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# ON QUASI-EXACT SEQUENCES

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Abstract. The notion of $U$-exact sequence (or quasi-exact sequence) of modules was introduced by Davvaz and Parnian-Garamaleky as a generalization of exact sequences. In this paper, we prove further results about quasi-exact sequences. In particular, we give a generalization of Schanuel's Lemma. Also we obtain some relationship between quasi-exact sequences and superfluous (or essential) submodules.

## 1. Introduction

Let $R$ be a ring and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of $R$-modules. Then $\operatorname{Im} f = \operatorname{Ker} g$ $(= g^{-1}(\{0\}))$. It is a natural question to ask: what does happen if we substitute a submodule $U$ of $C$ for the trivial submodule $\{0\}$ in the above definition? In [4], Davvaz and Parnian-Garamaleky introduced the concept of quasi-exact sequences and answered the above question. They generalized some results from the standard case to the modified case. In [3], Davvaz and Shabani-Solt introduced a generalization of some notions in homological algebra. They defined the concepts of chain $U$-complex, $U$-homology, chain $(U,U)$-map, chain $(U,U')$-homotopy and $U$-functor. They gave a generalization of the Lambek lemma, snake lemma, connecting homomorphism and exact triangle and they established new basic properties of the $U$-homological algebra. In [2], Anvariyeh and Davvaz studied $U$-split sequences and established several connections between $U$-split sequences and projective modules.

In this paper, we prove further results about quasi-exact sequences. In particular, we prove an analogue of Schanuel's Lemma (see [1]) for quasi-exact sequences. Also we obtain some relationship between quasi-exact sequences and superfluous (or essential) submodules.

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## 2. Basic definitions and examples

We recall some basic definitions from [4] and then give some examples to indicate how quasi-exact sequences occur naturally.

DEFINITION 2.1. Given a ring $R$ and a sequence of $R$-modules $A \xrightarrow{f} B \xrightarrow{g} C$. We say that this sequence is *quasi-exact* if there exists a submodule $U$ of $C$ such that $\operatorname{Im} f = g^{-1}(U)$. In this case, we say that the sequence is $U$-exact (at $B$).

DEFINITION 2.2. A sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is said to be *short $U$-exact* if $f$ is injective, $g$ is surjective, and $\operatorname{Im} f = g^{-1}(U)$.

EXAMPLE 2.3. Let $U$ and $V$ be two submodules of $M$ such that $V \subseteq U \subseteq M$. Then the sequence $0 \longrightarrow U \xrightarrow{\subseteq} M \xrightarrow{\pi} \frac{M}{V} \longrightarrow 0$ is short $\frac{U}{V}$-exact, where $\pi$ is the natural epimorphism. For example, the sequence $0 \longrightarrow 2\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_4 \longrightarrow 0$ is a short $\mathbb{Z}_2$-exact sequence.

EXAMPLE 2.4. A sequence $0 \longrightarrow A \xrightarrow{f} B$ is $U$-exact at $A$ if and only if $f(x) \in B - U$ for all $0 \neq x \in A$.

EXAMPLE 2.5. Let $(A_i)_{i\in I}$, $(B_i)_{i\in I}$ and $(C_i)_{i\in I}$ be three families of $R$-modules indexed by the same set $I$, where the sequence $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ is $U_i$-exact at $B_i$, for any $i \in I$. Then the sequence

$$\prod_{i\in I} A_i \xrightarrow{\prod_{i\in I} f_i} \prod_{i\in I} B_i \xrightarrow{\prod_{i\in I} g_i} \prod_{i\in I} C_i$$

is $\prod_{i\in I} U_i$-exact at $\prod_{i\in I} B_i$.

Also the dual notion of $U$-exact sequence can be defined as follows:

DEFINITION 2.6. A sequence of $R$-modules $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be *$V$-coexact* ($V$ a submodule of $A$) if $f(V) = \operatorname{Ker} g$.

EXAMPLE 2.7. A sequence $A \xrightarrow{g} B \longrightarrow 0$ is $V$-coexact if and only if $g(V) = B$.

EXAMPLE 2.8. Consider the following sequences:

$$0 \longrightarrow A \xrightarrow{f_0} B \xrightarrow{f} C \longrightarrow 0,$$

$$0 \longrightarrow C \xrightarrow{g} D \xrightarrow{g_0} E \longrightarrow 0,$$

$$0 \longrightarrow A \xrightarrow{f_0} B \xrightarrow{gf} D \xrightarrow{g_0} E \longrightarrow 0.$$

Then

i) If the first sequence is $U_1$-exact and the second sequence is $U_2$-exact, then the third sequence is $g(U_1)$-exact at $B$ and $U_2$-exact at $D$.
ii) If the first sequence is $V_1$-coexact and the second sequence is $V_2$-coexact, then the third sequence is $V_1$-coexact at $B$ and $f^{-1}(V_2)$-coexact at $D$.

## 3. Some properties of quasi-exact sequences.

We now turn our attention to quasi exact sequences.

PROPOSITION 3.1. *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a* $U$*-exact sequence and let* $B_1 \subseteq B_2 \subseteq \dots \subseteq B_n \subseteq \dots$ *be a chain of submodules of* $B$*. If we put*

$$C_i = g(B_i),\ A_i = f^{-1}(f(A) \cap B_i),$$
$$U_i = g(g^{-1}(U) \cap B_i),\ f_i = f|_{A_i},\ g_i = g|_{B_i},$$

*then* $0 \longrightarrow A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ *is a* $U_i$*-exact sequence, for each* $i \geq 1$.

*Proof.* Suppose that $y \in \operatorname{Im} f_i$. Then there exists $x \in A_i$ such that $f_i(x) = f(x) = y \in f(A) \cap B_i = g^{-1}(U) \cap B_i$ which implies $y \in g_i^{-1}(U_i)$.
Conversely, suppose that $y \in g_i^{-1}(U_i)$. Then $y \in g^{-1}(U) \cap B_i = f(A) \cap B_i$. So there exists $a \in A$ such that $f(a) = y \in B_i$. Therefore $a \in A_i$, and hence $y \in \operatorname{Im} f_i$. □

PROPOSITION 3.2. *Let the short* $U$*-exact sequence of* $R$*-modules* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be given. If* $M$ *is a set of generators of* $A$ *and* $N$ *is a set of generators of* $C$*, then* $f(M) \cup N'$ *is a set of generators of* $B$*, where* $N'$ *is a subset of* $B$ *with the property* $g(N') = N$*.*

*Proof.* The proof is similar to the proof of Corollary 4 in [5]. □

The following corollary is a direct consequence of Proposition 3.2.

COROLLARY 3.3. *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a short* $U$*-exact sequence. If* $A$ *and* $C$ *are finitely generated, then so is* $B$*.*

Assume that $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a quasi-exact sequence. Next we give an example where $A$ and $C$ are semi-simple but $B$ is not.

EXAMPLE 3.4. According to Example 2.3, the sequence

$$0 \longrightarrow \frac{3\mathbb{Z}}{9\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{9\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}/9\mathbb{Z}}{6\mathbb{Z}/9\mathbb{Z}} \longrightarrow 0$$

is short $\frac{3\mathbb{Z}/9\mathbb{Z}}{6\mathbb{Z}/9\mathbb{Z}}$-exact. We know that $\mathbb{Z}_3$ and $\mathbb{Z}_6$ are semi-simple but $\mathbb{Z}_9$ is not.

A submodule $K$ of $M$ is superfluous (or small) in $M$, abbreviated $K \ll M$, in case for every submodule $L \leqslant M$, $K+L=M$ implies $L=M$. Dually, a submodule $K$ of $M$ is essential (or large) in $M$, abbreviated $K \leqslant_e M$, in case for every submodule $L \leqslant M$, $K \cap L = 0$ implies $L=0$.
An epimorphism $\phi : K \to M$ is said to be superfluous in case $\operatorname{Ker}\phi \ll K$. A monomorphism $\phi : K \to M$ is essential in case $\operatorname{Im}\phi \leqslant_e M$.

LEMMA 3.5.(Lemma 5.18, [1])
*i) If $\phi : M \to N$ is a homomorphism and $K \ll M$, then $\phi(K) \ll N$.*
*ii) Let $\phi : M \to N$ be superfluous and $S$ a submodule of $N$. Then $S \ll N$ if and only if $\phi^{-1}(S) \ll M$.*
*iii) Let $\phi : M \to N$ be essential and $K$ a submodule of $M$. Then $K \leqslant_e M$ if and only if $\phi(K) \leqslant_e \phi(M)$.*

LEMMA 3.6.(Proposition 5.16 and Lemma 5.18, [1]) *Let $M$ be an $R$-module. Then*

i) *If $S \leqslant L \ll M$ then $S \ll M$.*
ii) *If $S \leqslant L \leqslant M$ and $S \leqslant_e M$ then $L \leqslant_e M$.*

COROLLARY 3.7. *Consider the sequence* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$.

i) *If the sequene is $U$-exact with $U \leqslant_e C$, then $f$ is essential.*
ii) *If the sequence is $V$-coexact with $V \ll A$, then $g$ is superfluous.*

THEOREM 3.8. *Consider the following commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0. \end{array}$$

i) *Let the first row be $U$-exact with $U \ll C$, the second row $U'$-exact, and $\alpha$ an epimorphism. If $g$ is superfluous, so is $g'$.*
ii) *Let the first row be $V$-coexact, the second row $V'$-coexact with $V' \leqslant_e A'$, and $\gamma$ a monomorphism. If $f'$ is essential, so is $f$.*

*Proof.* i) Since $g$ is superfluous and $U \ll C$, by Lemma 3.5 we have $\operatorname{Im} f = g^{-1}(U) \ll B$. Hence $\beta f(A) \ll B'$, and so $f'(\alpha(A)) = f'(A') = g'^{-1}(U') \ll B'$. Since $\operatorname{Ker} g' \leqslant g'^{-1}(U')$, by Lemma 3.6 we get $\operatorname{Ker} g' \ll B'$. Therefore $g'$ is superfluous.

ii) Since $f'$ is essential and $V' \leqslant_e A$, $f'(V') = \operatorname{Ker} g' \leqslant_e B'$. Hence $\beta^{-1} g'^{-1}(\{0\}) \leqslant_e B$, and so $(g'\beta)^{-1}(\{0\}) = (\gamma g)^{-1}(\{0\}) \leqslant_e B$. Since $\gamma$ is one to one, $f(V) = g^{-1}(\{0\}) \leqslant_e B$. By Lemma 3.6, we have $\operatorname{Im} f \leqslant_e B$. Therefore $f$ is essential. □

Now we shall give a generalization of Schanuel's Lemma.

THEOREM 3.9. (Generalization of Schanuel's Lemma) *Let*

$$0 \longrightarrow B \longrightarrow P_1 \xrightarrow{f} A \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow P_2 \xrightarrow{g} A \longrightarrow 0$$

*be $U$-exact sequences of $R$-modules with $P_1$ and $P_2$ projective. Then*

$$B \oplus P_2 \cong C \oplus P_1.$$

*Proof.* Let $W$ be the $R$-submodule of $P_1 \oplus P_2$ given by

$$W = \{(x, y) |\ f(x) - g(y) \in U\}.$$

The projection map $\pi : W \to P_1$ is onto. Indeed, if $x \in P_1$, then, since $g$ is onto, there exists $y \in P_2$ with $f(x) = g(y)$, and so $f(x) - g(y) = 0 \in U$. Therefore $(x, y) \in W$ and $\pi(x, y) = x$. Now $P_1$ is projective, so $\pi$ must split and therefore $W \cong \operatorname{Ker}(\pi) \oplus P_1$. But

$$\begin{aligned} \operatorname{Ker}(\pi) &= \{(x, y) |\ \pi(x, y) = 0\} \\ &= \{(x, y) |\ f(x) - g(y) \in U \text{ and } x = 0\} \\ &= \{(0, y) |\ g(y) \in U\} \\ &\cong g^{-1}(U) \cong C, \end{aligned}$$

and thus $W \cong C \oplus P_1$. Similary, $W \cong B \oplus P_2$. □

Let $A$ and $A'$ be $R$-modules. We write $A \sim_p A'$ if there exist projective $R$-modules $P$ and $P'$ with $A \oplus P \cong A' \oplus P'$.

Next we state a slight generalization of the preceding result.

PROPOSITION 3.10. *Suppose we are given the sequences*

$$0 \longrightarrow B \longrightarrow P \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow B' \longrightarrow P' \longrightarrow A' \longrightarrow 0,$$

*where the first sequence is $U$-exact, the second sequence is $U'$-exact, and $P$, $P'$ are projective. If $A \sim_p A'$, then $B \sim_p B'$.*

*Proof.* Since $A \sim_p A'$, we have $A \oplus P_1 \cong A' \oplus P_1'$ for suitable projective modules $P_1$ and $P_1'$. Thus we obtain the $U$-exact sequence

$$0 \longrightarrow B \longrightarrow P \oplus P_1 \longrightarrow A \oplus P_1 \longrightarrow 0$$

and the $U'$-exact sequence

$$0 \longrightarrow B' \longrightarrow P' \oplus P_1' \longrightarrow A' \oplus P_1' \longrightarrow 0$$

and since $A \oplus P_1 \cong A' \oplus P_1'$, Theorem 3.9 yields $B \oplus (P' \oplus P_1') \cong B' \oplus (P \oplus P_1)$. Hence $B \sim_p B'$ as required. □

We conclude this section with the following theorem.

THEOREM 3.11. (Generalization of the dual of Schanuel's Lemma). *Let*

$$0 \longrightarrow A \xrightarrow{f} Q_1 \xrightarrow{h} B \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{g} Q_2 \longrightarrow C \longrightarrow 0$$

*be $V$-coexact sequences of $R$-modules with $Q_1$ and $Q_2$ injective. Then*

$$B \oplus Q_2 \cong C \oplus Q_1.$$

*Proof.* Since $V \cong f(V) \subseteq Q_1$ and $V \cong g(V) \subseteq Q_2$, we assume that $V \subseteq Q_1$ and $V \subseteq Q_2$, and so $\overline{V} = \{(a,a) | a \in V\} \subseteq Q_1 \oplus Q_2$. Set $W := \frac{Q_1 \oplus Q_2}{\overline{V}}$. The map $V \oplus Q_2 \to Q_2$ which defined by $(a,q) \to -a + q$ is onto with kernel $\overline{V}$. Therefore $Q_2 \cong \frac{V \oplus Q_2}{\overline{V}} = V'$ and $\frac{W}{V'} \cong \frac{Q_1 \oplus Q_2}{V \oplus Q_2} \cong \frac{Q_1}{V}$. Since $V \cong kerh$ and $h$ is onto, $\frac{Q_1}{V} \cong B$. Therefore $\frac{W}{V'} \cong B$. Since $V'$ is injective and $V' \leqslant W$, we get $W \cong B \oplus Q_2$. Similarly, $W \cong C \oplus Q_1$. □

Let $A$ and $A'$ be $R$-modules. We write $A \sim_i A'$ if there exist injective $R$-modules $Q$ and $Q'$ with $A \oplus Q \cong A' \oplus Q'$.

PROPOSITION 3.12. *Suppose we are given the sequences*

$$0 \longrightarrow A \longrightarrow Q \longrightarrow B \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow Q' \longrightarrow B' \longrightarrow 0,$$

*where the first sequence is $V-$coexact, the second sequence is $V'-$coexact, and $Q$, $Q'$ are injective. If $A \sim_i A'$, then $B \sim_i B'$.*

*Proof.* The proof is similar to the proof of Proposition 3.10. □

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