+1)f(2x) = 2f((a+1)x), \ (a-1)f(2x) = 2f((a-1)x)$.

On the other hand, replacing y by ay in (2.4) we obtain that

$$f(x+y) + f(x-y) = f(x+ay) + f(x-ay). \tag{2.11}$$

Associating (2.11) from (1.4), we get

$$2f(x+ay) = (a+1)f(x+y) - (a-1)f(x-y). \tag{2.12}$$

Interchange x and y in (2.13) to have

$$2f(ax+y) = (a+1)f(x+y) + (a-1)f(x-y). \tag{2.13}$$

Combining (2.12) with (2.13), we get

$$f(ax+y) + f(x+ay) = (a+1)f(x+y), \tag{2.14}$$

which yields by virtue of (2.10)

$$\begin{aligned} &f(2ax+2y) + f(2x+2ay) \\ &= (a+1)f(2x+2y) = 2f((a+1)(x+y)). \end{aligned} \tag{2.15}$$

Replacing $2ax + 2y$ and $2x + 2ay$ by u and v, respectively, we see that the last equation implies that f satisfies the Jensen equation, and so f is additive since $f(0) = 0$.

Next we assume that f is a solution of the functional equation (1.4) and f is even, $f(0) = 0$. Replacing x by y in (1.4), we have

$$f(ax+y) + af(x-y) = f(ax-y) + af(x+y) \tag{2.16}$$

for all $x, y \in E_1$. Putting $y = ax$, $y = x$ in (2.16), respectively, one has that

$$\begin{aligned} &f(2ax) + af((a-1)x) = af((a+1)x), \\ &f((a+1)x) = f((a-1)x) + af(2x), \end{aligned} \tag{2.17}$$

from which we lead to $f(ax) = a^2 f(x)$ for all $x \in E_1$. Replacing y by $ax + y$ in (2.16), and then substituting $-y$ for y in the resulting relation, we have

$$\begin{aligned} f(2ax+y) + af(ax-(x-y)) &= f(y) + af(ax+(x+y)), \\ f(2ax-y) + af(ax-(x+y)) &= f(y) + af(ax+(x-y)). \end{aligned} \tag{2.18}$$

Adding two relations of (2.18) side by side, we get that

$$\begin{aligned} &f(2ax+y) + f(2ax-y) + 2a^2 f(y) \\ &= a^2 f(2x+y) + a^2 f(2x-y) + 2f(y). \end{aligned} \tag{2.19}$$

In turn, replacing y by $x + ay$ in (2.16), and then putting $-y$ instead of y in the resulting relation, we have

$$\begin{aligned} f(a(x+y)+x) + af(ay) &= f(a(x-y)-x) + af(2x+ay), \\ f(a(x-y)+x) + af(ay) &= f(a(x+y)-x) + af(2x-ay). \end{aligned} \tag{2.20}$$

Adding two relations of (2.20) side by side and utilizing (2.16), we get that

$$\begin{aligned} &f(2x+y) + f(2x-y) + 2(a^2-1)f(y) \\ &= f(2x+ay) + f(2x-ay). \end{aligned} \tag{2.21}$$

Replacing y, x by $2x, \frac{y}{2}$ in (2.21), respectively, one obtains that

$$\begin{aligned} &f(2x+y) + f(2x-y) + 2(a^2-1)f(2x) \\ &= f(2ax+y) + f(2ax-y). \end{aligned} \tag{2.22}$$

Combining (2.19) with (2.22), we obtain that

$$f(2x+y) + f(2x-y) = 2f(2x) + 2f(y), \tag{2.23}$$

which implies that f is quadratic.

That is, if $f : E_1 \to E_2$ satisfies the functional equation (1.4), then $f(x) = f_e(x) + f_o(x) = B(x,x) + A(x)$ for all $x \in E_1$, where B, A are functions stated in the theorem. Since we regard $f(x)$ as $f(x) - f(0)$, we get $f(x) = B(x,x) + A(x) + f(0)$ for all $x \in E_1$ and thus f satisfies the functional equation (1.3). The proof is complete. □

3. Stability of (1.4)

Throughout this section X and Y will be a real vector space and a real Banach space, respectively, unless we give any specific reference. Given $f : X \to Y$, we set

$$Df(x,y) = f(x+ay) + af(x-y) - f(x-ay) - af(x+y)$$

for all $x, y \in X$ and for any fixed integer a with $a \neq -1, 0, 1$.

Let $\varphi : X \times X \to [0, \infty)$ be a mapping satisfying one of the conditions $(\mathcal{A})$, $(\mathcal{B})$ and one of the conditions $(\mathcal{C})$, $(\mathcal{D})$:

$$\Phi_1(x,y) := \sum_{k=0}^{\infty} \frac{1}{a^{2k}} \varphi(a^k x, a^k y) < \infty \qquad (\mathcal{A})$$

$$\Phi_2(x,y) := \sum_{k=1}^{\infty} a^{2k} \varphi(\frac{x}{a^k}, \frac{y}{a^k}) < \infty \qquad (\mathcal{B})$$

$$\Psi_1(x,y) := \sum_{k=0}^{\infty} \frac{1}{|a|^k} \varphi(a^k x, a^k y) < \infty \qquad (\mathcal{C})$$

$$\Psi_2(x,y) := \sum_{k=1}^{\infty} |a|^k \varphi(\frac{x}{a^k}, \frac{y}{a^k}) < \infty \qquad (\mathcal{D})$$

for all $x, y \in X$.

Clearly the condition $(\mathcal{C})$ implies $(\mathcal{A})$ and $(\mathcal{B})$ implies $(\mathcal{D})$. One of the conditions $(\mathcal{A})$, $(\mathcal{B})$ will be needed to derive a quadratic function and one of the conditions $(\mathcal{C})$, $(\mathcal{D})$ will be required to derive an additive function in the following theorem.

THEOREM 3.1. *Assume that a function $f : X \to Y$ satisfies*

$$||Df(x,y))|| \leq \varphi(x,y) \tag{3.1}$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ satisfying (1.4) such that

$$\begin{aligned} ||f(x) - Q(x) - A(x) - f(0)|| \leq & \frac{1}{2|a|}\Big[\Phi_i(\frac{x}{2}, \frac{x}{2}) + \Phi_i(-\frac{x}{2}, -\frac{x}{2})\Big] \\ & + \frac{1}{2a^2}\Big[\Phi_i(\frac{ax}{2}, \frac{x}{2}) + \Phi_i(-\frac{ax}{2}, -\frac{x}{2})\Big] \\ & + \frac{1}{4|a|}\Big[\Psi_j(0,x) + \Psi_j(0,-x)\Big], \end{aligned}$$

$$\begin{aligned} ||\frac{f(x)+f(-x)}{2} - Q(x) - f(0)|| \leq & \frac{1}{2|a|}\Big[\Phi_i(\frac{x}{2}, \frac{x}{2}) + \Phi_i(-\frac{x}{2}, -\frac{x}{2})\Big] \\ & + \frac{1}{2a^2}\Big[\Phi_i(\frac{ax}{2}, \frac{x}{2}) + \Phi_i(-\frac{ax}{2}, -\frac{x}{2})\Big], \end{aligned}$$

and

$$||\frac{f(x)-f(-x)}{2} - A(x)|| \leq \frac{1}{4|a|}\Big[\Psi_j(0,x) + \Psi_j(0,-x)\Big]$$

for all $x \in X$ *and for* $i = 1$ *or* 2, $j = 1$ *or* 2.

The functions Q *and* A *are given by*

$$\begin{cases} Q(x) = \lim_{n\to\infty} \frac{f(a^n x)+f(-a^n x)}{2\cdot a^{2n}} & \text{if } (\mathcal{A}) \text{ holds} \\ Q(x) = \lim_{n\to\infty} a^{2n} \frac{f(\frac{x}{a^n})+f(-\frac{x}{a^n})-2f(0)}{2} & \text{if } (\mathcal{B}) \text{ holds} \\ A(x) = \lim_{n\to\infty} \frac{f(a^n x)-f(-a^n x)}{2\cdot a^n} & \text{if } (\mathcal{C}) \text{ holds} \\ A(x) = \lim_{n\to\infty} a^n \frac{f(\frac{x}{a^n})-f(-\frac{x}{a^n})}{2} & \text{if } (\mathcal{D}) \text{ holds} \end{cases}$$

for all $x \in X$

Proof. Let $f_1 : X \to Y$ be a function defined by $f_1(x) := (1/2)[f(x)+f(-x)] - f(0)$ for all $x \in X$. Then $f_1(0) = 0$, $f_1(x) = f_1(-x)$, and

$$\begin{aligned} &||Df_1(x,y)|| \\ &= ||f_1(x+ay) + af_1(x-y) - f_1(x-ay) - af_1(x+y)|| \\ &\leq (1/2)[\varphi(x,y) + \varphi(-x,-y)] \end{aligned} \tag{3.2}$$

for all $x, y \in X$. Putting $y = x$ in (3.2) yields

$$||f_1((a+1)x) - f_1((a-1)x) - af_1(2x)|| \leq \frac{[\varphi(x,x) + \varphi(-x,-x)]}{2} \tag{3.3}$$

for all $x \in X$. Replacing x by ay in (3.2) yields

$$||f_1(2ax) + af_1((a-1)x) - af_1((a+1)x)|| \leq \frac{[\varphi(ax,x) + \varphi(-ax,-x)]}{2} \tag{3.4}$$

for all $x \in X$. Multiplying $|a|$ on both sides of (3.3) and adding it to (3.4), we obtain that

$$\begin{aligned} &||f_1(2ax) - a^2 f_1(2x)|| \\ &\leq \frac{|a|[\varphi(x,x) + \varphi(-x,-x)]}{2} + \frac{[\varphi(ax,x) + \varphi(-ax,-x)]}{2} \end{aligned} \tag{3.5}$$

for all $x \in X$.

Case 1. Assume that φ satisfies the condition $(\mathcal{A})$.

Letting $\frac{x}{2}$ for x and dividing both sides of (3.5) by a^2, we have

$$\begin{aligned} \left\| \frac{f_1(ax)}{a^2} - f_1(x) \right\| \leq &\frac{1}{2|a|}\left[\varphi(\frac{x}{2},\frac{x}{2}) + \varphi(-\frac{x}{2},-\frac{x}{2})\right] \\ &+ \frac{1}{2|a|^2}\left[\varphi(\frac{ax}{2},\frac{x}{2}) + \varphi(-\frac{ax}{2},-\frac{x}{2})\right] \end{aligned} \tag{3.6}$$

for all $x \in X$. Replacing x by $a^{n-1}x$ and dividing by $|a|^{2n-2}$ in (3.6) we obtain

$$\begin{aligned}
&\left\|\frac{f_1(a^n x)}{a^{2n}} - \frac{f_1(a^{n-1}x)}{a^{2n-2}}\right\| \\
(3.7)\qquad &\leq \frac{1}{2\cdot|a|^{2n-1}}\Big[\varphi(\frac{a^{n-1}x}{2}, \frac{a^{n-1}x}{2}) + \varphi(-\frac{a^{n-1}x}{2}, -\frac{a^{n-1}x}{2})\Big] \\
&+ \frac{1}{2\cdot|a|^{2n}}\Big[\varphi(\frac{a^n x}{2}, \frac{a^{n-1}x}{2}) + \varphi(-\frac{a^n x}{2}, -\frac{a^{n-1}x}{2})\Big]
\end{aligned}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

An induction argument and triangle inequality imply easily that

$$\begin{aligned}
&\left\|\frac{f_1(a^n x)}{a^{2n}} - f_1(x)\right\| \\
(3.8)\qquad &\leq \frac{1}{2|a|}\sum_{i=0}^{n-1}\frac{1}{|a|^{2i}}\Big[\varphi(\frac{a^i x}{2}, \frac{a^i x}{2}) + \varphi(-\frac{a^i x}{2}, -\frac{a^i x}{2})\Big] \\
&+ \frac{1}{2|a|^2}\sum_{i=0}^{n-1}\frac{1}{|a|^{2i}}\Big[\varphi(\frac{a^{i+1}x}{2}, \frac{a^i x}{2}) + \varphi(-\frac{a^{i+1}x}{2}, -\frac{a^i x}{2})\Big]
\end{aligned}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

Thus it follows from (3.8) that

(3.9)

$$\begin{aligned}
&\left\|\frac{f_1(a^n x)}{a^{2n}} - \frac{f_1(a^m x)}{a^{2m}}\right\| \\
&\leq \frac{1}{a^{2m}}\left\|\frac{f_1(a^{n-m}a^m x)}{a^{2n-2m}} - f_1(a^m x)\right\| \\
&\leq \frac{1}{2|a|}\sum_{i=0}^{n-m-1}\frac{1}{|a|^{2m+2i}}\Big[\varphi(\frac{a^{m+i}x}{2}, \frac{a^{m+i}x}{2}) + \varphi(-\frac{a^{m+i}x}{2}, -\frac{a^{m+i}x}{2})\Big] \\
&+ \frac{1}{2|a|^2}\sum_{i=0}^{n-m-1}\frac{1}{|a|^{2m+2i}}\Big[\varphi(\frac{a^{m+i+1}x}{2}, \frac{a^{m+i}x}{2}) \\
&+ \varphi(-\frac{a^{m+i+1}x}{2}, -\frac{a^{m+i}x}{2})\Big]
\end{aligned}$$

for all $x \in X$ and for all $n, m \in \mathbb{N}$ with $n > m$. Since the right hand side of (3.9) tends to zero as $m \to \infty$, $\{\frac{f_1(a^n x)}{a^{2n}}\}$ is a Cauchy sequence for all $x \in X$ and thus converges by the completeness of Y. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} \frac{f_1(a^n x)}{a^{2n}}, \qquad x \in X.$$

Note that $Q(0) = 0$, $Q(-x) = Q(x)$ for all $x \in X$.

Replacing x, y in (3.2) by $a^n x, a^n y$ and dividing both sides by a^{2n}, and after then taking the limit in the resulting inequality, we have

$$Q(x + ay) + aQ(x - y) - Q(x - ay) - aQ(x + y) = 0. \tag{3.10}$$

Since Q is even and $Q(a^k x) = a^{2k} Q(x)$ for all $k \in \mathbb{N}$, the function Q is quadratic by Theorem 2.1.

Taking the limit in (3.8) as $n \to \infty$, we obtain

$$\begin{aligned} &\left\|\frac{f(x) + f(-x)}{2} - f(0) - Q(x)\right\| \\ &\le \frac{1}{2|a|}\left[\Phi_1(\frac{x}{2}, \frac{x}{2}) + \Phi_1(-\frac{x}{2}, -\frac{x}{2})\right] \\ &+ \frac{1}{2a^2}\left[\Phi_1(\frac{ax}{2}, \frac{x}{2}) + \Phi_1(-\frac{ax}{2}, -\frac{x}{2})\right] \end{aligned} \tag{3.11}$$

for all $x \in X$.

To prove the uniqueness, let Q' be another quadratic function satisfying (3.11). Then $Q'(0) = 0$, $Q'(a^n x) = a^{2n} Q'(x)$, and $Q'(-x) = Q'(x)$ for all $x \in X$. Thus we have

$$\begin{aligned} &||Q(x) - Q'(x)|| \\ &\le \frac{1}{a^{2n}}\Big\{||Q(a^n x) - f_1(a^n x)|| + ||f_1(a^n x) - Q'(a^n x)||\Big\} \\ &\le \frac{1}{|a|^{2n+1}}\left[\Phi_1(\frac{a^n x}{2}, \frac{a^n x}{2}) + \Phi_1(-\frac{a^n x}{2}, -\frac{a^n x}{2})\right] \\ &+ \frac{1}{a^{2n+2}}\left[\Phi_1(\frac{a^{n+1} x}{2}, \frac{a^n x}{2}) + \Phi_1(-\frac{a^{n+1} x}{2}, -\frac{a^n x}{2})\right]. \end{aligned}$$

Taking the limit as $n \to \infty$, we conclude that $Q(x) = Q'(x)$ for all $x \in X$.

Case 2. Assume that φ satisfies the condition $(\mathcal{B})$ (and hence implies $(\mathcal{D})$). The proof is analogous to that of Case 1. In fact, replacing x by

$\frac{x}{2a}$ in (3.5) we get

$$\begin{aligned} &\|f_1(x) - a^2 f_1(\frac{x}{a})\| \\ (3.12) \qquad &\leq \frac{|a|[\varphi(\frac{x}{2a}, \frac{x}{2a}) + \varphi(-\frac{x}{2a}, -\frac{x}{2a})]}{2} \\ &+ \frac{[\varphi(\frac{x}{2}, \frac{x}{2a}) + \varphi(-\frac{x}{2}, -\frac{x}{2a})]}{2} \end{aligned}$$

for all $x \in X$.

An induction argument implies by (3.12) that

$$\begin{aligned} &\left\|f_1(x) - a^{2n} f_1(\frac{x}{a^n})\right\| \\ (3.13) \qquad &\leq \frac{1}{2|a|}\sum_{i=1}^{n} |a|^{2i}\Big[\varphi(\frac{x}{2a^i}, \frac{x}{2a^i}) + \varphi(-\frac{x}{2a^i}, -\frac{x}{2a^i})\Big] \\ &+ \frac{1}{2|a|^2}\sum_{i=1}^{n} |a|^{2i}\Big[\varphi(\frac{x}{2a^{i-1}}, \frac{x}{2a^i}) + \varphi(-\frac{x}{2a^{i-1}}, -\frac{x}{2a^i})\Big] \end{aligned}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

Similarly as that of Case 1 it follows from (3.13) that $\{a^{2n} f_1(\frac{x}{a^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $Q : X \to Y$ by

$$Q(x) = \lim_{n\to\infty} a^{2n} f_1(\frac{x}{a^n}), \ x \in X.$$

Note that $Q(0) = 0$, $Q(-x) = Q(x)$ for all $x \in X$. By (3.2) we have

$$(3.14) \qquad Q(x + ay) + aQ(x - y) - Q(x - ay) - aQ(x + y) = 0$$

for all $x, y \in X$ and thus Q is quadratic by Theorem 2.1.

Taking the limit in (3.13) as $n \to \infty$, we obtain

(3.15)

$$\begin{aligned} \left\|\frac{f(x) + f(-x)}{2} - f(0) - Q(x)\right\| &\leq \frac{1}{2|a|}\Big[\Phi_2(\frac{x}{2}, \frac{x}{2}) + \Phi_2(-\frac{x}{2}, -\frac{x}{2})\Big] \\ &+ \frac{1}{2a^2}\Big[\Phi_2(\frac{ax}{2}, \frac{x}{2}) + \Phi_2(-\frac{ax}{2}, -\frac{x}{2})\Big] \end{aligned}$$

for all $x \in X$.

Using the similar argument to that of Case 1, we see easily the uniqueness of Q satisfying (3.15).

Now let $f_2 : X \to Y$ be a function defined by $f_2(x) := (1/2)[f(x) - f(-x)]$ for all $x \in X$. Then $f_2(0) = 0$, $f_2(-x) = -f_2(x)$, and the relation (3.1) gives rise to

(3.16)
$$\begin{aligned}||Df_2(x,y)|| &= ||f_2(x+ay)+af_2(x-y)-f_2(x-ay)-af_2(x+y)|| \\ &\le \frac{1}{2}[\varphi(x,y)+\varphi(-x,-y)]\end{aligned}$$

for all $x, y \in X$. Putting $x = 0$ in (3.16) yields

$$||f_2(ay)-af_2(y)|| \le \frac{1}{4}[\varphi(0,y)+\varphi(0,-y)] \tag{3.17}$$

for all $y \in X$.

Case 3. Assume that φ satisfies the condition $(\mathcal{C})$ (and hence implies $(\mathcal{A})$).

Dividing the last inequality by $|a|$ we have

$$||\frac{f_2(ax)}{a} - f_2(x)|| \le \frac{1}{4|a|}[\varphi(0,x)+\varphi(0,-x)] \tag{3.18}$$

for all $x \in X$.

It follows by an induction argument that

$$\left\|\frac{f_2(a^n x)}{a^n} - f_2(x)\right\| \le \frac{1}{4|a|}\sum_{i=0}^{n-1}\frac{[\varphi(0,a^i x)+\varphi(0,-a^i x)]}{|a|^i} \tag{3.19}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

By the similar argument to that of Case 1, we see that $\{\frac{f_2(a^n x)}{a^n}\}$ is a Cauchy sequence for all $x \in X$ and thus converges in Y. Therefore we can define a function $A : X \to Y$ by

$$A(x) = \lim_{n\to\infty}\frac{f_2(a^n x)}{a^n}, \qquad x \in X.$$

Note that $A(0) = 0$, $A(-x) = -A(x)$ for all $x \in X$.

By (3.16) we get

$$A(x+ay)+aA(x-y)-A(x-ay)-aA(x+y) = 0$$

for all $x, y \in X$ and thus A is additive by Theorem 2.1.

Taking the limit in (3.19) as $n \to \infty$, we obtain

$$||f_2(x)-A(x)|| \le \frac{1}{4|a|}[\Psi_1(0,x)+\Psi_1(0,-x)] \tag{3.20}$$

for all $x \in X$.

If A' is another additive function satisfying the inequality (3.20), then $A'(0) = 0$, $A'(-x) = -A'(x)$ and $A'(a^n x) = a^n A'(x)$ for all $x \in X$. Thus one obtains that by (3.20)

$$\begin{aligned} ||A(x) - A'(x)|| &\le \frac{1}{|a|^n}\Big\{||A(a^n x) - f_2(a^n x)|| + ||f_2(a^n x) - A'(a^n x)||\Big\} \\ &\le \frac{\Psi_1(0, a^n x) + \Psi_1(0, -a^n x)}{2|a|^{n+1}}. \end{aligned}$$

Taking the limit as $n \to \infty$, we conclude that $A(x) = A'(x)$ for all $x \in X$.

Case 4. Assume that φ satisfies the condition $(\mathcal{D})$. The proof is similar to that of Case 3.

Replacing y by $\frac{x}{a}$ in (3.17) we get

$$||f_2(x) - af_2(\frac{x}{a})|| \le \frac{1}{4}[\varphi(0, \frac{x}{a}) + \varphi(0, -\frac{x}{a})] \tag{3.21}$$

for all $x \in X$.

An induction argument implies that

$$||f_2(x) - a^n f_2(\frac{x}{a^n})|| \le \frac{1}{4|a|}\sum_{i=1}^{n} |a|^i \Big[\varphi(0, \frac{x}{a^i}) + \varphi(0, -\frac{x}{a^i})\Big] \tag{3.22}$$

holds for all $x \in X$ and for all $n \in \mathbb{N}$.

Using the inequality (3.22), we see easily that $\{a^n f_2(\frac{x}{a^n})\}$ is a Cauchy sequence for all $x \in X$ and thus converges. Therefore we can define a function $A : X \to Y$ by

$$A(x) = \lim_{n\to\infty} a^n f_2(\frac{x}{a^n}), \qquad x \in X.$$

Note that $A(0) = 0$, $A(-x) = -A(x)$ for all $x \in X$ and thus A is additive by Theorem 2.1.

Taking the limit in (3.22) as $n \to \infty$, we obtain

$$||f_2(x) - A(x)|| \le \frac{1}{4|a|}[\Psi_2(0, x) + \Psi_2(0, -x)] \tag{3.23}$$

for all $x \in X$.

Similarly we can show easily that A is a unique additive mapping subject to (3.23). This completes the proof. □

From the main Theorem 3.1, we obtain the following corollary concerning the stability of the equation (1.4).

COROLLARY 3.2. *Let p, q, ϵ be real numbers such that $\epsilon \ge 0$, $p > 0$ and one of the three cases $p, q < 1$ or $1 < p, q < 2$ or $p, q > 2$ holds. Assume that a function $f : X \to Y$ satisfies the inequality*

$$||Df(x, y))|| \le \epsilon(||x||^p + ||y||^q) \tag{3.24}$$

for all $x, y \in X$ ($y \in X \backslash \{0\}$ if $q < 0$). Then there exist a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ which satisfy (1.4) and the inequalities

$$||f(x) - Q(x) - A(x) - f(0)|| \leq \frac{(|a| + |a|^p)\epsilon||x||^p}{2^p\Big||a|^2 - |a|^p\Big|} + \frac{(|a| + |a|^q)\epsilon||x||^q}{2^q\Big||a|^2 - |a|^q\Big|} + \frac{\epsilon||x||^q}{2\Big||a| - |a|^q\Big|},$$

$$||\frac{f(x) + f(-x)}{2} - Q(x) - f(0)|| \leq \frac{(|a| + |a|^p)\epsilon||x||^p}{2^p\Big||a|^2 - |a|^p\Big|} + \frac{(|a| + |a|^q)\epsilon||x||^q}{2^q\Big||a|^2 - |a|^q\Big|},$$

and

$$||\frac{f(x) - f(-x)}{2} - A(x)|| \leq \frac{\epsilon||x||^q}{2\Big||a| - |a|^q\Big|}$$

for all $x \in X$ ($x \in X \backslash \{0\}$ if $q < 0$).

Proof. Let $\varphi(x, y) := \epsilon(||x||^p + ||y||^q)$ for all $x, y \in X$.
If $1 < p, q < 2$, then we have the condition $(\mathcal{D})$,

$$\sum_{n=1}^{\infty} |a|^n \varphi(a^{-n}x, a^{-n}y) = \sum_{n=1}^{\infty} \epsilon\Big(\frac{|a|^n||x||^p}{|a|^{pn}} + \frac{|a|^n||x||^q}{|a|^{qn}}\Big) < \infty,$$

and the condition $(\mathcal{A})$,

$$\sum_{n=0}^{\infty} \frac{\varphi(a^n x, a^n y)}{|a|^{2n}} = \sum_{n=0}^{\infty} \frac{\epsilon(|a|^{np}||x||^p + |a|^{nq}||y||^q)}{|a|^{2n}} < \infty$$

for all $x, y \in X$. If $p, q > 2$, then we have the condition $(\mathcal{B})$ (hence $\mathcal{D}$),

$$\sum_{n=1}^{\infty} |a|^{2n} \varphi(a^{-n}x, a^{-n}y) = \sum_{n=1}^{\infty} \epsilon\Big(\frac{|a|^{2n}||x||^p}{|a|^{pn}} + \frac{|a|^{2n}||x||^q}{|a|^{qn}}\Big) < \infty$$

for all $x, y \in X$. If $p, q < 1$, then we have the condition $(\mathcal{C})$ (hence $\mathcal{A}$),

$$\sum_{n=0}^{\infty} \frac{\varphi(a^n x, a^n y)}{|a|^n} = \sum_{n=0}^{\infty} \frac{\epsilon(|a|^{np}||x||^p + |a|^{nq}||y||^q)}{|a|^n} < \infty$$

for all $x, y \in X$ ($x, y \in X \backslash \{0\}$ if $q < 0$).

Thus applying Theorem 3.1 for the three cases $p, q < 1$, $1 < p, q < 2$ and $p, q > 2$, we obtain easily the results. □

The following corollary is an immediate consequence of Theorem 3.1.

COROLLARY 3.3. *Assume that for some $\theta > 0$, a function $f : X \to Y$ satisfies the inequality*

$$||Df(x,y))|| \leq \theta \tag{3.25}$$

for all $x, y \in X$. Then there exist a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ which satisfy (1.4) and the inequalities

$$||f(x) - Q(x) - A(x) - f(0)|| \leq \frac{3\theta}{2(|a|-1)},$$
$$||\frac{f(x)+f(-x)}{2} - Q(x) - f(0)|| \leq \frac{\theta}{|a|-1},$$

and

$$||\frac{f(x)-f(-x)}{2} - A(x)|| \leq \frac{\theta}{2(|a|-1)}$$

for all $x \in X$

Proof. Putting $\varphi(x,y) := \theta$, we get immediately the results. □

Let X be a normed linear space and let $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be mappings such that

$$\varphi_0(\lambda) > 0, \quad \text{for all} \quad \lambda > 0$$
$$\varphi_0(|a|) < |a|, \quad a \neq 0, -1, 1 \in \mathbb{Z}$$
$$\varphi_0(|a|\lambda) \leq \varphi_0(|a|)\varphi_0(\lambda), \quad \text{for all} \quad \lambda > 0$$
$$H(\lambda t, \lambda s) \leq \varphi_0(\lambda)H(t,s), \quad \text{for all} \quad t, s \in \mathbb{R}_+, \quad \lambda > 0.$$

We consider the following in the next corollary. Let

$$\varphi(x,y) := H(||x||, ||y||).$$

Then

$$\begin{aligned}\varphi(a^k x, a^k y) &= H(|a|^k||x||, |a|^k||y||)\\ &\leq \varphi_0(|a|^k)H(||x||, ||y||)\\ &\leq (\varphi_0(|a|))^k H(||x||, ||y||),\end{aligned}$$

and according to $\varphi_0(|a|) < |a|$ we have

$$\begin{aligned}\Phi_1(x,y) &\leq \sum_{k=0}^{\infty} \frac{(\varphi_0(|a|))^k H(||x||, ||y||)}{|a|^{2k}}\\ &= \frac{|a|^2 H(||x||, ||y||)}{|a|^2 - \varphi_0(|a|)},\end{aligned}$$

and

$$\begin{aligned}\Psi_1(x,y) &\leq \sum_{k=0}^{\infty} \frac{(\varphi_0(|a|))^k H(||x||,||y||)}{|a|^k} \\ &= \frac{|a|H(||x||,||y||)}{|a|-\varphi_0(|a|)}.\end{aligned}$$

Hence, we see that the following corollary holds.

COROLLARY 3.4. *Assume that a function* $f : X \to Y$ *satisfies*

$$||Df(x,y)|| \leq H(||x||,||y||)$$

for all $x, y \in X$. *Then there exist a unique additive function* $A : X \to Y$ *and a unique quadratic function* $Q : X \to Y$ *which satisfy (1.4) and the inequalities*

$$\begin{aligned}&||f(x) - Q(x) - A(x) - f(0)|| \\ &\leq \frac{|a|\varphi_0(\frac{1}{2})H(||x||,||x||) + \varphi_0(\frac{1}{2})H(|a|||x||,||x||)}{|a|^2 - \varphi_0(|a|)} \\ &+ \frac{H(0,||x||)}{2(|a| - \varphi_0(|a|))},\end{aligned}$$

$$\begin{aligned}&||\frac{f(x)+f(-x)}{2} - Q(x) - f(0)|| \\ &\leq \frac{|a|\varphi_0(\frac{1}{2})H(||x||,||x||) + \varphi_0(\frac{1}{2})H(|a|||x||,||x||)}{|a|^2 - \varphi_0(|a|)},\end{aligned}$$

and

$$||\frac{f(x)-f(-x)}{2} - A(x)|| \leq \frac{H(0,||x||)}{2(|a| - \varphi_0(|a|))}$$

for all $x \in X$.

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