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EVERY DEFINABLE C^r MANIFOLD IS AFFINE

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Abstract. Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$ of the field of real numbers. We prove that if $2 \leq r < \infty$, then every n-dimensional definable C^r manifold is definably C^r imbeddable into $\mathbb{R}^{2n+1}$. Moreover we prove that if $1 < s < r < \infty$, then every definable C^s manifold admits a unique definable C^r manifold structure up to definable C^r diffeomorphism.

1. Introduction

M. Shiota proved that if $0 < r < \infty$, then every C^r Nash manifold is affine ([7]). Let $\mathcal{M} = (\mathbb{R}, +, \cdot, >, \dots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, >)$ of $\mathbb{R}$. Note that if $\mathcal{M} = \mathcal{R}$, then a definable C^r manifold is a C^r Nash manifold. Definable C^r categories based on $\mathcal{M}$ are generalizations of the C^r Nash category.

General references on o-minimal structures are [1], [2], see also [8]. The term "definable" means "definable with parameters in $\mathcal{M}$" and any manifold in this paper does not have boundary.

If r is a non-negative integer and $\mathcal{M}$ is polynomially bounded (resp. exponentially bounded), then every definable C^r manifold is affine ([6]) (resp. [4]). We have the following theorem as a generalization of these results.

Theorem 1.1. *If $2 \leq r < \infty$, then every n-dimensional definable C^r manifold X is definably C^r imbeddable into $\mathbb{R}^{2n+1}$.*

The above theorem is the definable version of Whitney's imbedding theorem (e.g. 2.14, [3]). Even in the Nash category (i.e. $\mathcal{M} = \mathcal{R}$), we cannot take $r = \infty$ in Theorem 1.1 ([7]).

By Theorem 1.1 and 1.3 in [4], we have the following theorem.

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THEOREM 1.2. *Let $1 < s < r < \infty$, then every definable C^s manifold admits a unique definable C^r manifold structure up to definable C^r diffeomorphism.*

2. Proof of result

PROPOSITION 2.1. *Let X be an affine definable C^r manifold, V a definable subset closed in X and r a non-negative integer. Then there exists a non-negative definable C^r function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = V$.*

Proof. By 3.2 in [6], X is definably C^r diffeomorphic to a definable C^r submanifold of some $\mathbb{R}^l$ which is closed in $\mathbb{R}^l$. We identify X with its image. Thus V is closed in $\mathbb{R}^l$. By [2], there exists a definable C^r function $\phi : \mathbb{R}^l \to \mathbb{R}$ such that $\phi^{-1}(0) = V$. Hence restricting ϕ, we have a definable C^r function $\psi : X \to \mathbb{R}$ such that $\psi^{-1}(0) = V$. Therefore $f := \psi^2 : X \to \mathbb{R}$ is the required function. □

The following is a definable C^r partition of unity.

PROPOSITION 2.2. *Let r be a non-negative integer and $\{U_i\}_{i=1}^k$ a definable open covering of a definable C^r manifold X. Then there exist definable C^r functions $\lambda_i : X \to \mathbb{R}$ $(1 \le i \le k)$ such that $0 \le \lambda_i \le 1$, $\operatorname{supp} \lambda_i \subset U_i$ and $\sum_{i=1}^k \lambda_i = 1$.*

If X is affine, then Proposition 2.2 is known in 4.8 ([5]).

Proof. We now prove that there exists a definable open covering $\{V_i\}_{i=1}^k$ of X such that $\overline{V_i} \subset U_i$, $(1 \le i \le k)$, where $\overline{V_i}$ denotes the closure of V_i in X.

We proceed by induction on k. If $k = 1$, then there is nothing to prove. Assume that there exists a definable open covering $\{V_i\}_{i=1}^{k-1} \cup \{U_k\}$ of X such that $\overline{V_i} \subset U_i$, $(1 \le i \le k-1)$.

Let $X_{k-1} := \bigcup_{i=1}^{k-1} V_i$. By the inductive hypothesis, there exists a definable open covering $\{W_i\}_{i=1}^{k-1}$ of X_{k-1} such that $cl\ W_i \subset V_i$, where $cl\ W_i$ means the closure of W_i in X_{k-1}.

We may assume that U_k is affine. Let $Z_k := U_k \cap \bigcup_{i=1}^{k-1} V_i$ and $Cl\ Z_k$ denote the closure of Z_k in U_k. By Proposition 2.1, there exists a non-negative definable C^r function $\phi_k : U_k \to \mathbb{R}$ such that $\phi_k^{-1}(0) = Cl\ Z_k$. Since $cl\ W_1 \subset V_1$, ϕ_k is extensible to a non-negative definable C^r function $\phi_k^1 : U_k \cup W_1 \to \mathbb{R}$ such that $\phi_k^{1^{-1}}(0) = Cl\ Z_k \cup W_1$. Inductively, we have a non-negative definable C^r function $\phi : X \to \mathbb{R}$ such that $\phi^{-1}(0) = Cl\ Z_k \cup W_1 \cdots \cup W_{k-1}$. Let $V_k := \{x \in U_k | \phi(x) > 0\}$. Then

$V_k = \{x \in X | \phi(x) > 0\}$, $\overline{V_k} \subset U_k$ and $\{V_i\}_{i=1}^k$ is the required definable open covering of X.

By Proposition 2.1, we have a non-negative definable C^r function $\mu_i : U_i \to \mathbb{R}$ such that $\mu_i^{-1}(0) = U_i - V_i$. Thus μ_i is extensible to a non-negative definable C^r function $\mu_i' : X \to \mathbb{R}$ such that ${\mu_i'}^{-1}(0) = X - V_i$. Therefore $\lambda_i := \mu_i' / \sum_{i=1}^k \mu_i'$ is the required definable C^r partition of unity. □

Proof of Theorem 1.1. Let $\{\phi_i : U_i \to \mathbb{R}^n\}_{i=1}^k$ be a definable C^r atlas of X. By Proposition 2.2, we have definable C^r functions $\lambda_i : X \to \mathbb{R}$, $(1 \le i \le k)$ such that $0 \le \lambda_i \le 1$, $\operatorname{supp} \lambda_i \subset U$ and $\sum_{i=1}^k \lambda_i = 1$. Thus the map $F : X \to \mathbb{R}^{nk} \times \mathbb{R}^k$ defined by $F(x) = (\lambda_1(x)\phi_1(x), \dots \lambda_k(x)\phi_k(x), \lambda_1(x), \dots, \lambda_k(x))$ is a definable C^r imbedding. Hence X is affine. Thus it is compact or compactifiable by 1.2 [5]. Hence we may assume that X is affine and compact at the beginning. A similar argument of the proof of 1.4 ([9]), every definable C^r map $f : X \to \mathbb{R}^{2n+1}$ can be approximated in the C^r topology by an injective definable C^r immersion $h : X \to \mathbb{R}^{2n+1}$. Since X is compact, h is the required definable C^r imbedding. □

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